Mathematical Methods for the Physical Sciences

Two Semester Course

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PUBLISHED ONLINE

Special thanks to Mathias Legrand (legrand.mathias@gmail.com) with modifications by Vel(vel@latextemplates.com) for creating and modifying the template used for these lecture notes. Also, thank you to the students in the Mathematical Methods I-II class at Kent State University in the 2015-2016 academic year including K. Bittinger, R. Dovishaw, T. Dubensky, M. Grose, K. Khanal, J. Krusinski, E. McMasters, J. Paltani, J. Sobieski, J. Taylor, C. Zickel, B. Zimmerman.

First published online, February 2016

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1. Fundamentals of Complex Numbers

1.1 Introduction

The two course sequence Mathematical Methods in the Physical Sciences I and II are designed to condense many courses in higher level mathematics into the essential information needed to study upper level physics undergraduate courses. Our main focus is to develop mathematical intuition for solving real world problems while developing our tool box of useful methods. Topics in this course are derived from five principle subjects in Mathematics

(i) **Complex Numbers** (Math 42048, Boas Ch. 2) $\rightarrow$ Quantum Mechanics

(ii) **Linear Algebra** (Math 21001, Boas Ch. 3) $\rightarrow$ Transformations, Change of Coor., Stability

(iii) **Multivariable Calculus** (Math 22005, Boas Ch. 4-6) $\rightarrow$ Forces, Inertia, Volume, Area

(iv) **Introduction to Ordinary Differential Equations** (Math 32044, Boas Ch. 7-8) $\rightarrow$ Particle Motion, Dynamics

(v) **Introduction to Partial Differential Equations** (Math 42045, Boas Ch. 13) $\rightarrow$ Signal Analysis, Heat Conduction, Waves, Equilibrium Physics

Each class individually goes deeper into the subject, but we will cover the basic tools needed to handle problems arising in physics, materials sciences, and the life sciences. Your upper level curses will introduce the physical motivation for the problems, but here we will develop the solution methods for solving those problems. In Math Methods 1 we will cover Chapters 2 - 5 or the first half of our list.

Recall the first place you most likely saw a complex number: solving a quadratic equation $ax^2+bx+c=0$ with the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

When the so-called *discriminant*, $b^2 - 4ac$ is negative the square root produces an imaginary number. For example if one wants to solve $x^2 + 1 = 0$ the quadratic formula gives $\pm\sqrt{(-1)} = \pm i$. A quadratic equation must have two roots or solutions and the imaginary number i was introduced to handle such cases.

Consider some easy examples for how to handle negative square roots.

■ **Example 1.1** i) $\sqrt{-64} = 8\sqrt{-1} = 8i$
ii) $\sqrt{-5} = \sqrt{5}\sqrt{-1} = \sqrt{5}i$
iii) Powers of i: $i = \sqrt{-1}$, $i^2 = \sqrt{-1}\sqrt{-1} = -1$, $i^3 = i^2 i = -i, i^4 = i^2 i^2 = 1$. Any other power of i can be found by dividing the exponent by 4 and only considering the remainder $i^5 3 = i^{4(13)} i^1 = i$. ■

Just as a refresher solve the following quadratic equation using the quadratic formula

■ **Example 1.2** Solve $x^2 - x + 1 = 0$. The quadratic formula gives

$$x = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

■

We have this built up intuition from the past, but what exactly is a complex number. Let's make an analogy with negative numbers (thanks Kalid Azad for the insight). Imagine a time before negative numbers were accepted around the 1700's in Europe. Given two numbers 7 and 8 we can easily write $8 - 7 = 1$. Starting with 8 sheep, if I give you 7 I will only have one left. What about $7 - 8$? *How can I have less than nothing?* The problem is trying to think about this problem with concrete objects. The easiest way to understand this is with money. If I owe you \$50 and I am paid only \$10 to teach this course, then at the end of the day I have lost \$40 hence the negative sign. In this case -40 represents a debt or something I owe. The negative sign was invented to keep track of which direction I am (positive I earned money or negative I owe money).

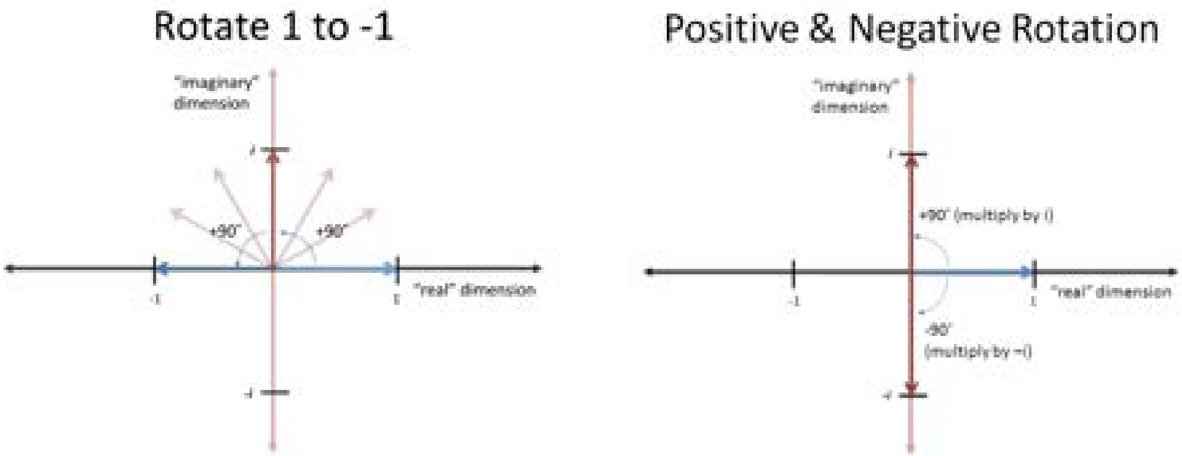


Figure 1.1: Thanks to Kalid Azad for the figures

Now, how to we handle a square root of a number less than zero? Suppose we want to solve $x^2 = 9$. This mean finding a number such that $1 \times x \times x = 9$. What can I apply to 1 twice so that I receive 9? The answers are 3 and -3. We can scale 1 by 3 and then scale by 3 again or we can scale 1 by negative 3 (scale by 3 and reflect it to negative side) and do the same again.

Now, try to solve $x^2 = -1$, or $1 \times x \times x = -1$. What can we apply twice to turn 1 into -1? We cannot multiply by a positive or negative number twice, because the result will be negative. What if we rotated it by 90° (see Figure'1.1)? This works, but what does it mean. **Summary:**

Fun Fact	Negative Numbers (-x)	Complex Numbers (a +bi)
Invented to answer	"What is 3 – 4?"	"What is sqrt(-1)?"
Strange because...	*How can you have less than nothing?*	*How can you take the square root of less than nothing?*
Intuitive meaning	"Opposite"	"Rotation"
Considered absurd until	1700s	Today ☺
Multiplication cycle [& general pattern]	`1, -1, 1, -1...` `X, -X, X, -X...`	`1, i, -1, -i...` `X, Y, -X, -Y...`
Use in coordinates	Go backwards from origin	Rotate around origin
Measure size with	Absolute value $\sqrt{(-x)^2}$	Pythagorean Theorem $\sqrt{a^2+b^2}$

Figure 1.2: Thanks to Kalid Azad for the figures

1) i can be thought of as a "new dimension" to measure a number

2) Multiplying by i is a rotation of a number by 90° counter-clockwise

3) Multiplying by $-i$ is a rotation of a number by 90° clockwise

Complex numbers are very similar to real numbers. Can we make sense of arithmetic operations of complex number (e.g., $+,-,\times,\div$)? What about functions of complex numbers such as e^i or $\sin(iz)$ and $\cos(iz)$?. Also, in the upcoming sections we will consider graphing, power series of complex functions/radius of convergence, and distances or magnitudes of complex numbers.

1.2 Real and Imaginary Parts of a Complex Number

So according to the last section numbers can be two-dimensional! Can a number be both real and imaginary? YES! Take for example the solution to $x^2 - x + \frac{1}{2} = 0$, which is $1+i$. This number has both a real part 1 and a purely imaginary part i, but together they form a complex number of the form $x+yi$.

Definition 1.2.1 A **complex number** is any number of the form $z = x + iy$ where x and y are real numbers. x is called the *real part* and y is called the *imaginary part*.

R Notice the imaginary part y of a complex number $z = x + iy$ is in fact real! It is the real number coefficient for i.

■ **Example 1.3** Find the real and imaginary parts of:

i) $5+6i$, $\text{Re}\{5+6i\} = 5$ and $\text{Im}\{5+6i\} = 6$.

ii) $-1+3i$, $\text{Re}\{-1+3i\} = -1$ and $\text{Im}\{-1+3i\} = 3$.

iii) $6i$, $\text{Re}\{6i\} = 0$ and $\text{Im}\{6i\} = 6$.

iii) 7, $\text{Re}\{7\} = 7$ and $\text{Im}\{7\} = 0$. ■

All real numbers are complex numbers with zero imaginary part. Therefore the real numbers are a subset of the complex numbers; however, there is a more useful observation. All complex numbers can be written as $z = x + yi$. If we associate this with the point (x, y) in two-dimensional space, then we can plot complex numbers (see Figure 1.3.2). In the next section we will investigate graphing complex numbers further.

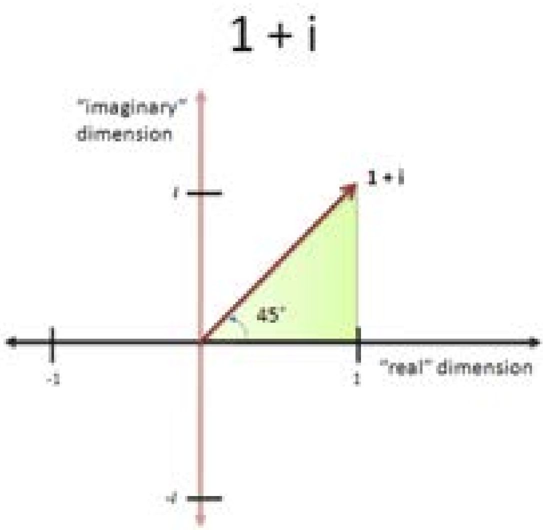
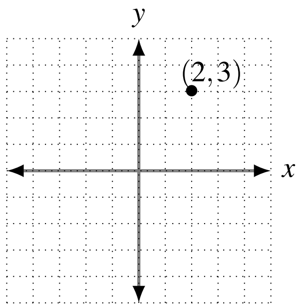


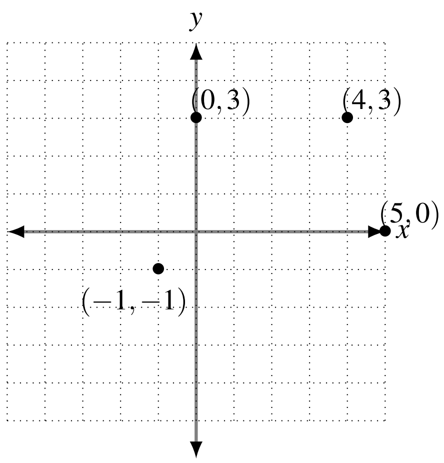
Figure 1.3: Thanks to Kalid Azad for the figures

1.3 The Complex Plane

As mentioned briefly before, the two-dimensional real space can be thought of as equivalent to the complex plane, $\mathbb{R}^2 \simeq \mathbb{C}$. Any complex number can be associated to a point we can plot in the xy-plane with a traditional Cartesian coordinate system. Consider the complex number $z = 2 + 3i \rightarrow (2,3)$



■ **Example 1.4** Plot $4+3i$, $3i$, 5, -1-i



■

Recall from calculus another form on coordinates in two dimensions, *polar coordinates* $(x,y) \mapsto (r,\theta)$. Can we use the same idea to identify complex numbers with their associated polar coordinates? Yes!

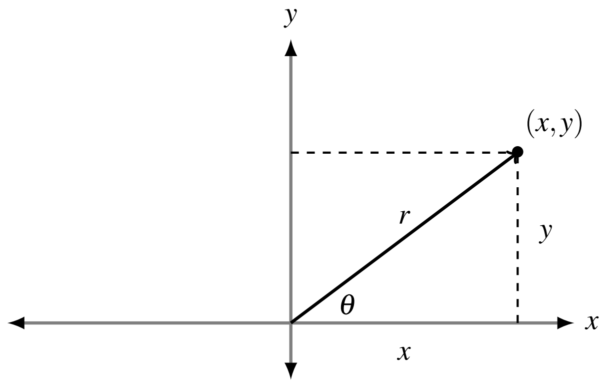
Definition 1.3.1 *(Polar Coordinates of Complex Numbers)* Any complex number $z = x + iy$ can be written in polar form using the same relations from two dimensional Cartesian coordinates

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}(y/x)$$

or

$$x = r\cos(\theta)$$
$$y = r\sin(\theta).$$

Thus, $z = x + iy = rcos(\theta) + i\sin(\theta) = r[\cos(\theta) + i\sin(\theta)]$. **NOTE: That all the quantities involved are real (e.g., x, y, r, θ)!**



We can actually simplify this expression further with the help of Euler's Identity

Definition 1.3.2 *(Euler's Identity)* The polar form of a complex number can be written as

$$e^{i\theta} = \cos(\theta) + i\sin(\theta). \tag{1.1}$$

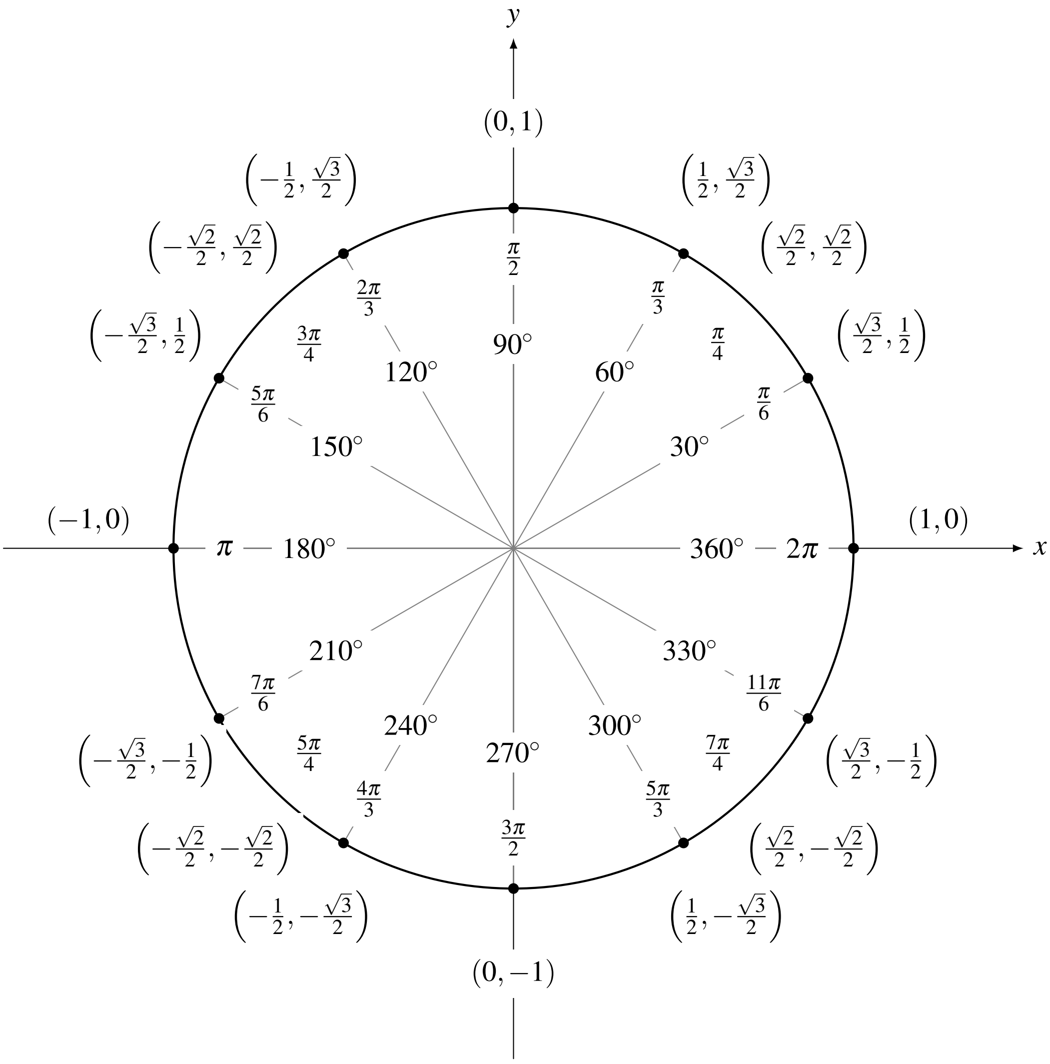
This will be taken as a fact for now and will be shown explicitly in a few sections when we study complex power series.

R Traditionally if asked for the polar form of a complex number z the expectation is that it is written $z = re^{i\theta}$.

Key idea: For solving a lot of problems with complex numbers the main task is to identify whether it would be easier to tackle the problem in Cartesian (x, y) or polar (r, θ) coordinates.

1.3.1 Review of Unit Circle in Radians

You need to be very familiar with the standard right triangles, 45-45-90 and 30-60-90 in each quadrant in order to effectively use the polar form of a complex number. Even though the unit circle is familiar we must also be able to scale to any size right triangle of these two forms.



■ **Example 1.5** For each of the following find the polar form and plot the result in two-dimensions.

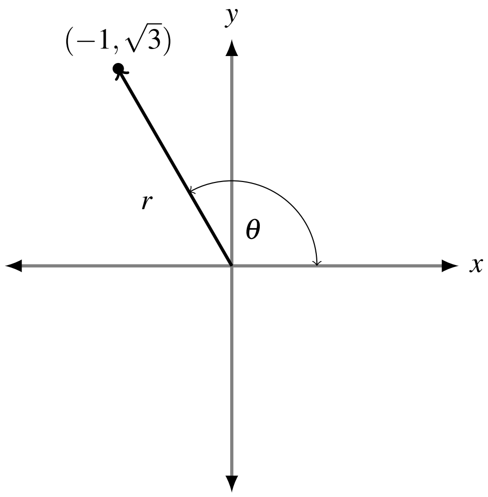
i) $z = -1 + \sqrt{3}i$

Step 1: Find $r = \sqrt{x^2 + y^2} = \sqrt{1+3} = 2$.

Step 2: Find $\theta = \tan^{-1}(\sqrt{3}/1)$. Recall that tangent is opposite over adjacent. Thus, we need a triangle where the opposite side has length $\sqrt{3}$ and the adjacent side is along -1. If $x < 0$ and $y > 0$ we are in quadrant II with a 30-60-90 right triangle. Therefore $\theta = \frac{2\pi}{3}$.

Step 3: Write in polar form $z = 2e^{i\frac{2\pi}{3}}$.

Step 4: Plot the result!



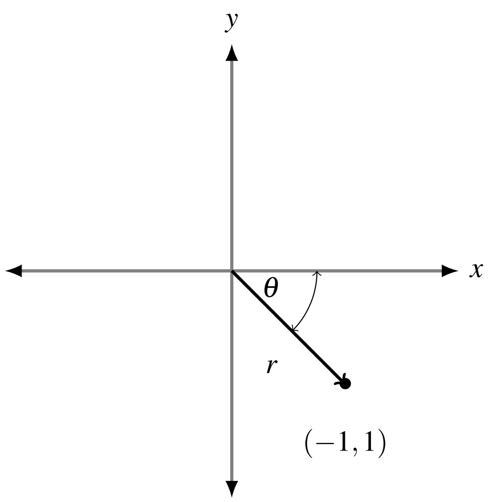
ii) $z = 1 - i$

Step 1: Find $r = \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2}$.

Step 2: Find $\theta = \tan^{-1}(-1/1)$. The tangent is opposite over adjacent. Thus, we need a triangle where the opposite side is along -1 and the adjacent side is along 1. If $x > 0$ and $y < 0$ we are in quadrant IV with a 45-45-90 right triangle. Therefore $\theta = \frac{3\pi}{4}$ or $-\frac{\pi}{4}$. Note that it is important to observe the sign of both x and y to be in the correct quadrant.

Step 3: Write in polar form $z = \sqrt{2} e^{i\frac{7\pi}{4}}$.

Step 4: Plot the result!



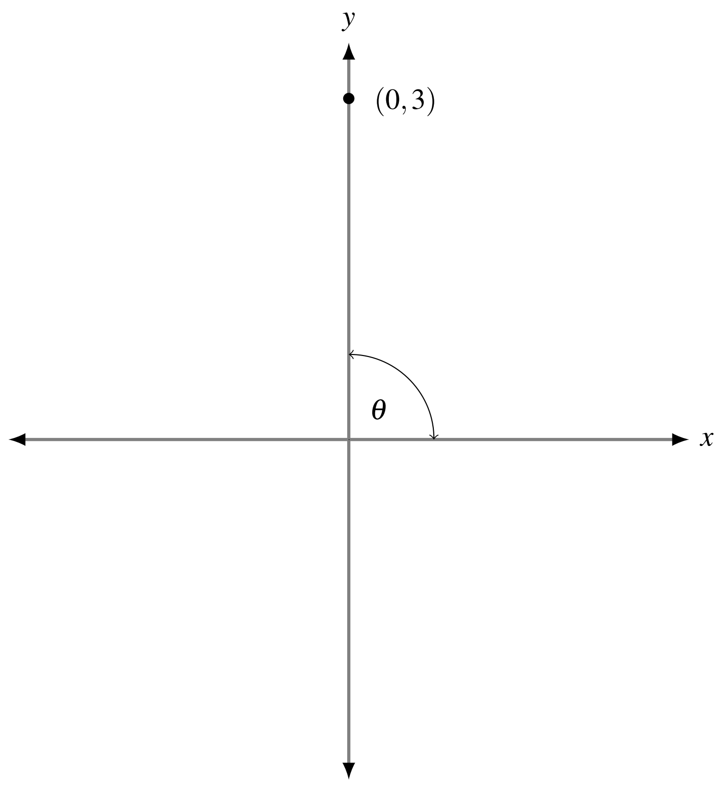
iii) $z = 3i$

Step 1: Find $r = \sqrt{x^2 + y^2} = \sqrt{0+9} = 3$.

Step 2: Find $\theta = \tan^{-1}(5/0)$. We are looking for an angle whose tangent is ∞. Recall that tangent is sine over cosine. the tangent is infinite if $\cos(\theta) = 0$ or $\theta = \pi/2$ or $\theta = -\pi/2$. Since $y > 0$, then $\theta = \pi/2$.

Step 3: Write in polar form $z = 3e^{i\frac{\pi}{2}}$.

Step 4: Plot the result!



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1.3.2 Going Deeper: Understanding Euler's Identity

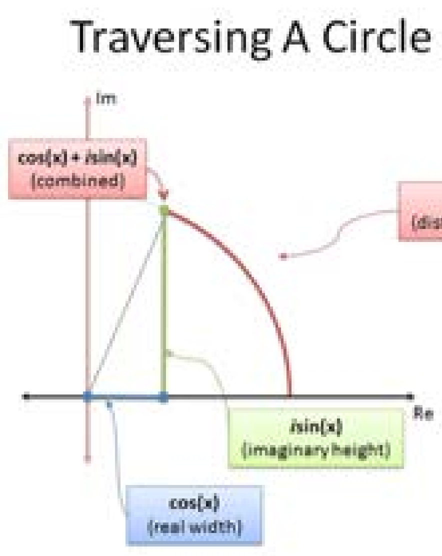


Figure 1.4: Thanks to Kalid Azad for the figures.

Euler's Identity $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ is a formula, which explains how to move around the unit circle. Consider a point confined to the unit circle traveling x radians. The horizontal distance traveled is $\cos(x)$ and the vertical distance traveled is $\sin(x)$. To take these two coordinates and combine them into one number we make it complex! $z = \cos(x) + i\sin(y)$. Thus, the right side of Euler's formula/identity describes motion on a circle.

The left-hand side of Euler's Identity contains the exponential function e. In real number the function e^x arises in problems involving growth or decay at a fast rate. Here what do we mean by imaginary growth ($e^{i\theta}$)?!? Imaginary growth is different than normal exponential growth. The growth is in a different direction, instead of going forward we growth along the imaginary axis (y-direction or 90°). Instead of speeding up or slowing down a point begins to rotate (multiplying a number by i does not change its magnitude it only rotates it).

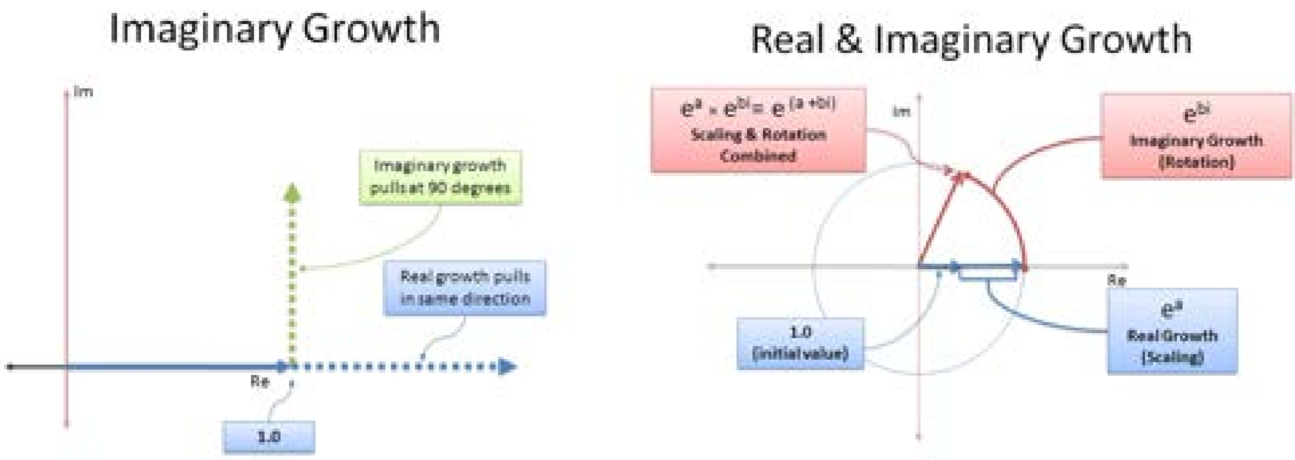


Figure 1.5: Thanks to Kalid Azad for the figures.

Thinking Question In real numbers the exponential function keeps growing larger and larger, so in the case of "imaginary growth" should we rotate faster and faster?

Since we are constrained to the unit circle instead of growing larger and larger, a point moves further along the circle. For example if we compare $e^{i\theta}$ and $e^{2i\theta}$. The magnitude does not change (still 1), but we rotate twice as far (or travel twice as long if θ is thought of as time).

Interesting Case: Complex Growth What if the growth rate is complex e^{x+iy}?

The real part e^x grows like normal while the imaginary part e^{iy} rotates. Thus, one can expect a spiral shape. This will be seen later when finding complex solutions to equations of motion!!

1.4 Terminology and Notation

In this class we will always use i to denote the complex (pure imaginary) number $i := \sqrt{-1}$. Be aware that in many physics textbooks j is also used. Often this is seen when studying electricity where current is denoted as i to avoid confusion. An additional point regarding notation is that a complex number $z = x + iy$ is one number so when labeling points in the complex plane usually a single letter is used (e.g., A, B, P etc.).

Recall from the last lecture the polar from of a complex number using Euler's identity.

$$z = x + iy = r(\cos(\theta) + i\sin(\theta) = re^{i\theta} \tag{1.2}$$

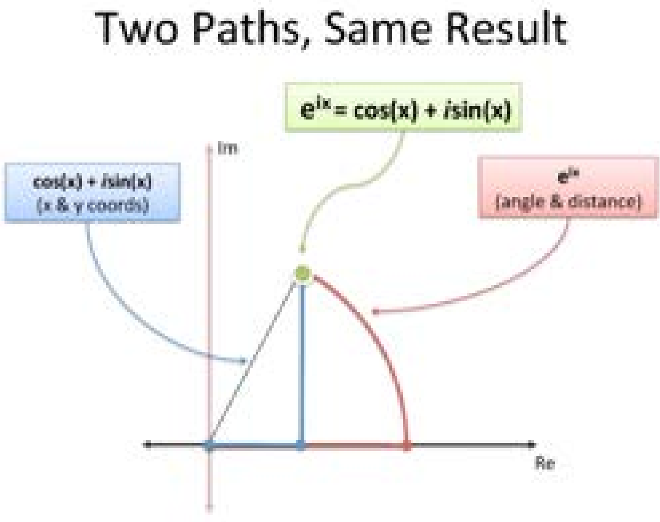


Figure 1.6: Thanks to Kalid Azad for the figures.

where the real part of z, $\mathrm{Re}\{z\} = x$, and the imaginary part of z, $\mathrm{Im}\{z\} = y$. In addition, the magnitude or length associated with the complex number z is $r = |z| = \sqrt{x^2+y^2}$ and the angle $\theta = \tan^{-1}(y/x)$.

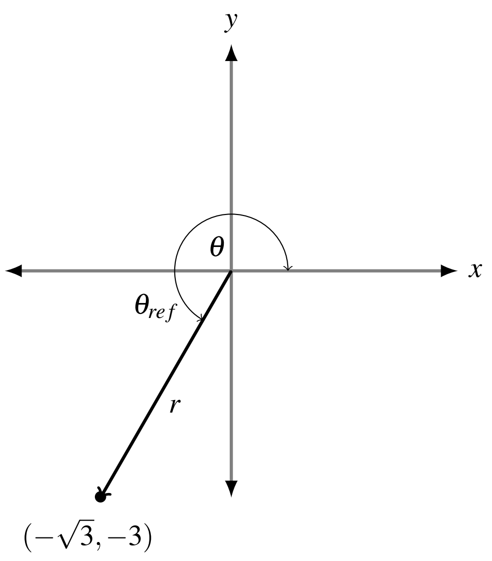
■ **Example 1.6** Write $z = -\sqrt{3} - 3i$ in polar form and plot it.

Step 1: Find $r = \sqrt{x^2+y^2} = \sqrt{3+9} = 2\sqrt{3}$.

Step 2: Find $\theta = \tan^{-1}(-3/-\sqrt{3}) = \tan^{-1}(-\sqrt{3}/-1)$. Recall that tangent is opposite over adjacent. Thus, we need a triangle where the opposite side along $-\sqrt{3}$ and the adjacent side is along -1. If $x < 0$ and $y < 0$ we are in quadrant III with a 30-60-90 right triangle. Therefore $\theta = \frac{4\pi}{3}$.

Step 3: Write in polar form $z = 2\sqrt{3}e^{i\frac{4\pi}{3}}$.

Step 4: Plot the result!



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Note that due to periodicity the true answer for the angle is $\theta = \frac{4\pi}{3} + 2\pi n$ where n is an integer. The first component, $\theta_{principle} := \frac{4\pi}{3}$, is known as the principle angle and must be between the standard interval of $0 \leq \theta_p < 2\pi$. Another angle of importance is the reference angle, $0 \leq \theta_{ref} \leq \frac{\pi}{2}$, which gives the magnitudes of the sides of the 30-60-90 or 45-45-90 right triangle. Observe that the reference angle has nothing to do with the sign of each side of the triangle, because it is independent of the quadrant it is in.

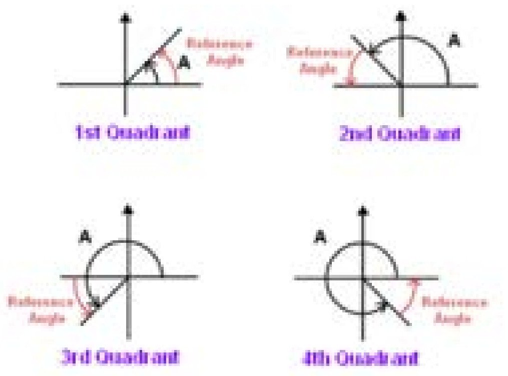


Figure 1.7: Difference between principle angle θ_p (black) and the reference angle θ_{ref} (red).

R When working with complex numbers make sure that the angle θ you find is in the same quadrant as the complex number itself.

1.4.1 Complex Conjugation

Consider two complex numbers $z_1 = x + iy$ and $z_2 = x - iy$. The only difference is the sign of the imaginary part, $\pm iy$. These two complex numbers are known as complex conjugates. Specifically, z_2 is the complex conjugate of z_1 denoted by $z_2 = \bar{z}_1$. Where the bar indicates “complex conjugate" (in some textbooks the notation of a $\star$ is used, $z_2 = z_1^\star$. Given any complex number we can always find its conjugate by changing the sign of the imaginary part. You may have come across these pairs before when solving quadratic equations, because complex solutions to equations always come as conjugate pairs. In other words, if $z = 2+3i$ is a solution, then $z = 2-3i$ must also be a solution.

■ **Example 1.7** Find the complex conjugate of each of the following complex numbers:

i) $z = 1+i$, then $\bar{z} = 1-i$

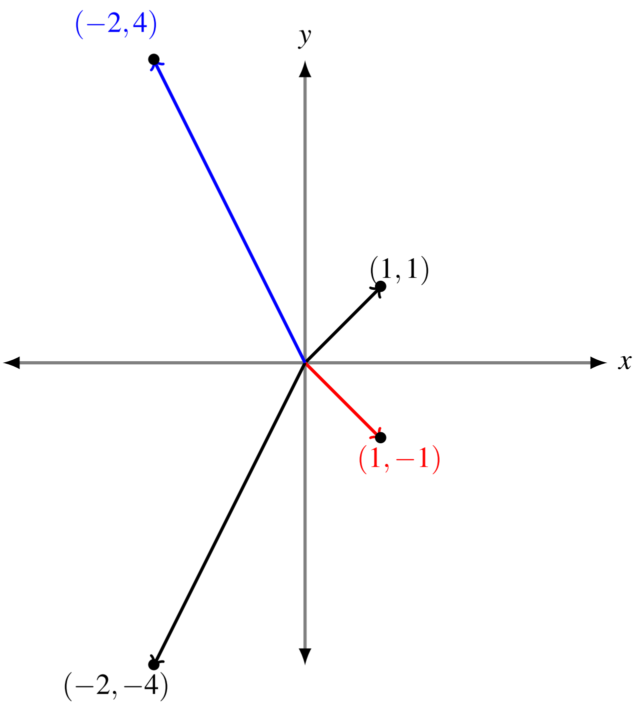
ii) $z = -2-4i$, then $\bar{z} = -3+4i$

iii) $z = -5i$, then $\bar{z} = 5i$

iv) $z = -3$, then $\bar{z} = -3$

v) $z = 0$, then $\bar{z} = 0$ ■

It is easy to blindly remember to change the sign of the imaginary part, but let's look at a pair of complex conjugates plotted on the same coordinate plane to see if there is any relationship. Lets look back at Example 3 i) and ii) .



The complex conjugate of a number is just its reflection across the x-axis (real axis).

Thinking Question: A complex conjugate is just a reflection across the x-axis so the change in (x,y) coordinates is simple $y \mapsto -y$. How does a complex conjugate effect the polar form of a complex number?

Answer: The magnitude of a complex number and its conjugate are identical so r remains the same. However, $\theta \mapsto -\theta$. What does this mean in terms of the principle angle and the reference angle? The reference angle θ_{ref} remains unchanged, but the principle angle changes sign. So if $\theta = \frac{\pi}{4}$, then $\theta_{\bar{z}} = -\frac{\pi}{4}$ or $\frac{7\pi}{4}$.

We can also directly see this from the polar form of a complex number

$$\bar{z} = \overline{x+iy} = \overline{r[\cos(\theta)+i\sin(\theta)]} = r[\cos(-\theta)+i\sin(-\theta)] = r[\cos(\theta)-i\sin(\theta)] = x-iy. \quad (1.3)$$

1.5 Complex Algebra

With real numbers we can perform various algebraic operations to combine them into something new. These include, but are not limited to

i) **Basic Operations:** Addition $+$, Subtraction $-$, Multiplication $\times$, and Division $\div$

ii) **Magnitude and Distance** $|\cdot|$, $v = |\mathbf{v}|$, or $|a-b|$.

iii) **Solving Equations** $4x = 6$

Now we will consider the complex analogue of each of these as well as some physical applications for complex numbers.

1.5.1 Simplifying to Standard Form $x+iy$

As we have seen before complex numbers can be written in two equivalent forms $z = x+iy = re^{i\theta}$. The first form is referred to as *standard form* and will be useful for the basic operations.

Addition of Complex Numbers

Definition 1.5.1 Given two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, their sum $z_1 + z_2$ is defined as

$$(x_1+iy_1)+(x_2+iy_2) = (x_1+x_2)+i(y_1+y_2) \quad (1.4)$$

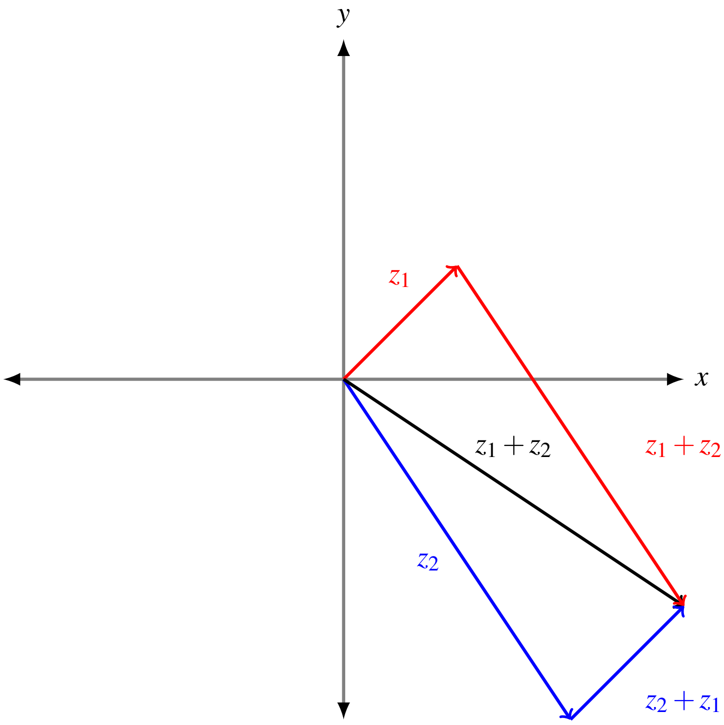
Just add the real parts and add the imaginary parts.

■ **Example 1.8** i) $(1+i)+(2-3i) = (1+2)+i(1+(-3)) = 3-2i$

ii) $(1+5i)+(-4i) = (1+0)+i(5+(-4)) = 1+i$

iii) $(1+0i)+(0+2i) = (1+0)+i(0+2) = 1+2i$ ■

Now, let's visualize Example 2i).



This visualization shows two key ideas:

1. The addition of complex numbers behaves exactly as vector addition in two-dimensions (recall the analogy between $\mathbb{R}^2 \cong \mathbb{C}$).

2. Addition of real numbers is commutative, $a+b=b+a$. Here we see the addition of complex numbers is also commutative. In others words the order in which one adds them does not matter.

Subtraction of Complex Numbers

Definition 1.5.2 Given two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, their difference $z_1 - z_2$ is defined as

$$(x_1+iy_1)-(x_2+iy_2) = (x_1-x_2)+i(y_1-y_2) \tag{1.5}$$

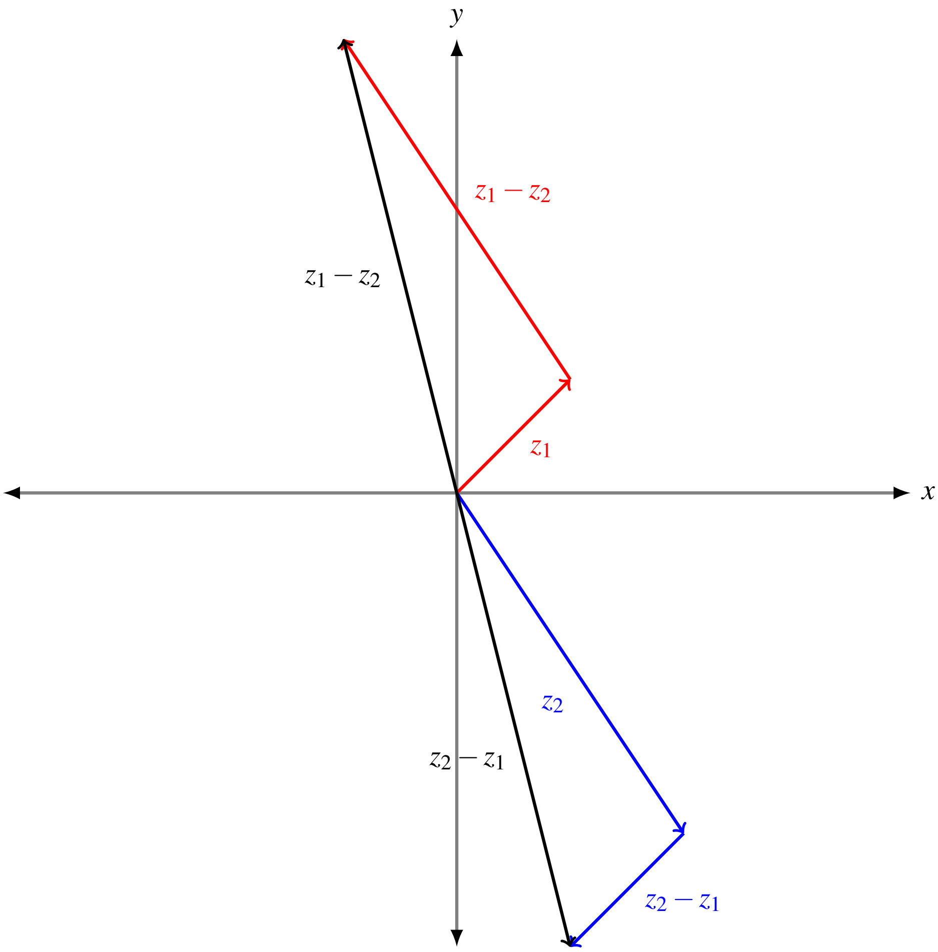
Just subtract the real parts and the imaginary parts.

■ **Example 1.9** i) $(1+i)-(2-3i) = (1-2)+i(1-(-3)) = -1+4i$

ii) $(1+5i)-(-4i) = (1-0)+i(5-(-4)) = 1+9i$

iii) $(1+0i)-(0+2i) = (1-0)+i(0-2) = 1-2i$ ■

Now, let's visualize Example 4i).



This visualization shows two key ideas:

1. The subtraction of complex numbers behaves exactly as vector subtraction in two-dimensions.

2. Subtraction of real numbers is **NOT** commutative, $a-b=b-a$. The results are the same except for the sign. Here we see also that the subtraction of complex numbers is **NOT** commutative. In others words the order in which one subtracts does matter!

Multiplication of Complex Numbers

Definition 1.5.3 Given two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, their product $z_1 z_2$ is defined as

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_2 y_1 + x_1 y_2) \tag{1.6}$$

Just FOIL as you would any product of binomials! Once the four terms are found just combine the two real numbers into x and the two imaginary numbers into y

R The most common mistake made is that one forgets $i^2 = -1$ so when multiplying the two imaginary parts you receive a real number and a sign change.

■ **Example 1.10** ii) $(2+3i)^2 = (2+3i)(2+3i)$ $4+6i+6i-9 = -5+12i$

ii) $(1+i)(2-3i) = 2-3i+2i+3 = 5+i$

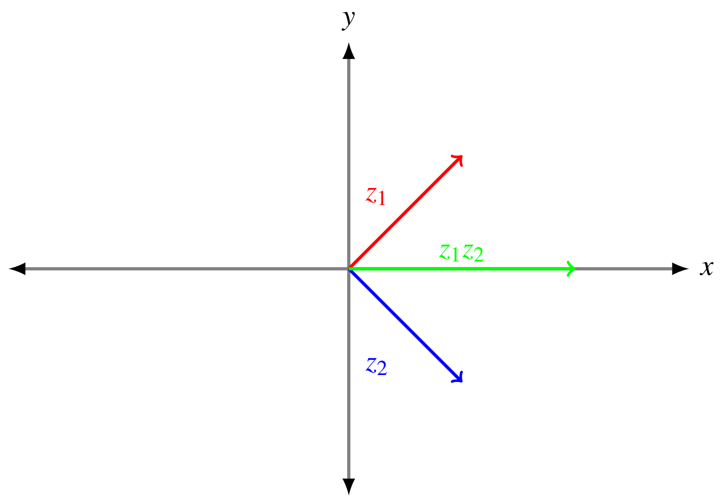
iii) $(1+5i)(-4i) = -4i - 20i^2 = 20 - 4i$

iv) $(1-i)^2 = 1 - i - i + i^2 = -2i$

v) $(1+i)(1-i) = 1 - i + i - i^2 = 2$

vi) $(1-i)(1+2i)(2-i) = [1+2i+i+2i^2](2-i) = [-1+3i](2-i) = -2+i+6i-3i^2 = 1+7i$ ■

Now, let's visualize Example 7 v).



This visualization shows two key ideas:

1. What is special about multiplying two complex conjugates? Example 7 v) shows that this always results in a real number

$$(x_1+iy_1)(x_1-iy_1) = (x_1x_1+y_1y_1)+i(x_1y_1-x_1y_1) = x_1^2+y_1^2 = r^2 \tag{1.7}$$

2. The multiplication of complex numbers behaves exactly as FOIL in the case of two binomials.

3. Multiplication of real numbers is commutative, $ab = ba$. Here we see also that the multiplication of complex numbers is commutative. In others words the order in which one multiplies is not important.

In some cases multiplication may be easier to carry out in *polar form* (while polar form is clearly not a good choice for addition/subtraction). To multiply two complex numbers in polar form we simply multiply the magnitudes, r, and add the angles

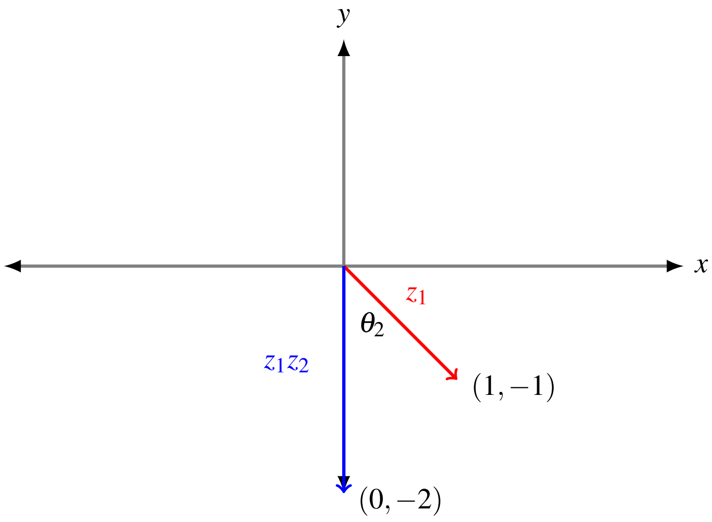
$$z_1z_2 = r_1e^{i\theta_1}r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}. \tag{1.8}$$

Thus, visually multiplication amounts to rotating the first complex number z_1 by angle θ_2 and extending its length by a factor of r_2.

■ **Example 1.11** i) $(1-i)^2 = (1-i)(1-i) = 1-i-i-1 = -2i$

In Polar Form: $(1-i) = \sqrt{2}e^{-i\frac{\pi}{4}}$

$(1-i)^2 = \sqrt{2}e^{-i\frac{\pi}{4}}\sqrt{2}e^{-i\frac{\pi}{4}} = 2e^{-i\frac{pi}{2}} = -2i$



■

Division of Complex Numbers

Definition 1.5.4 Given two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, their quotient z_1/z_2 is defined as

$$\frac{x_1 + iy_1}{x_2 + iy_2} \tag{1.9}$$

To write this in standard form, $x + iy$, one must follow two steps:

Step 1: Multiply the top and bottom by the complex conjugate of the denominator (resulting in a real number in the denominator).

Step 2: Separate the real and imaginary parts to find x and y in the standard form.

■ **Example 1.12** i) $\frac{1+i}{2-3i} = \frac{1+i}{2-3i}\frac{2+3i}{2+3i} = \frac{2+3i+2i+3i^2}{4+6i-6i-9i^2} = \frac{2+5i-3}{4+9} = \frac{-1+5i}{13} = -\frac{1}{13} + \frac{5}{13}i$

CHECK: $(2-3i)(-\frac{1}{13} + \frac{5}{13}i) = -\frac{2}{13} + \frac{10}{13}i + \frac{3}{13}i - \frac{15}{13}i^2 = 1 + i$

ii) $\frac{1+5i}{-4i} = \frac{1+5i}{-4i}\frac{4i}{4i} = \frac{4i+20i^2}{-16i^2} = \frac{-20+4i}{16} = -\frac{5}{4} + \frac{1}{4}i$

ii) $\frac{2-i}{2+i} = \frac{2-i}{2+i}\frac{2-i}{2-i} = \frac{4-2i-2i+i^2}{4+2i-2i-i^2} = \frac{3-4i}{5} = \frac{3}{5} - \frac{4}{5}i$ ■

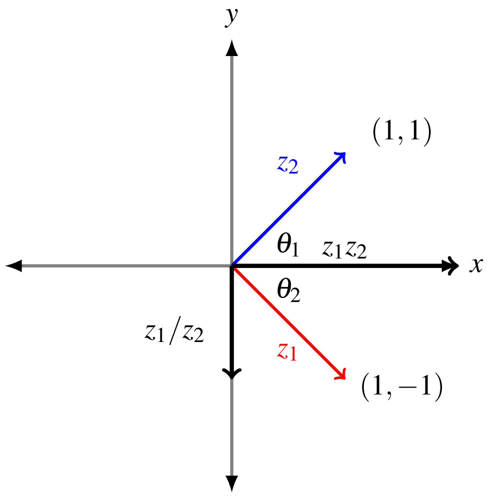
Unlike in multiplication, using the polar form is not a good choice unless the angle of both the numerator and denominator are easy to find (e.g., $30-60-90$ or $45-45-90$ right triangle). To divide two complex numbers in polar form we simply divide the magnitudes, r, and subtract the angle in the denominator from the angle in the numerator

$$z_1/z_2 = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}. \tag{1.10}$$

Thus, visually division amounts to rotating the first complex number z_1 by angle $-\theta_2$ and reducing its length by a factor of r_2.

■ **Example 1.13** i) $\frac{1-i}{1+i} =$ Method 1: $\frac{1-i}{1+i}\frac{1-i}{1-i} = \frac{1-i-i+i^2}{1-i+i-i^2} = \frac{-2i}{2} = -i$

Method 2: $\frac{\sqrt{2}e^{-i\frac{\pi}{4}}}{\sqrt{2}e^{i\frac{\pi}{4}}} = e^{i(-\frac{\pi}{4} - \frac{\pi}{4})} = e^{-i\frac{\pi}{2}} = \cos(\frac{-\pi}{2}) + i\sin(-\frac{\pi}{2}) = -i$



■

R Multiplication and Division amount to rotating complex numbers in one direction or another and scaling them. Most times it is useful to avoid the polar form and carry out these operations on the standard form. No matter which method is used the final answer should always be reported in standard form, $x+iy$.

1.5.2 Complex Conjugation of an Expression

Definition 1.5.5 The conjugate of a sum of two complex numbers is the sum of the conjugates. Given $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, their sum

$$\overline{z_1+z_2} = \overline{(x_1+x_2)+i(y_1+y_2)} = (x_1+x_2)-i(y_1+y_2) = (x_1-iy_1)+(x_2-iy_2) = \bar{z}_1+\bar{z}_2. \quad (1.11)$$

In addition, the conjugate of a difference, product, or quotient is equal to the difference, product, or quotient of the conjugates (e.g., $\overline{z_1-z_2} = \bar{z}_1-\bar{z}_2$, $\overline{z_1z_2} = \bar{z}_1\bar{z}_2$, and $\overline{z_1/z_2} = \bar{z}_1/\bar{z}_2$).

■ **Example 1.14** i) $\overline{(1+i)(2-3i)} = (1-i)(2+3i) = 2+3i-2i-3i^2 = 2+i+3 = 5+i$
OR: $\overline{2-3i+2i-3i^2} = \overline{5-i} = 5+i$.

ii) $\overline{(1+i)/(3-4i)} = \frac{1-i}{3+4i} = \frac{1-i}{3+4i}\frac{3-4i}{3-4i} = \frac{3-4i-3i+4i^2}{9+16} = \frac{-1-7i}{25} = -\frac{1}{25}-\frac{7}{25}i$
OR: $\overline{\frac{1+i}{3-4i}\frac{3+4i}{3+4i}} = \overline{\frac{3+4i+3i+4i^2}{9+16}} = \overline{\frac{-1+7i}{25}} = \overline{\frac{-1}{25}+\frac{7}{25}i} = -\frac{1}{25}-\frac{7}{25}i$. ■

Notice $z = f+ig$ where f,g are complex numbers, then $\bar{z} = \bar{f}+i\bar{g}$, NOT: $f-ig$.

■ **Example 1.15** Let $f = 1+i$ or $g = 2-i$, then find $\bar{z} = \overline{f+ig}$.

Thus, $\bar{z} = \overline{(1+i)}+\overline{i(2-i)} = (1-i)-i(2+i) = 2-3i$. As a check find $z = (1+i)+i(2-i) = 2+3i$, then $\bar{z} = 2-3i \neq f-ig = -i$. ■

■ **Example 1.16** Show the conjugate of the quotient is the quotient of the conjugates.

Then $\overline{\frac{z_1}{z_2}} = \overline{\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}}} = \overline{\frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}} = \frac{r_1}{r_2}e^{i(\theta_2-\theta_1)} = \frac{r_1e^{-i\theta_1}}{r_2e^{i\theta_1}} = \frac{\bar{z}_1}{\bar{z}_2}$. ■

Observe that if it works for a sum then it automatically works for a difference $A-B = A+(-B)$. If it works for a quotient, then it will work for a product $A/B = A\times(1/B)$.

1.5.3 Finding the Absolute Value $|z|$

Definition 1.5.6 The magnitude or length of a complex number z is $r = |z| = \sqrt{x^2+y^2}$. We take the positive square root since distance is positive.

■ **Example 1.17** Given a complex number $z = x+iy$ find $z\bar{z}$.

$z\bar{z} = (x+iy)(x-iy) = x^2 - ixy + ixy - i^2y^2 = x^2+y^2 = r^2 = |z|^2$ or in polar form $z\bar{z} = re^{i\theta}re^{-i\theta} = r^2 = |z|^2$. Thus, $|z| = r = \sqrt{z\bar{z}}$. ■

■ **Example 1.18** Find:

i) $|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$.

ii) $|4i| = \sqrt{0^2+4^2} = \sqrt{16} = 4$.

iii) $|1+2i| = \sqrt{1^2+2^2} = \sqrt{5}$. ■

(R) The absolute value of a product or quotient is the product or quotient of the absolute values.

■ **Example 1.19** i) $\left|\frac{1+i}{1-i}\right| = \left|\frac{1+i}{1-i}\frac{1+i}{1+i}\right| = \left|\frac{1+i+i+i^2}{1+1}\right| = \left|\frac{2i}{2}\right| = |i| = 1$

OR: $\frac{|1+i|}{|1-i|} = \frac{\sqrt{1^2+1^2}}{\sqrt{1^2+(-1)^2}} = \frac{\sqrt{2}}{\sqrt{2}} = 1$.

ii)$\left|\frac{2-3i}{5+6i}\right| = \frac{|2-3i|}{|5+6i|} = \frac{\sqrt{2^2+(-3)^2}}{\sqrt{5^2+6^2}} = \frac{\sqrt{13}}{\sqrt{61}}$.

ii)$\left|\frac{2+4i}{1+i}\right| = \frac{|2+4i|}{|1+i|} = \frac{\sqrt{2^2+4^2}}{\sqrt{1^2+1^2}} = \frac{\sqrt{20}}{\sqrt{2}} = \sqrt{10}$. ■

1.5.4 Complex Equations

The main idea is to remember that a complex number is associated with a pair of real numbers (the real and imaginary parts). Thus, a complex equation really contains two equations, one for the real parts and one for the imaginary parts.

Definition 1.5.7 Two complex numbers, z_1 and z_2, are equal *if and only if* $x_1 = x_2$ (real parts) and $y_1 = y_2$ (imaginary parts).

For example, $2+i \neq 2-i$. What does this say about complex equations? When solving a complex equation we really need to solve two equations at once (for the real and imaginary parts). Knowing that an equations is complex gives a relationship between each part.

■ **Example 1.20** Find $z = x+iy$ if $z^2 = 4i$.

$$\begin{aligned} & (x+iy)^2 = 4i \\ \text{FOIL} \quad & x^2+2ixy-y^2 = 4i \\ \text{Split into two equations} \quad & \text{Real: } x^2-y^2 = 0, \quad \text{Imaginary: } 2xy = 4. \end{aligned}$$

Solving the first equation gives $x^2 = y^2$. Either $x = -y$ or $x = y$. In the first case ($x = -y$), the second equation gives $-2x^2 = 4$, which implies $x = \pm = \sqrt{-2} = \pm 2i$, but we know x must be a real number so this case cannot hold!

In the second case ($x = y$), the second equation gives $2x^2 = 4$ or $x = \pm\sqrt{2}$. Thus, the two solutions are $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$. ■

■ **Example 1.21** i) #39 Solve $x+iy=y+ix$.

Matching real and imaginary parts, this is always true if $x=y$. So there are infinitely many solutions that lie on the line $y=x$ in the complex plane.

ii) #44 Solve $x+iy=(1-i)^2$.

FOIL the left-hand side.

$$x+iy=1-i-i+i^2=1-2i-1=-2i \qquad \Rightarrow \qquad x=0,\ \ y=-2.$$

iii) #45 Solve $(x+iy)^2=(x-iy)^2$.

FOIL both sides.

$$\begin{aligned} x^2+2xyi-y^2 &= x^2-2xyi-y^2 \\ 2xyi &= -2xyi \\ xy &= -xy. \end{aligned}$$

Thus, $x=-x$ or $y=-y$. So either $x=0$ or $y=0$. Therefore, $z=x$ or $z=y$. ■

Now for a harder example! If you can solve this you can handle most quadratic equations and have demonstrated you follow all the necessary steps.

■ **Example 1.22** $\frac{x+iy+2+3i}{x+iy-3}=i+2$. Let $z=x+iy$ and rewrite the equation as $\frac{z+2+3i}{z-3}=i+2$.

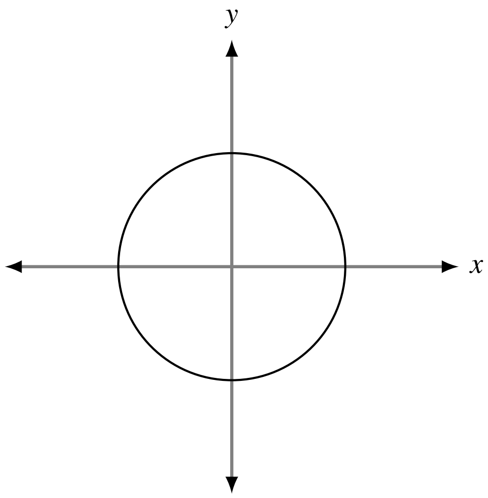
$$\begin{aligned} z+2+3i &= (i+2)(z-3) \\ z+2+3i &= zi+-3i+2z-6 \\ \text{Rearrange terms with } z: \quad z-(2+i)z &= -3i-6-(2-3i) \\ (-1-i)z &= -8-6i. \end{aligned}$$

Thus, $z=\frac{-8-6i}{-1-i}=\frac{-8-6i}{-1-i}\frac{-1+i}{-1+i}=\frac{8-8i+6i-6i^2}{1+1}=\frac{14-2i}{2}=7-i$. ■

1.5.5 Graphs of Complex Equations

What is the curve made up of points in the complex plane satisfying $|z|=1$?

In others words, find x and y such that $x^2+y^2=1$. This is just the equation of a circle of radius 1.



■ **Example 1.23** Describe the plots of each of these complex equations:

i) $|z-3| = 4$, square both sides $|z-3|^2 = 16 \Rightarrow (x-3)^2 + y^2 = 16$. This is a circle centered at (3,0) with radius 4.

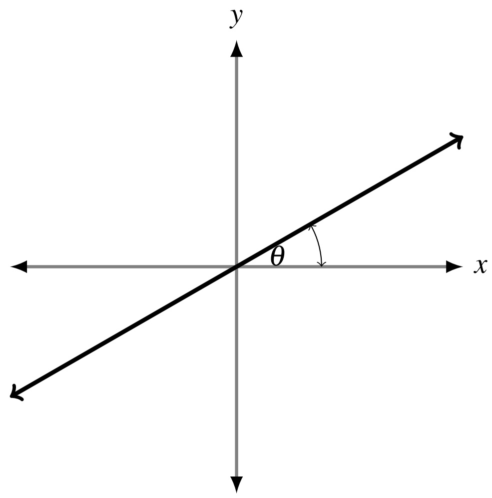
ii) $|z-3| \geq 4$. This is the area outside the circle centered at (3,0) of radius 4 including the circle boundary.

iii) $|z-3| < 4$. This is the interior of the circle centered at (3,0) of radius 4. ■

Recall the three basic conic sections and their equations

$a)$ Circle centered at (a,b) of radius r	$(x-a)^2 + (y-b)^2 = r^2$
$b)$ Ellipse centered at (a,b) with radius c in x and radius d in y	$\frac{(x-a)^2}{c^2} + \frac{(y-b)^2}{d^2} = 1$
$c)$ Hyperbola centers at (a,b)	$\frac{(x-a)^2}{c^2} - \frac{(y-b)^2}{d^2} = 1$

Plot the solution to the equation $\theta = \pi/6$



■ **Example 1.24** Plot each of the following:

i) $\text{Re}\{z\} < 2$

ii) $\text{Re}\{z\} \geq -1$

iii) $\text{Im}\{z\} \geq 3$. ■

1.5.6 Physical Applications

Complex Equations appear in all sorts of physical applications. Primarily adding analysis for problems in two-dimensions. Classical examples are 2D fluid flow, superconductivity in a wire, among others. Understanding how complex equations work will provide the basis for learning advanced methods. If interested further look up the techniques of conformal mapping or contour integration, which are very useful in physics.

■ **Example 1.25** A particle moves in the (x, y) plane so that its position as a function of time t is given by

$$z = x + iy = \frac{i+3t}{t-2i}.$$

Find the magnitudes of the velocity and the acceleration as a function of time.

Answer: First recall the definitions of position, velocity, and acceleration as well as their relationships
Position: $z = x + iy$
Velocity: $\frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}$
Acceleration: $\frac{d^2z}{dt^2} = \frac{d^2x}{dt^2} + i\frac{d^2y}{dt^2}$

First find the velocity using the Quotient Rule, $\frac{dz}{dt} = \frac{3(t-2i)-(i+3t)}{(t-2i)^2)} = \frac{3t-6i-i-3t}{(t-2i)^2} = \frac{-7i}{(t-2i)^2}$. We need to find the magnitude so consider $\left|\frac{dz}{dt}\right| = \frac{|-7i|}{|t^2-4it-4|} = \frac{7}{\sqrt{(t^2-4)^2+(4t)^2}} = \frac{7}{\sqrt{t^4-8t^2+16+16t^2}} = \frac{7}{\sqrt{t^4+8t^2+16}} = \frac{7}{\sqrt{(t^2+4)^2}} = \frac{7}{t^2+4}$.

Now find the acceleration by taking one more derivative. $\frac{d^2z}{dt^2} = \frac{14i}{(t-2i)^3}$. Last find its magnitude

$$a = \left|\frac{d^2z}{dt^2}\right| = \sqrt{\frac{14i}{(t-2i)^3}\frac{-14i}{(t+2i)^3}} = \frac{14}{\sqrt{(t^3-6it^2-12t+8i)(t^3+6it^2-12t-8i)}},$$

then $a = \frac{14}{\sqrt{(t^2+4)^3}} = \frac{14}{(t^2+4)^{3/2}}$. ∎

1.6 Complex Infinite Series

Recall that for a function of a real variable, $f(x)$, we can find an approximation locally using a *Taylor Expansion*. Any function of one variable can be expanded about a point $x = a$ as follows:

$$f(x+a) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \ldots$$

Key Questions: Does this series *converge*? If so, then for what values of x is the expansion valid? We will explore these questions for complex Taylor Series.

Convergence in Section 2.6

Radius/Disk of Convergence in Section 2.7

For a real series we can define a partial sum of the first n terms, $S_n := \sum_{k=1}^{n} f^{(k)}(a)(x-a)^k$. We say the series *converges* if $\lim_{n\to\infty} S_n = S$ where S is the sum.

R In future courses, you may see convergence defined different. Rigorously, a series is said to converge if the partial sums get closer and closer together, $|S_m - S_n| \to 0$ as $m, n \to \infty$.

Analogously, for complex numbers we say that the partial sum $S_n = X_n + iY_n$ consisting of a sum in the real parts and a sum in the imaginary parts. The sum converges if both expressions approach some limit!

$$\lim_{n\to\infty}[X_n + iY_n] = \lim_{n\to\infty} X_n + i\lim_{n\to\infty} Y_n = X + iY.$$

Thus, $X_n \to \infty$ and $Y_n \to \infty$. In other words, the real and imaginary parts of the series each converge as a series of real numbers.

First, let's review the definition of absolute convergence and convergence tests for series of real numbers.

Definition 1.6.1 If the series of absolute values $\sum_{n=1}^{\infty} |z_n| < \infty$, then the series is called *absolutely convergence*.

There is also a special type of series, which converges known as a geometric series.

Definition 1.6.2 A geometric series has the form: $\sum_{i=1}^{\infty} ar^i$. If $|r| < 1$ this series converges. Useful formulas for the infinite sum and all partial sums are

$$\sum_{i=1}^{n} ar^i = a\left(\frac{1-r^n}{1-r}\right), \qquad \sum_{i=1}^{\infty} ar^i = \frac{a}{1-r}.$$

1.6.1 Review from Calculus: Tests for Convergence

For more detail please consult Chapter 1 in the textbook by Boas.

Convergence Test

If the terms $X_i \nrightarrow 0$, then the series must diverge.

■ **Example 1.26** $\sum_{i=1}^{\infty} \frac{i}{i+1}$ must diverge since each term $X_i \to 1 \neq 0$. ■

Comparison Test

Consider two series $a_1 + a_2 + a_3 + \ldots$ and $b_1 + b_2 + b_3 + \ldots$. If $|a_n| \leq |b_n|$ for all n and the series $\sum b_n$ converges, then the series for a_n is absolutely convergent OR if $|a_n| \geq d_n$ and the series for d_n diverges, then the series for a_n diverges.

■ **Example 1.27** $\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{6} + \ldots$. Let $b_n = \frac{1}{2^n}$, then $|a_n| \leq b_n$ and $\sum b_n < \infty$ (geometric series). Thus a_n converges! ■

Integral Test

If $0 < a_{n+1} \leq a_n$ for $n > N$, then $\sum_{n=1}^{\infty} a_n$ converges/diverges if $\int_0^{\infty} a_n dn$ converges/diverges.

■ **Example 1.28** $\sum_{n=1}^{\infty} \frac{1}{n}$. Using the Integral Test: $\int_1^{\infty} \frac{1}{n} dn = \ln(n)\Big|_1^{\infty} = \ln(\infty) - 0 \to \infty$. So the original series diverges! ■

Ratio Test

Take the ratio of two consecutive terms in the series: $\rho_n = \left|\frac{a_{n+1}}{a_n}\right|$ and consider $\lim_{n\to\infty} \rho_n$. If:
i) $\rho < 1$ the series converges
ii) $\rho > 1$ the series diverges
iii) $\rho = 1$ there is not enough info to conclude if the series converges or diverges.

■ **Example 1.29** $\sum_{n=1}^{\infty} \frac{1}{n!}$, then $\rho_n = \left|\frac{1}{(n+1)!}\frac{n!}{1}\right| = \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0$. Thus, the original series converges. ■

Root Test

Consider the nth root of the summand $L := \lim_{n\to\infty} \sqrt[n]{|a_n|}$. If:
i) $L < 1$ the series is absolutely convergent
ii) $L > 1$ the series is divergent
iii) $L = 1$ there is not enough info to conclude if the series converges or diverges.

■ **Example 1.30** $\sum_{n=0}^{\infty} \left(\frac{5n-3n^3}{7n^3+2}\right)^n$, then $L = \left|\frac{5n-3n^3}{7n^3+2}\right| = \left|\frac{-3}{7}\right| = \frac{3}{7} < 1$. The series converges! ■

Alternating Series

An alternating series is a series where the terms have the form $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$. An alternating series converges if the limit of the absolute value of the terms converges to zero and the terms are decreasing: $|a_{n+1}| < |a_n|$ and $\lim_{n\to\infty} a_n = 0$.

■ **Example 1.31** $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$, converges by the alternating series test. ■

Definition 1.6.3 If a series converges, but not absolutely, then it is said to be *conditionally convergent*. This is a weaker form of convergence. In particular, the terms in the sum can be rearranged to form any total. In contrast, for a series that is absolutely convergent, rearranging the terms does not change the sum.

1.6.2 Examples with Complex Series

■ **Example 1.32** i) $1 + \frac{(i+2)}{3} + \frac{(i+2)^2}{9} + \frac{(i+2)^3}{27} + \ldots + \frac{(2+i)^n}{3^n} + \ldots$.

By the Ratio Test: $\lim_{n\to\infty} |\rho_n| = \lim_{n\to\infty} \left| \frac{(2+i)^{n+1}}{3^{n+1}} \frac{3^n}{(2+i)^n} \right| = \left| \frac{2+i}{3} \right| = \frac{\sqrt{2^2+1^2}}{3} = \frac{\sqrt{5}}{3} < 1$. The series converges!

ii) $\sum_{n=1}^{\infty} \frac{i^n}{\sqrt{n}}$.

Consider the Real Part: $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{2n}}$ and Imaginary Part: $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{2n+1}}$. Both series converge by the alternating series test. Thus, the complex series converges!

iii) $\sum_{n=0}^{\infty} (z+1)^n$.

Using the Root Test: $L := \lim_{n\to\infty} |z+1|$ converges for $|z+1| < 1$ or $\sqrt{(x+1)^2 + y^2} < 1$ or $(x+1)^2 + y^2 < 1$. Thus, the series converges for z inside the circle centered at $(-1,0)$ of radius 1 not including the boundary. ■

1.7 Complex Power Series and Disk of Convergence

Recall from calculus a power series for a function of a real variable (centered at zero)

$$f(x) = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n,$$

or centered at point $x = a$

$$f(x) = \sum_{n=1}^{\infty} b_n (x-a)^n = \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

Definition 1.7.1 *(Interval of Convergence)* The values of x where the series converges.

■ **Example 1.33** Given the power series $\sum x^n$. By the ratio test $\rho := \left| \frac{x^{n+1}}{x^n} \right| = |x|$. For convergence we need $\rho < 1$. Thus, $|x| < 1$ is the interval of convergence. ■

Before defining a complex power series, let's discuss some facts for real power series (review from Calculus):

1. A power series can be differentiated or integrated term by term. The resulting series converges to the derivative or integral of the original function within the same interval of convergence.

■ **Example 1.34** Consider the function $f(x) = e^x$, which has power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

a) Differentiating term by term: $\sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} \to_{k:=n-1} \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$.

b) Integrating term by term: $\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \to_{k:=n+1} \sum_{k=1}^{\infty} \frac{x^k}{k!} = e^x - 1$. Note $\int e^x = e^x + C$.

c) The Interval of Convergence (I.O.C.) can be found using the ratio test $\rho := \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| \to 0$. Thus, the interval of convergence is all real numbers. ■

2. Two power series can be added, subtracted, multiplied. The result converges in the common interval of convergence.

3. One series can be substituted into another if the substituted series values are in the interval of convergence of the series it is being plugged into.

4. The power series of a function is unique! Only one power series of the form $\sum_n a_n x^n$ converges to a given function.

R Properties 1.-4. still hold for complex power series!

Definition 1.7.2 A **complex power series** has the form $\sum_n a_n z^n$ where $z = x + iy$. The real power series just a special case of the complex power series when $y = 0$.

■ **Example 1.35** i) $1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \ldots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

ii) $1 - i(z+1) + \frac{(i[z+1])^2}{2} + \frac{(i[z+1])^3}{6} + \frac{(i[z+1])^4}{24}$

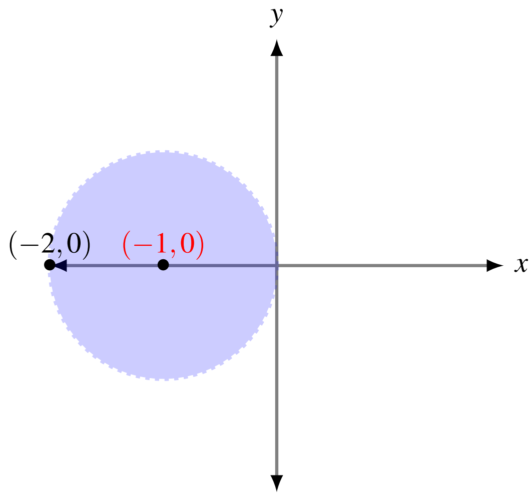
iii) $\sum_{n=0}^{\infty} \frac{(z-2+2i)^n}{6^n n^3}$. ■

Definition 1.7.3 The complex analogue of the radius of convergence is the disk of convergence (in the 2D complex plane).

■ **Example 1.36** Find the Disk of Convergence (D.O.C.) for each complex power series in the previous example.

i) For $\sum_{n=1}^{\infty} \frac{z^n}{n!}$, use the ratio test. $\rho := \lim_{n\to\infty} \left| \frac{z^{n+1}}{(n+1)!} \frac{n!}{z^n} \right| = \lim_{n\to\infty} \left| \frac{z}{n+1} \right| \to 0$. Thus, the series converges for all z in the complex plane. Therefore, the disk of convergence is the entire complex plane, $\mathbb{C}$.

ii) For $1 + \sum_{n=1}^{\infty} \frac{(i[z+1])^n (-1)^n}{n}$. By the ratio test $\rho = \lim_{n\to\infty} \left| \frac{(i[z+1])^{n+1}(-1)^{n+1}}{n+1} \frac{n}{(i[z+1])^n(-1)^n} \right| = \lim_{n\to\infty} \left| \frac{i(z+1)(-1)n}{n+1} \right| \to \frac{|z+1|}{1}$. Thus, $\rho < 1$ if $|z+1| < 1$. Thus, the disk of convergence is the interior of the circle centered at $(-1, 0)$ with radius 1.



iii) $\sum_{n=0}^{\infty} \frac{(z-2+2i)^n}{6^n n^3}$. Use the ratio test, $\rho = \lim_{n\to\infty} \left| \frac{(z-2+2i)^{n+1}}{6^{n+1}(n+1)^3} \frac{6^n n^3}{(z-2+2i)^n} \right| = \lim_{n\to\infty} \left| \frac{(z-2+2i)n^3}{6(n+1)^3} \right| \to \left| \frac{z-2+2i}{6} \right|$. Thus, the series converges if $\rho < 1$ or $|z-2+2i| < 6$. The disk of convergence is centered at $(2,-2)$ of radius $\sqrt{6}$. ■

■ **Example 1.37** iv) $1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \ldots$. General Form: $\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}$. Then by the ratio test, $\rho = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} z^{2(n+1)}}{[2(n+1)+1]!} \frac{(2n+1)!}{(-1)^n z^{2n}} \right| = \lim_{n\to\infty} \left| \frac{(-1)z^2}{(2n+3)(2n+2)} \right| \to 0$. Thus, the disk of convergence is the entire complex plane, $\mathbb{C}$.

v) $\sum_{n=0}^{\infty} 2^{n+1}(z+i-3)^{2(n+1)}$. Then by the ratio test, $\rho = \lim_{n\to\infty} \left| \frac{2^{n+1}(z+i-3)^{2(n+1)}}{2^n(z+i-3)^{2n}} \right| = \lim_{n\to\infty} |2(z+i-3)^2| = |2(z+i-3)^2|$. the disk of convergence is where $|(z+i-3)|^2 < \frac{1}{2}$ or the disk centered at $(3,-1)$ of radius $1/\sqrt{2}$. ■

1.8 Elementary Functions of Complex Numbers

In principle we can consider any function we have traditionally used (e.g., exponential, trig functions, polynomials). In the previous section we saw complex polynomials in the form of power series. We start this section with the next level of complexity, rational functions (ratios of polynomials):

$$f(z) = \frac{a_0 + a_1 z + a_2 z^2 + \ldots + a_N z^N}{b_0 + b_1 z + b_2 z^2 + \ldots + b_M z^M}.$$

■ **Example 1.38** Given the complex function $f(z) = \frac{z^3-1}{z+2}$, find $f(i-1)$.

Step 1: Substitute value of z into the function

$$f(i-1) = \frac{(i-1)^3 - 1}{(i-1)-2}.$$

Step 2: Simplifying

$$f(i-1) = \frac{(i-1)(i^2-2i+1)-1}{i+1} = \frac{(i-1)(-2i)-1}{i+1} = \frac{-2i^2+2i-1}{i+1} = \frac{1+2i}{i+1}$$

Step 3: Rationalize the denominator (Multiply by the complex conjugate over itself)

$$f(i-1) = \frac{1+2i}{i+1}\frac{1-i}{1-i} = \frac{1+2i-i-2i^2}{2} = \frac{3}{2} + \frac{1}{2}i$$

More Examples in Class! ■

Recall the power series for $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ in real numbers. Can we make a similar definition for the complex exponential function?

Definition 1.8.1 Using the definition of the power series expansion of the exponential function of real variables, replace x with z:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{1.12}$$

Where does it converge (Disk of Convergence)?

By the Ratio Test: $\rho = \lim_{n\to\infty} \left| \frac{z^{n+1}}{(n+1)!} \frac{n!}{z^n} \right| = \lim_{n\to\infty} \left| \frac{z}{n+1} \right| \to 0$. Thus, $\rho > 1$ for all z in the Complex Plane and the disk of convergence must be $\mathbb{C}$.

Operations with Complex Exponential Functions

■ **Example 1.39** i) $e^{z_1}e^{z_2} = \left[(1+z_1+\frac{z_1^2}{2}+...)(1+z_2+\frac{z_2^2}{2}+...)\right] = \left[1+(z_1+z_2)+\frac{(z_1+z_2)^2}{2}\right] = e^{z_1+z_2}$

ii) Note: $\frac{d}{dz}[z^n] = nz^n$ (just like normal derivatives of real numbers!

$\frac{d}{dz}[e^z] = \frac{d}{dz}\left(1+z+\frac{z^2}{2}+...+\frac{z^n}{n!}+...\right) = 0+1+z+\frac{z^2}{2}+...+\frac{nz^{n-1}}{n!}+...$
$= 0+1+z+\frac{z^2}{2}+...+\frac{z^{n-1}}{(n-1)!}+... = e^z$ ■

1.9 Euler's Formula

Recall the Taylor expansion for the basic trig functions of one real variable

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$

Can we do something similar for complex trig functions? First, consider the complex Taylor series for the exponential function

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + ... \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + ... \\ &= \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + ...\right] + i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + ...\right] \\ &= \cos(\theta) + i\sin(\theta). \end{aligned}$$

This result gives us Euler's Formula!

$$\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta)} \tag{1.13}$$

We have been using this formula since Section 3.2, but now we can see why it holds. We also have verified

$$\boxed{z = x + iy = r(\cos(\theta) + i\sin(\theta) = re^{i\theta}.} \tag{1.14}$$

■ **Example 1.40** Find the values of $3e^{i\pi/3}, e^{i\pi/2}, 2e^{-i\pi/6}, e^{2n\pi i}$.

i) $3e^{i\pi/3} \Rightarrow r = 3, \theta = \pi/3$. Recall from polar coordinates $x = r\cos(\theta) = 3\cos(\pi/3) = 3\left(\frac{1}{2}\right) = 3/2$ and $y = r\sin(\theta) = 3\sin(\pi/3) = 3\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2}$. Thus, $z = \frac{3}{2} + \frac{3\sqrt{3}}{2}i$.

ii) $e^{i\pi/2} \Rightarrow r = 1, \theta = \pi/2$. Recall from polar coordinates $x = r\cos(\theta) = \cos(\pi/2) = 0$ and $y = r\sin(\theta) = \sin(\pi/2) = 1$. Thus, $z = i$.

iii) $2e^{-i\pi/6} \Rightarrow r = 2, \theta = -\pi/6$. Recall from polar coordinates $x = r\cos(\theta) = 2\cos(-\pi/6) = 2\cos(\pi/6) = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$ and $y = r\sin(\theta) = 2\sin(-\pi/6) = -2\sin(\pi/6) = 2\left(-\frac{1}{2}\right) = -1$. Thus, $z = \sqrt{3} - i$.

iv) $e^{2n\pi i} \Rightarrow r = 1, \theta = 2n\pi$. Recall from polar coordinates $x = r\cos(\theta) = 1$ and $y = r\sin(\theta) = 0$. Thus, $z = 1$ for all n. ■

Recall that Euler's Formula/Identity is especially useful for multiplying and dividing complex numbers

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$
$$z_1/z_2 = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

■ **Example 1.41** Evaluate $\frac{(1-i)^2}{1+i}$.

Step 1: Write in Polar Form: $z_1 = (1-i)^2 = \left[\sqrt{2}e^{-i\pi/4}\right]^2 = 2e^{-i\pi/2}$ and $z_2 = 1+i = \sqrt{2}e^{i\pi/4}$.

Step 2: Carry out the multiplication or division:

$$\frac{z_1}{z_2} = \frac{2e^{-i\pi/2}}{\sqrt{2}e^{i\pi/4}} = \frac{2}{\sqrt{2}} e^{i(-\pi/2-\pi/4)} = \sqrt{2}e^{i3\pi/4}.$$

Step 3: Write in Standard Form, $z = x+iy$. $z = \sqrt{2}e^{i3\pi/4} = -1-i$. ■

■ **Example 1.42** Evaluate $(2+2\sqrt{3}i)(1+i)$.

Step 1: Write in Polar Form: $z_1 = (2+2\sqrt{3}i) = 4e^{i\pi/3}$ and $z_2 = 1+i = \sqrt{2}e^{i\pi/4}$.

Step 2: Carry out the multiplication or division:

$$z_1 z_2 = 4\sqrt{2}e^{i(\pi/3+\pi/4)} = 4\sqrt{2}e^{i7\pi/12}.$$

Step 3: Write in Standard Form, $z = x+iy$, if an easy angle (e.g., 30-60-90, 45-45-90). Here $7\pi/12$ is not an angle that can be handled easily by hand, so we will leave it in polar form. ■

■ **Example 1.43** Evaluate $(1+i)(1-i)$.

Step 1: Write in Polar Form: $z_1 = 1+i = \sqrt{2}e^{i\pi/4}$ and $z_2 = 1-i = \sqrt{2}e^{-i\pi/4}$.

Step 2: Carry out the multiplication or division:

$$z_1 z_2 = 2e^{i(\pi/4-\pi/4)} = 2e^{i0} = 2$$

Step 3: Write in Standard Form, $z = x+iy$, if an easy angle (e.g., 30-60-90, 45-45-90). Notice $(1+i)(1-i) = 1+i-i-i^2 = 2$. ■

1.10 Powers and Roots of Complex Numbers

We have clear definitions for powers and roots (fractional powers) of real numbers. Can we define the analogous notions for complex numbers?

Given a complex number z, consider it raised to the nth power.

Definition 1.10.1 To raise a complex number to the nth power one needs to raise the modulus, r, to the nth power and multiply the angle by n.

$$z^n = \left[re^{i\theta}\right]^n = r^n e^{in\theta} \tag{1.15}$$

Another useful idea using this definition is Demoivre's Theorem:

Theorem 1.10.1 *(Demoivre's Theorem)* When $r = 1$, the nth power can be expressed int he following way:

$$\left(e^{i\theta}\right)^n = (\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin)n\theta). \tag{1.16}$$

■ **Example 1.44** Evaluate $(1+i)^4$.

Step 1: Write in Polar Form: $1+i = \sqrt{2}e^{i\pi/4}$

Step 2: Carry out the calculation using the definition.

$$(1+i)^4 = \left[\sqrt{2}e^{i\pi/4}\right]^4 = (\sqrt{2})^4 e^{i\pi} = 4\left[\cos(\pi) + i\sin(\pi)\right] = 4[-1+0] = -4. \tag{1.17}$$

■

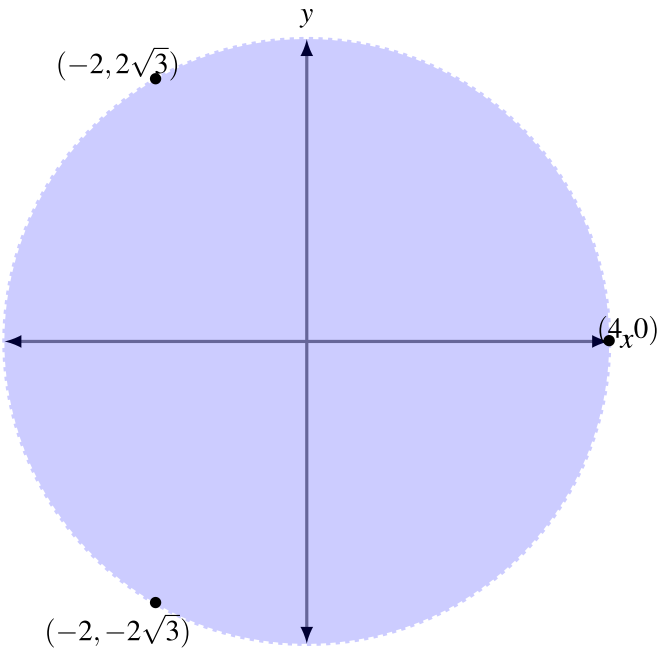
Now we want to consider taking the nth root. Recall that taking the nth root of a real number is equivalent to raising that number to the $\frac{1}{n}$ power. Similarly, for a complex number, $\sqrt[n]{z} = z^{1/n}$.

Definition 1.10.2 To take the nth root of a complex number one needs to take the nth root of the modulus, r, and divide the angle by n.

$$\sqrt[n]{z} = z^{1/n} = \left[re^{i\theta}\right]^{1/n} = r^{1/n}e^{i\theta/n} = \sqrt[n]{r}\left[\cos(\theta/n) + i\sin(\theta/n)\right]. \tag{1.18}$$

■ **Example 1.45** Find the cube roots of 64. In other words, find z so that $z^3 = 64$. Let's attack this problem using the polar form of the complex number, $z = 64$. Thus, $r = 64$ and $\theta = 0, 2\pi, 4\pi, \ldots, 2\pi n$. Now, by the definition of the root: $z^{1/3} = r^{1/3}e^{i\theta/3} = r^{1/3}e^{i(2\pi n+\theta)/3} \Rightarrow r = 4, \theta = 0, 2\pi/3, 4\pi/3, 6\pi/3, \ldots$. Observe that $6\pi/3 = 2\pi = 0$ (on the complex plane). Thus, the three roots are: $4e^{i0} = 4, 4e^{i2\pi/3}, 4e^{4\pi/3}$ or $z = 4, -2+2\sqrt{3}i, -2-2\sqrt{3}i$.

As a check: $(-2+2\sqrt{3}i)^3 = (-2+2\sqrt{3}i)(4-8\sqrt{3}i-12) = (-2+2\sqrt{3}i)(-8-8\sqrt{3}i) = (16+16\sqrt{3}i-16\sqrt{3}i+48) = 64$.



■

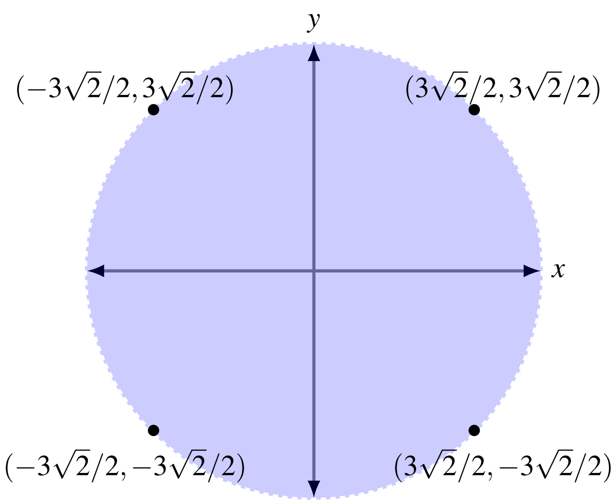
R Always remember that complex numbers always come in conjugate pairs!

■ **Example 1.46** Find the 4th roots of -81. In other words, find z so that $z^4 = -81$. Use the polar form of the complex number, $z = -81$. Thus, $r = 81$ and $\theta = \pi, 3\pi, 5\pi, 7\pi, ..., \pi + 2\pi n$.

Now, by the definition of the root: $z^{1/4} = r^{1/4}e^{i\theta/4} = r^{1/4}e^{i(2\pi n+\theta)/4} \Rightarrow r = 3, \theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4, ...$ Observe that $9\pi/4 = \pi/4$ (on the complex plane).

Thus, the four roots are: $3e^{i\pi/4}, 3e^{i3\pi/4}, 3e^{i5\pi/4}, 3e^{i7\pi/4}$ or $z = \frac{3\sqrt{2}}{2} + i\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2} - i\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2} + i\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2} - i\frac{3\sqrt{2}}{2}$.

As a check: $(\frac{3\sqrt{2}}{2} + i\frac{3\sqrt{2}}{2})^4 = (\frac{18}{4} + \frac{36}{4}i - \frac{18}{4})(\frac{18}{4} + \frac{36}{4}i - \frac{18}{4}) = (9i)(9i) = -81$.



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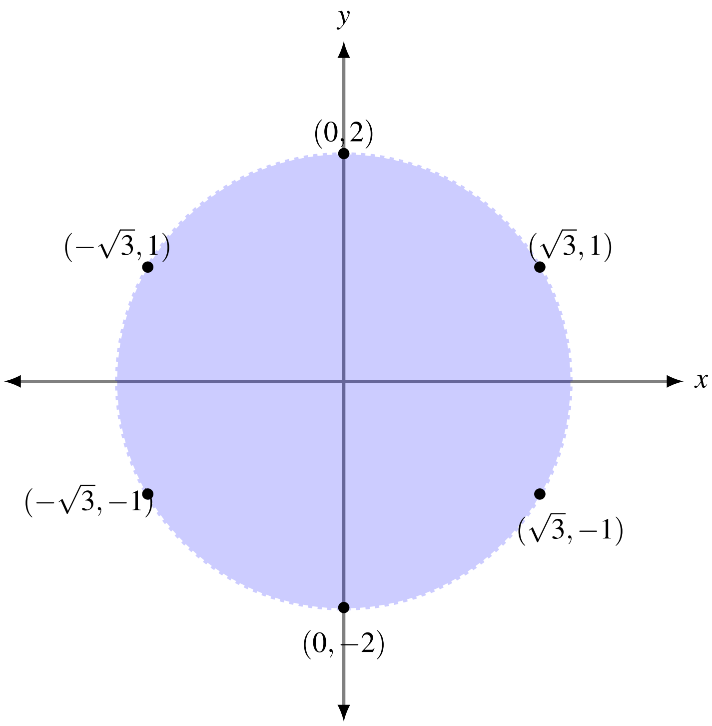
■ **Example 1.47** Find and plot the values of $\sqrt[6]{-64}$.

Thus, we need to find r, θ such that $\left(re^{i\theta}\right)^6 = -64$. Consider the polar form of -64, where $r = 64$ and $\theta = \pi, 3\pi, 5\pi, 7\pi, 9\pi, 11\pi$.

Now, by the definition of the root: $z^{1/6} = r^{1/6}e^{i\theta/6} = r^{1/6}e^{i(2\pi n+\theta)/6} \Rightarrow r = 2, \theta = \pi/6, \pi/2, 5\pi/6, 7\pi/6, 3\pi/2, 11\pi/6, \ldots$ Observe that $13\pi/6 = \pi/6$ (on the complex plane).

Thus, the six roots are: $2e^{i\pi/6}, 2e^{i\pi/2}, 2e^{i5\pi/6}, 2e^{i7\pi/6}, 2e^{i3\pi/2}, 2e^{i11\pi/6}$ or $z = \sqrt{3}+i, 2i, -\sqrt{3}+i, -\sqrt{3}-i, -2i, \sqrt{3}-i$.



■

1.11 The Exponential and Trigonometric Functions

Recall from a previous section the power series expansion for $e^z = 1 + z + \frac{z^2}{2!} + \ldots \frac{z^n}{n!} + \ldots$. This can be written in another form:

$$e^z = e^{x+iy} = e^x e^{iy} = e^x\left[\cos(y) + i\sin(y)\right].$$

This new form using Euler's Formula may be easier to use in some instances.

■ **Example 1.48** i) $e^{3+i\pi/2} = e^3 e^{i\pi/2} = e^3\left[\cos(\pi/2) + i\sin(\pi/2)\right] = e^3\left[0+i\right] = e^3 i$.

ii) $e^{3\ln 3 - i\pi/2} = e^{\ln(27)}e^{-i\pi/2} = 27\left[\cos(-\pi/2) + i\sin(-\pi/2)\right] = 27[0-i] = -27i$. ■

Recall Euler's Formula:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{1.19}$$

$$e^{-i\theta} = \cos(\theta) - i\sin(\theta) \tag{1.20}$$

Subtracting (8.114) from (8.54):

$$e^{i\theta} - e^{-i\theta} = 2i\sin(\theta) \qquad \Rightarrow \qquad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Adding (8.54) to (8.114) gives:

$$e^{i\theta} + e^{-i\theta} = 2\cos(\theta) \qquad \Rightarrow \qquad \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

These expressions hold for real θ, but can be extended to all complex numbers, z, by replacing $\theta \mapsto z$.

Definition 1.11.1 *(Complex Trigonometric Functions)*

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2} \qquad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}. \tag{1.21}$$

The remaining trigonometric functions can be derived using the usual relations:

$$\tan(z) = \frac{\sin(z)}{\cos(z)}, \quad \cot(z) = \frac{\cos(z)}{\sin(z)}, \quad \csc(z) = \frac{1}{\sin(z)}, \quad \sec(z) = \frac{1}{\cos(z)}. \tag{1.22}$$

■ **Example 1.49** Find $\sin(i)$.

Using the definition:

$$\sin(i) = \frac{e^{i^2} - e^{-i^2}}{2i} = \frac{e^{-1} - e^{1}}{2i} \approx 1.1752i.$$

■

R One interesting difference from real numbers is the range for sine and cos. For real x, $|\sin(z), \cos(z)| \leq 1$. This bound does not hold for the complex forms of sine and cosine as seen by the previous example.

We can recover some of the same calculus trig identities for the complex versions.

■ **Example 1.50** Does $\sin^2(z) + \cos^2(z) = 1$?

Check: $\sin^2(z) = \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2 = \frac{e^{2iz}-2+e^{-2iz}}{-4}$.

Check: $\cos^2(z) = \left(\frac{e^{iz}+e^{-iz}}{2i}\right)^2 = \frac{e^{2iz}+2+e^{-2iz}}{4}$.

So, $\sin^2(z) + \cos^2(z) = \frac{4}{4} = 1$. ■

■ **Example 1.51** Show the double angle formula: $\sin(2z) = 2\cos(z)\sin(z)$.

$\sin(2z) = \frac{e^{2iz}-e^{-2iz}}{2i} = \frac{(e^{iz}+e^{-iz})(e^{iz}-e^{-iz})}{2i} = 2\left[\frac{e^{iz}+e^{-iz}}{2}\right]\left[\frac{e^{iz}-e^{-iz}}{2i}\right] = 2\cos(z)\sin(z)$. ■

What about the derivatives of the sine and cosine? Are they the same or very different?

■ **Example 1.52** i) $\frac{d}{dz}\sin(z) = \frac{d}{dz}\left[\frac{e^{iz}-e^{-iz}}{2i}\right] = \frac{ie^{iz}+ie^{-iz}}{2i} = \frac{e^{iz}+e^{-iz}}{2} = \cos(z)$. Same!

ii) $\frac{d}{dz}\cos(z) = \frac{d}{dz}\left[\frac{e^{iz}+e^{-iz}}{2}\right] = \frac{ie^{iz}-ie^{-iz}}{2} = -\left[\frac{e^{iz}-e^{-iz}}{2i}\right] = -\sin(z)$. Same! ■

1.12 Hyperbolic Functions

What do sine and cosine look like when a complex number is purely imaginary, $z = iy$?

$$\sin(iy) = \frac{e^{i(iy)} - e^{-i(iy)}}{2i} = \frac{e^{-y} - e^{y}}{2i} = i\frac{e^{y} - e^{-y}}{2}$$
$$\cos(iy) = \frac{e^{i(iy)} + e^{-i(iy)}}{2} = \frac{e^{-y} + e^{y}}{2} = \frac{e^{y} + e^{-y}}{2}.$$

These are special functions and come up when solving dynamic problems (differential equations, more in Math Methods II!).

Definition 1.12.1 *(Hyperbolic Trig Functions)*

$$\sinh(z) = \frac{e^{z} - e^{-z}}{2} \qquad \cosh(z) = \frac{e^{z} + e^{-z}}{2}. \tag{1.23}$$

Similarly,

$$\tanh(z) = \frac{\sinh(z)}{\cosh(z)}, \quad \coth(z) = \frac{\cosh(z)}{\sinh(z)}, \quad \text{sech}(z) = \frac{1}{\cosh(z)}, \quad \text{csch}(z) = \frac{1}{\sinh(z)}. \tag{1.24}$$

Thus, observe that $\sin(iy) = i\sinh(y)$ and $\cos(iy) = \cosh(y)$. Now consider some trig identities with hyperbolic trig functions.

■ **Example 1.53** Show: $\cosh^2(z) - \sinh^2(z) = 1$

Using the definition, $\cosh^2(z) = \left[\frac{e^{z}+e^{-z}}{2}\right]^2 = \frac{e^{2z}+2+e^{-2z}}{4}$

Also, using the definition: $\sinh^2(z) = \left[\frac{e^{z}-e^{-z}}{2}\right]^2 = \frac{e^{2z}-2+e^{-2z}}{4}$. So, $\cosh^2(z) - \sinh^2(z) = \frac{4}{4} = 1$.
■

We also can consider the derivatives of the hyperbolic trig functions:

■ **Example 1.54** i) $\frac{d}{dz}\sinh(z) = \frac{d}{dz}\left[\frac{e^{z}-e^{-z}}{2}\right] = \frac{e^{z}+e^{-z}}{2} = \cosh(z)$.

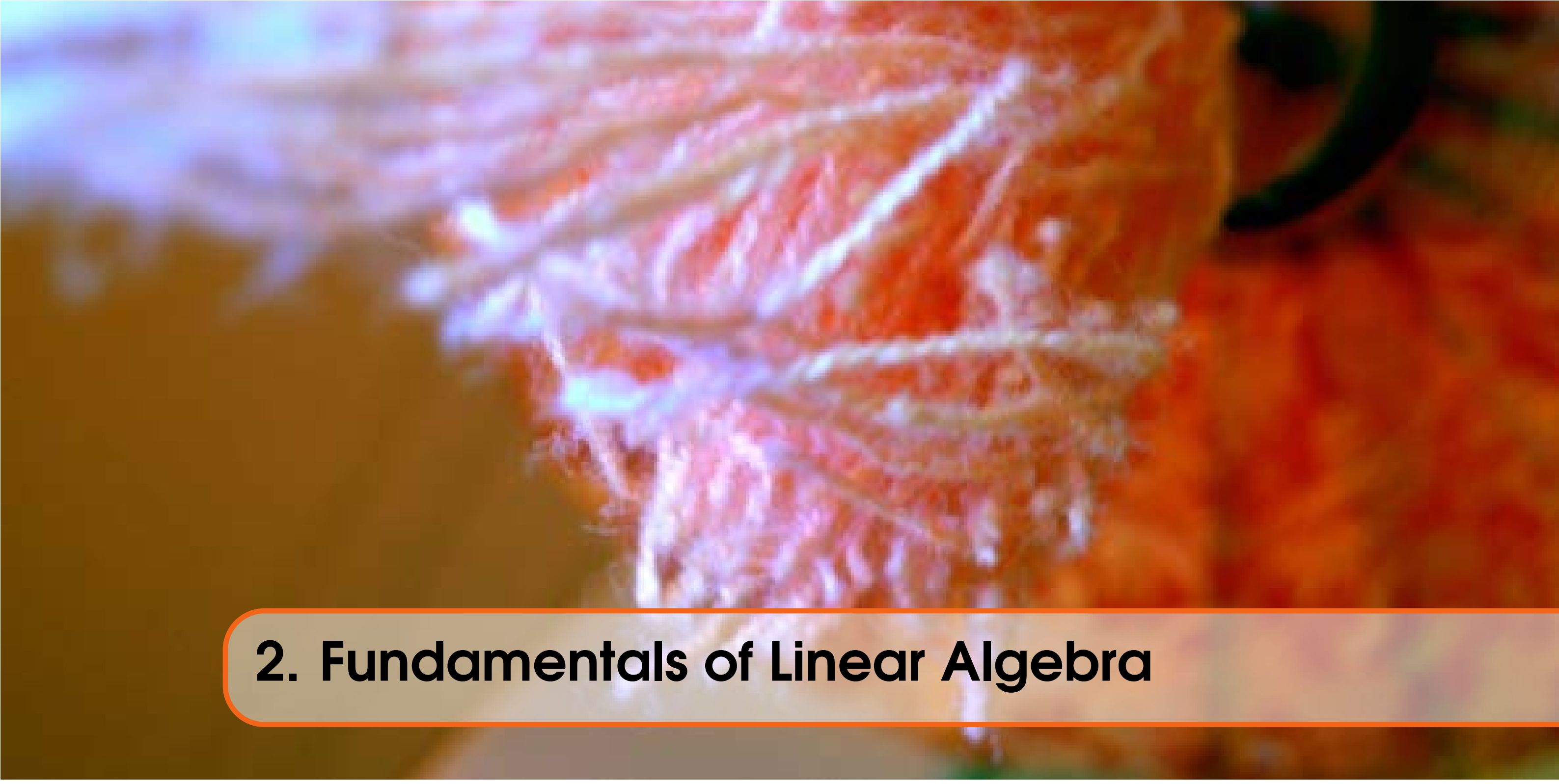
ii) $\frac{d}{dz}\cosh(z) = \frac{d}{dz}\left[\frac{e^{z}+e^{-z}}{2}\right] = \frac{e^{z}-e^{-z}}{2} = \sinh(z)$. ■

R Observe that there is no sign change when taking the derivative of the hyperbolic cosine. This is in contrast to normal trig functions where $\frac{d}{dz}\cos(z) = -\sin(z)$.

Exercise 1.1 Why are complex roots of quadratic equations always found in pairs?

Hint: Look at the Quadratic Formula, which is valid for any quadratic equation. ■

Part Two: Linear Algebra



2. Fundamentals of Linear Algebra

Linear Algebra basically refers to linear relationships between objects. Can we think of examples of linear functions we have seen in the past?

Definition 2.0.2 A function $f(x)$ is linear if:

1. $f(x+y) = f(x) + f(y)$.
2. $f(cx) = cf(x)$ for any real number c.

Linear algebra takes this idea to the next level of abstraction by introducing the idea of a linear operation. The idea is to take a system of linear equations and solve them simultaneously using object called matrices. This section will start by introducing the relationship between matrices and systems of linear equations. After the basic definitions are known we will begin to explore how to work with these objects to solve real problems.

2.1 Systems of Linear Equations

First we must define what is meant by a single *Linear Equation*.

Definition 2.1.1 *(Linear Equation)* A linear equation is any equation of the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b,$$

where $a_1, \ldots, a_n, b$ are constant real numbers and $x_1, \ldots, x_n$ are the unknown variables.

■ **Example 2.1** Are the following equations linear?

i) $4x_1 - 5x_2 + 2 = x_1$

If we rearrange we find: $3x_1 - 5x_2 = -2$, so yes!

ii) $x_2 = 2(\sqrt{6} - x_1) + x_3$

If we rearrange we find: $2x_1 + x_2 - x_3 = 2\sqrt{6}$, so yes! ■

Sometime it is easier to see if an equation meets any of these easy cases for being **nonlinear** to rule out linearity:

i) *Products of Variables:* $x_1 x_2 + x_3 = 4$ is Nonlinear

ii) *Trig. Functions:* $\sin(x_1) + x_2 = 2$ is Nonlinear

iii) *Powers/Roots:* $x^2, x^{1/2}$ are Nonlinear.

Definition 2.1.2 *(A System of Linear Equations)* A system of linear equations is a collection of one or more linear equations involving the same set of variables (e.g., $x_1, \ldots, x_n$).

Definition 2.1.3 *(Solution of a Linear System)* A list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation in the system true when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$ respectively.

■ **Example 2.2** Possible solutions for two equations in two variables:

i) One Unique Solution (Consistent):

$$\begin{aligned} x_1 + x_2 &= 10 \\ -x_1 + x_2 &= 0 \end{aligned}$$

ii) No Solution (Inconsistent):

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 8 \end{aligned}$$

iii) Infinitely Many Solutions (Consistent):

$$\begin{aligned} x_1 + x_2 &= 3 \\ -2x_1 - 2x_2 &= -6 \end{aligned}$$

■

Definition 2.1.4 *(Equivalent Systems)* Two linear systems with the same solution set.

2.1.1 Matrix Notation

Given a linear system in **Standard Form**:

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

we can define two associated matrices. The first is the *coefficient matrix* made up of the coefficients of each variable:

$$\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix},$$

and the *augmented matrix* that is composed of the coefficients and the righthand side

$$\begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 3 \end{bmatrix}.$$

2.1.2 Elementary Row Operations

There are three basic operations we can perform on an augmented matrix without changing its solution set:
1. *(Replacement)* Add one row to a multiple of another row.
2. *(Interchange)* Switch two rows.
3. *(Scaling)* Multiply all entries in a row by a nonzero constant.

These three operations are your only valid tools in attacking problems involving matrices. We will learn more advanced methods throughout the course, but for now we will stick with these three rules.

Definition 2.1.5 *(Row Equivalent Matrices)* Two matrices where one matrix can be transformed into the other matrix by a sequence of elementary row operations.

Fact: If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Definition 2.1.6 *(Size of a Matrix)* We say a matrix with m rows and n columns is an $m \times n$ matrix. Thus, a 2×3 matrix has two rows and three columns. In fact the Matrix A is composed of elements (numbers) a_{ij} where i corresponds to the row and j the column.

■ **Example 2.3** Use the three elementary row operations to solve the following linear system:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9, \end{aligned}$$

Step 1: Put the equations into Standard Form
Step 2: Find the augmented matrix for the system of linear equations
Step 3: Row reduce using elementary row operations
Step 4: Back Substitute the values into the system to solve.

Solution: (29,16,3)

Final Step: Check by plugging the solution back into the original system. ■

2.1.3 Fundamental Questions In Linear Algebra

1. Is the system consistent? (Does a solution exist?)

2. If a solution exists, is it **unique**? (Is there one and only one solution)?
These questions are answered during the course of elementary row operations.

If the augmented matrix ever has a row with all zeros except the last element is nonzero, $[00...0b]$, then the system is inconsistent and there is no solution!

More on uniqueness in the next section (hint: it will have to do with the concept of pivot variables).

2.2 Row Reduction and Echelon Forms

Definition 2.2.1 *(Echelon Form).* A matrix is in *echelon form* if:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry (e.g., left most entry, also called *pivot*) of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero.

Definition 2.2.2 *(Reduced Echelon Form).* A matrix is in *reduced echelon form* if in addition to 1.-3.:

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

■ **Example 2.4** i) Row reduce the following matrix to echelon form and locate the pivot columns (columns which contain a pivot).

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Row reduce to see that the pivot columns are 1, 2, and 4. There can be no more than 1 pivot in any row.

ii) Row reduce the following matrix to reduced echelon form.

$$\begin{bmatrix} 0 & -3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 12 \end{bmatrix}$$

■

2.2.1 Solutions of Linear Systems

Definition 2.2.3 A **basic variable** is any variable that corresponds to a pivot column in the augmented matrix of a system.

A **free variable** is any variable that is not a basic variable.

The final step in solving any linear system is writing all the basic variables in terms of any free variables.

■ **Example 2.5**

$$\begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -8 & 5 & \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \Rightarrow \begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 8x_4 \\ x_4 \text{ is free} \\ x_5 = 7. \end{cases}$$

■

Definition 2.2.4 The **general solution** of a system of linear equations provides a parametric description of the solution set.

Thinking Question: The above example has infinitely many solutions. Why is it true?

Definition 2.2.5 The **Transpose** of a matrix denoted A^T is the matrix formed when the rows and columns of A are switched. $(A^T)_{ij} = A_{ji}$, and thus the transpose of an $m \times n$ matrix is an $n \times m$ matrix.

■ **Example 2.6** Given a matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, find its transpose.

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$ ■

Definition 2.2.6 *(Rank of a Matrix)* The number of nonzero rows remaining when a matrix has been row reduced is the rank of the matrix.

If we know the rank of a matrix we know how many solutions to expect based on the original size of the matrix, M is $m \times n$ and A is the row-reduced augmented matrix. Consider the general problem of solving m equations in n unknowns:
1. If $rank(M) < rank(A)$, the equations are inconsistent .
2. If $rank(M) = rank(A) = n$ (the number of unknowns), there is exactly one solution.
3. If $rank(M) = rank(A) = R < n$, then there are R basic variables and $n - R$ free variables resulting in infinitely many solutions.

2.3 Determinants and Cramer's Rule

Recall in the last lecture we introduced the concept of a matrix to provide a convenient and organized way to solve a system of linear equations. The main concepts from the last section were the elementary row operations and deciding whether a matrix has none, one or infinitely many solutions. An arbitrary $m \times n$ matrix is just a display of coefficients for m equations in n unknowns. Through this next section we focus on a special type of matrix called a *square matrix*. In this case the matrix has the exact same number of rows and columns (e.g., $n \times n$).

Now that we have these matrices we want to use them to do more advanced things then simply solving linear systems. The first basic quantity associated to any square matrix that will be useful going forward is the determinant. The determinant can be though of as a measure of volume in some sense (more to come on this later). The loose analog for real numbers is the absolute value, $|\cdot|$.

Start with the simplest case of a 2×2 matrix of the form

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{2.1}$$

Definition 2.3.1 The determinant of a 2×2 matrix A, denoted $|A|$, is defined to be

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \tag{2.2}$$

Note that the determinant of a 1×1 matrix, $A = a$ is a trivial extension of this idea $|A| = a$.

■ **Example 2.7** i) Find the determinant of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

$det(A) = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1(4) - 2(3) = 4 - 6 = -2.$

ii) Find the determinant of $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

$det(A) = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 1(4) - 2(2) = 4 - 4 = 0.$ ■

In order to introduce a formula for the determinant of an arbitrary $n \times n$ matrix we first must introduce some notation. Given a matrix A we can define a sub-matrix $A_i j$ where the ith row and jth column have been deleted.

■ **Example 2.8**

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}, \qquad A_{23} = \begin{bmatrix} 1 & 2 & 4 \\ 9 & 10 & 12 \\ 13 & 14 & 16 \end{bmatrix}.$$

■

Definition 2.3.2 *(Determinant of an $n \times n$ Matrix)* For $n \geq 2$, the **determinant** of an $n \times n$ matrix A is given by

$$det(A) = a_{11}det(A_1 1) - a_{12}det(A_{12}) + \ldots + (-1)^{1+n}a_{1n}det(A_{1n}) = \sum_{j=1}^{n}(-1)^{1+j}a_{1j}det(A_{1j}). \tag{2.3}$$

This process is call *Cofactor Expansion*.

Thus, for an $n \times n$ matrix we keep applying cofactor expansion until all the remaining determinants are 2×2.

■ **Example 2.9** i) Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

Solution: 1

ii) Compute the determinant of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Solution: 6 ■

R Cofactor expansion can actually be done about any row or column not just the first one

$$|A| = det(A) = \begin{cases} \sum_{j=1}^{n}(-1)^{i+j}a_{ij}det(A_{ij}) & \text{Expand about row } i \\ \sum_{i=1}^{n}(-1)^{i+j}a_{ij}det(A_{ij}) & \text{Expand about column } j. \end{cases} \tag{2.4}$$

■ **Example 2.10** i) Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$ using cofactor expansion about the third column

Solution: 1

ii) Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{bmatrix}$

Solution: 14 ■

2.3.1 Special Case: Upper and Lower Triangular Matrices

Definition 2.3.3 An $n \times n$ matrix A is said to be *upper triangular* if all the elements below the main diagonal, a_{ij} for $i < j$, are zero. An $n \times n$ matrix A is said to be *lower triangular* if all the elements above the main diagonal, a_{ij} for $i > j$, are zero. Finally, a matrix is said to be diagonal if the only nonzero elements are in positions (i,i) for $i = 1, \ldots, n$.

FACT: If a matrix A is one of the three cases of triangular matrices (e.g., upper, lower, diagonal), then the determinant is just the product of the diagonal elements.

■ **Example 2.11** i) Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Solution: -1

ii) Compute the determinant of $A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

Solution: -24 ■

2.3.2 Properties of Determinants

Facts:
1. If each element of one row or one column of a determinant is multiplied by a number k, the value of the determinant is multiplied by k.

2. The value of a determinant is zero if one of the following occurs:
(a) All elements of one row or column are zero.
(b) Two rows or two columns are identical.
(c) Two rows or two columns are proportional.

3. If two rows or two columns of a determinant are interchanged, the value of the determinant changes sign.

4. The value of a determinant is unchanged if:
(a) Row are written as columns and columns as rows (e.g., $det(A) = det(A^T)$).
(b) We add to each element of one row, k times the corresponding element of another row, where k is any number (and a similar statement for columns).

■ **Example 2.12** i) Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{bmatrix}$.

Solution: -10.

ii) Find the determinant of $A = \begin{bmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{bmatrix}$.

Solution: -40.

iii) Find the determinant of $A = \begin{bmatrix} 2 & 3 & 0 & 1 \\ 4 & 7 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 1 & 2 & 0 & 4 \end{bmatrix}$.

Solution: -12. ■

■ **Example 2.13** (Application) Find the equation of a plane through $(0,0,0),(1,0,1),(1,2,0)$. Recall the equation of a plane has the form $ax+by+cz+d=0$. Treat a,b,c,d as the unknowns. We can setup the following determinant problems to find the equation of the plane

$$\begin{vmatrix} x & y & z & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 0 & 1 \end{vmatrix} = 0. \tag{2.5}$$

Using cofactor expansion about the first row and simplifying we find $-2x+y+2z=0$ is the equation for the plane. ■

2.3.3 Cramer's Rule

Recall the original purpose of linear algebra was to solve linear systems more efficiently. Can we use the determinant to solve a system of linear equations? The answer is yes and the solution method is called *Cramer's Rule*

Theorem 2.3.1 *(Cramer's Rule)* Given the following linear system with n unknowns, $x_1,\dots,x_n$ and coefficients a_{ij}

$$a_{11}x_1 + \dots + a_{1n}x_n = b_1$$

$$\vdots a_{n1}x_1 + \dots + a_{nn}x_n \qquad = b_n.$$

Also, define $D := det(A)$ is the determinant of the coefficient matrix consisting of the a_{ij} and $D_j = det(A_j)$ where A_j is the matrix where the jth column is replaced by the righthand side $b_1,\dots,b_n$. Using these quantities we can find the solution:

$$x_1 = \frac{D_1}{D},\dots,x_j = \frac{D_j}{D},\dots,x_n = \frac{D_n}{D}. \tag{2.6}$$

Observe that if the determinant $D=0$ there is no solution.

■ **Example 2.14** Use Cramer's Rule to solve the following linear systems: i)

$$x+2y=1$$
$$-x+3y=4$$

Solution: $x=1$, $y=-1$.

i)

$$x+y+z=3$$
$$3y-z=-2$$
$$2x-z=0$$

Solution: $x=1$, $y=0$, $z=2$. ■

2.4 Vectors

This section should be a review from calculus or a brief introduction to vectors if you are unfamiliar. In terms of matrices, a vector is a matrix with only one column.

Definition 2.4.1 An n-dimensional vector has the form:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

where u_i is referred to as the ith component of the vector $\mathbf{u}$.

A vector describes physical quantities such as velocity which require a magnitude and a direction.

Definition 2.4.2 The magnitude or norm of a vector is denoted by

$$|\mathbf{u}| = \|\mathbf{u}\| = \sqrt{u_1^2+\ldots+u_n^2}.$$

Definition 2.4.3 *(Unit Vector)* There is a special kind of vector which will be useful in the coming lectures that has length 1. Any vector with this property is called a **unit vector**. If a vector does not have unit length it can easy be scaled to have length 1 by dividing each element of the vector $\mathbf{v}$ by its norm, $|\mathbf{v}|$, $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$.

In 2D, we have two basis vector from which all other vectors can be constructed, $\hat{\mathbf{i}} = [1,0], \hat{\mathbf{j}} = [0,1]$. In 3D, we have three basis vectors, $\hat{\mathbf{i}} = [1,0,0], \hat{\mathbf{j}} = [0,1,0], \hat{\mathbf{k}} = [0,0,1]$. This idea can be extended to any dimension n, resulting in n basis vectors $\hat{\mathbf{e}}_i$ having a zero in every component except for the ith component which is 1.

Definition 2.4.4 *(Zero Vector)* There is another special kind of vector which has magnitude zero. The zero vector $\mathbf{0} = [0,0,\ldots,0]$.

There are two basic operations among vectors:

1. Vector addition, $\mathbf{u}+\mathbf{v} = (u_1+v_1, u_2+v_2, \ldots, u_n+v_n)$

2. Scalar Multiplication, $c\mathbf{u} = (cu_1, c_u2, \ldots, cu_n)$.

R Vector addition is *commutative*, in other words $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ and *associative* $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$.

■ **Example 2.15** Let $\mathbf{u}=\begin{bmatrix}1\\2\end{bmatrix}$. Express $\mathbf{u}, 2\mathbf{u}$, and $-\frac{3}{2}\mathbf{u}$ on a graph. ■

2.4.1 Scalar Product

Unlike in the previous section on complex numbers, the notation of multiplication of vectors is not well defined. Instead of a traditional product of vectors the useful notion of a *scalar product* was introduced.

Definition 2.4.5 *(Scalar Product)* Given two vectors of the same length, $\mathbf{u},\mathbf{v}$, the scalar product is defined as

$$\mathbf{u}\cdot\mathbf{v}=u_1v_1+u_2v_2+\ldots+u_nv_n.$$

The scalar product has the following properties:
1. $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$
2. $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w}=\mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}$.

■ **Example 2.16** i) Let $\mathbf{u}=[1,2,3]$ and $\mathbf{v}=[-1,0,1]$. Then $\mathbf{u}\cdot\mathbf{v}=1(-1)+2(0)+3(1)=2$.

ii) Let $\mathbf{u}=[1,4]$ and $\mathbf{v}=[-1,-2]$. Then $\mathbf{u}\cdot\mathbf{v}=1(-1)+4(-2)=-9$.

iii) In 3D, $\hat{\mathbf{i}}\cdot\hat{\mathbf{j}}=1(0)+0(1)+0(0)=0$. ■

The scalar product is a very useful quantity we can give information about the angle between the two vectors involved and the magnitude. This will be crucial for application that require one to determine when vectors are parallel or perpendicular.

Theorem 2.4.1 Given two vectors of equal length $\mathbf{u},\mathbf{v}$, then the scalar product can also be expressed as

$$\mathbf{u}\cdot\mathbf{v}=|\mathbf{u}|\mathbf{v}|\cos(\theta), \tag{2.7}$$

where θ is the angle between the two vectors. In particular we see that if the two vectors are parallel (e.g., $\theta=0$), then the scalar product is just the product of the magnitudes (and positive!). If the two vectors are perpendicular (e.g., $\theta=\pi/2$), then the scalar product is zero.

Notice that in the previous example all the unit basis vectors in any dimension are perpendicular.

R If two vectors are parallel, then every component of one of the vectors is proportional to the same component in the other vector (e.g., $u_1/v_1=u_2/v_2=u_3/v_3$). In other words, they are scalar multiples of each other. For example, $\mathbf{u}=[1,1]$ and $\mathbf{v}=[2,2]$.

■ **Example 2.17** i) Take the scalar product of a vector with itself, $\mathbf{v}\cdot\mathbf{v}=|\mathbf{v}|^2\cos(0)=|\mathbf{v}|^2$. In particular, we find an alternate definition for the norm of a vector: $|\mathbf{v}|=\sqrt{\mathbf{v}\cdot\mathbf{v}}$.

ii) Find the angle between $\mathbf{u}=[1,0]$ and $\mathbf{v}=[1,1]$. Using the alternate definition of the scalar product $\mathbf{u}\cdot\mathbf{v}=|\mathbf{u}||\mathbf{v}|\cos(\theta)$. Plugging in the appropriate values we find $1=\sqrt{2}\cos(\theta)$. Thus, $\cos(\theta)=1/\sqrt{2}$ and therefore $\theta=\pi/4$. ■

2.4.2 Vector (Cross) Product

In addition to the scalar product we can define another form of product that results in a vector.

Definition 2.4.6 *(Cross Product)* To find a vector which is perpendicular to two given three dimensional vectors, denoted $\mathbf{w} = \mathbf{u} \times \mathbf{v}$.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = \hat{\mathbf{i}}(u_y v_z - u_z b_y) + \hat{\mathbf{j}}(u_z v_x - u_x v_z) + \hat{\mathbf{k}}(u_x v_y - u_y v_x). \tag{2.8}$$

Similar to the scalar product there is an alternate way to find the magnitude of the cross product $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin(\theta)$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

The resulting vector $\mathbf{w}$ is perpendicular to the plane containing $\mathbf{u}$ and $\mathbf{v}$. Its direction is determined by the “righthand rule".

■ **Example 2.18** Let $\mathbf{u} = [0,1,0]$ and $\mathbf{v} = [1,0,0]$. Find $\mathbf{u} \times \mathbf{v}$.

Solution: $\mathbf{u} \times \mathbf{v} = [0,0,1]$. ■

There are a few special cases of the cross product we should highlight before moving on:
1. If $\mathbf{u} \times \mathbf{v} = 0$, the $\mathbf{u}$ and $\mathbf{v}$ are parallel or anti-parallel (opposite directions).
2. $\mathbf{u} \times \mathbf{u} = |\mathbf{u}|^2 \sin(\theta) = 0$, since $\theta = 0$.
3. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.
4. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$.

■ **Example 2.19** i) Let $\mathbf{u} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{v} = 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$. Find $\mathbf{u} \times \mathbf{v}$.

Solution: $[-4,-2,6]$.

ii) Let $\mathbf{u} = [0,3,-1]$ and $\mathbf{v} = [1,2,3]$. Find $\mathbf{u} \times \mathbf{v}$.

Solution: $[11,-1,-3]$. ■

2.4.3 Orthogonality

Definition 2.4.7 *(Orthogonal)* If two vectors are perpendicular we say they are *orthogonal*. Orthogonal vectors are characterized by vectors whose scalar product is zero. If, in addition, the vectors have unit length then they are called *orthonormal*.

Next, we want to define the notion of distance between vectors.

Definition 2.4.8 The distance between $\mathbf{u}$ and $\mathbf{v}$ is:

$$dist(\mathbf{u},\mathbf{v}) = ||\mathbf{u} - \mathbf{v}|| = \sqrt{(u_1 - v_1)^2 + \ldots + (u_n - v_n)^2}. \tag{2.9}$$

In general, $||\mathbf{u} - \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2\mathbf{u} \cdot \mathbf{v}$ and $||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 + 2\mathbf{u} \cdot \mathbf{v}$.

If $||\mathbf{u} - \mathbf{v}||^2 = ||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$, then $\mathbf{u} \cdot \mathbf{v} = 0$ or the vectors are orthogonal.

Returning to Matrices

Not all linear systems $A\mathbf{x} = \mathbf{b}$ have solutions. For example

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}. \tag{2.10}$$

The solution to this system are multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and the righthand side is not a multiple. Thus, many times when we solve linear systems in physical applications we have to find the closest solution to the real thing, $\hat{\mathbf{x}}$. This is defined to be the point whose distance from the solutions is minimized, $\|A\hat{\mathbf{x}} - \mathbf{b}\|$.

In particular, the orthogonal projection of the righthand side onto the solution will be the best approximation.

Definition 2.4.9 *(Orthogonal Projection)* The projection of vector $\mathbf{u}$ onto $\mathbf{v}$ is

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{(\mathbf{u} \cdot \mathbf{v})}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}. \tag{2.11}$$

Using the orthogonal projection, the closest right hand side which has a solution is $[1.4, 2.8]$ and the $\hat{\mathbf{x}}$ which produces this is $\hat{\mathbf{x}} = [1.4, 0]$.

2.5 Lines, Planes, and Geometric Applications

A common problem in physics is finding the vector between two points. This can be done by taking the difference of the vectors (the direction of the result will depend on which vector is subtracted).

■ **Example 2.20** Find the vector from $\mathbf{u} = [1, 2]$ to $\mathbf{v} = [1, 0]$.

Solution: Compute $\mathbf{v} - \mathbf{u} = [1-1, 0-2] = [0, -2]$. Notice $\mathbf{u} - \mathbf{v} = [1-1, 2-0] = [0, 2]$, has the same magnitude, but the opposite sign. This indicated that it is the vector from $\mathbf{v}$ point toward $\mathbf{u}$. ■

In two dimension another common problem is finding the line from a point (x_0, y_0) in the direction of a given vector $\mathbf{v} = [a, b]$. This general line has the form

$$\mathbf{x} - \mathbf{x}_0 = \hat{\mathbf{i}}(x - x_0) + \hat{\mathbf{j}}(y - y_0).$$

If the line must be parallel to the vector $\mathbf{v}$, then the components of the line must be proportional to the components of $\mathbf{v}$ (Recall from 3.3 if vectors are parallel their components are proportional).

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} \quad \Rightarrow \quad \frac{y - y_0}{x - x_0} = \frac{b}{a} \quad \Rightarrow \quad y = \frac{b}{a}(x - x_0) + y_0. \tag{2.12}$$

This is exactly the familiar *slope intercept form* of a line. Another way to write this is in *parametric form* where $\mathbf{x} - \mathbf{x}_0$ is a scalar multiple of $\mathbf{v}$

$$\mathbf{x} - \mathbf{x}_0 = \mathbf{v}t \qquad \Rightarrow \qquad \mathbf{x} = \mathbf{x}_0 + \mathbf{v}t. \tag{2.13}$$

This form has a physical meaning: $\mathbf{x}_0$ is the starting point of a particle and $\mathbf{x}$ is the location of the particle at time t if it moves with velocity $\mathbf{v}$.

We can preform and analogous procedure in three dimensions. Find the line that passes through $\mathbf{x} = (x_0, y_0, z_0)$ in the direction of $\mathbf{v} = (a, b, c)$. Using the first approach of parallel vectors we known the components are proportional:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \qquad (\text{if } a, b, c \neq 0). \tag{2.14}$$

For example, if $c = 0$ then $z = z_0$. Using the second approach we can find the parametric form:

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{v}t = \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases} . \tag{2.15}$$

■ **Example 2.21** Given the point $(1,0,1)$ find the equation for the line through this point and parallel to $(1,2,3)$.

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{v}t = \begin{cases} x = 1 + t \\ y = 2t \\ z = 1 + 3t \end{cases} . \tag{2.16}$$

■

It possible to ask a similar question, given a point find the line through this point , but perpendicular to a vector $\mathbf{v} = (a,b)$. Recall, that two line are perpendicular (*orthogonal*) if their scalar product is zero

$$(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{v} = 0 \quad \Rightarrow \quad a(x - x_0) + b(y - y_0) = 0 \quad \Rightarrow \quad y = -\frac{a}{b}(x - x_0) + y_0. \tag{2.17}$$

■ **Example 2.22** Given the point $(1,1)$ find the equation for the line through this point and orthogonal to $(1,2)$.

$$y = -\frac{1}{2}(x - 1) + 1. \tag{2.18}$$

■

In three dimensions, this can be used to find the equation of a plane $(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{v} = a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ or after rearranging $ax + by + cz = d = ax_0 + by_0 + cz_0$.

■ **Example 2.23** Find the equation of the plane through $\mathbf{u} = (1,0,0)$, $\mathbf{v} = (1,1,1)$, and $\mathbf{w} = (0,1,1)$.

First we need to find a normal vector! To do this we find the vector from $\mathbf{u}$ to $\mathbf{v}$, $\mathbf{v} - \mathbf{u} = (0,1,1)$ and the vector from $\mathbf{u}$ to $\mathbf{w}$, $\mathbf{w} - \mathbf{u} = (-1,1,1)$. Now that we have two vectors in the plane containing $\mathbf{u}, \mathbf{v}, \mathbf{w}$ we can compute the normal to this plane using the cross product.

$$\mathbf{N} = (\mathbf{v} - \mathbf{u}) \times (\mathbf{w} - \mathbf{u}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = \hat{\mathbf{i}}(1 - 1) + \hat{\mathbf{j}}(-1 - 0) + \hat{\mathbf{k}}(0 + 1) = (0,-1,1). \tag{2.19}$$

So the equation of the plane is

$$0(x - x_0) - 1(y - y_0) + (z - z_0) = 0. \tag{2.20}$$

Plugging in one of the three points for (x_0, y_0, z_0) (e.g., $\mathbf{v}$) we find

$$-1(y - 1) + (z - 1) = 0 \qquad \Rightarrow \qquad -y + z = 0. \tag{2.21}$$

■

■ **Example 2.24** Find the equation of the line through $\mathbf{u} = (-1,1,0)$, and orthogonal to the plane in the previous example.

Since $\mathbf{N} = (0,-1,1)$ is perpendicular to the plane, then it must be parallel to the vector we want. Thus, returning to the previous set of examples we need to find the equation for the line through $\mathbf{u}$ and parallel to $\mathbf{N}$.

$$\mathbf{x} = \mathbf{u} + \mathbf{N}t = \begin{cases} x = -1 \\ y = 1 - t \\ z = t \end{cases} . \tag{2.22}$$

■

■ **Example 2.25** Find the closest distance from a point $P = (1,2,3)$ to the plane defined by $x + 2y + 2z - 1 = 0$.

What we need to solve this problem is the normal vector $\mathbf{n} = (1,2,2)$ and a point on the plane (e.g., $Q = (1,0,0)$. We now construct the vector from Q to P, $\overline{PQ} = P - Q = (0,2,3)$. The distance is then

$$\text{dist} = \left| \overline{PQ} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| . \tag{2.23}$$

Now we find $|\mathbf{n}| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$. Then $\frac{\mathbf{n}}{|\mathbf{n}|} = (1/3, 2/3, 2/3)$ and

$$\text{dist} = \left| \overline{PQ} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| = |0(1/3) + (2)(2/3) + (3)2/3| = |4/3 + 6/3| = |10/3| = 10/3.$$

Or there is an alternative method using the cross product instead of the scalar product. The distance from a point to a plane will be the magnitude of the vector from the point P to the nearest spot on the plane denoted R. Thus,

$$|\overline{PR}| = |\overline{PQ}| \sin(\theta) = \left| \overline{PQ} \times \frac{\mathbf{v}}{|\mathbf{v}|} \right| , \tag{2.24}$$

where θ is the angle between $\overline{PQ}$ and $\overline{RQ}$ and $\mathbf{v} = \overline{RQ}$. To find R we must use the previous example to find the line through P, but orthogonal to the plane

$$\mathbf{x} = P + \mathbf{n}t = \begin{cases} x = 1 + t \\ y = 2 + 2t \\ z = 3 + 2t \end{cases} . \tag{2.25}$$

Now plug the expressions for x, y, z into the equation of the plane and solve for t

$$\begin{aligned} (1 + t) + 2(2 + 2t) + 2(3 + 2t) - 1 = 0 \\ 9t + 10 = 0 \\ t = -10/9. \end{aligned}$$

Substitution of this value of t into (6.202) gives $R = (-1/9, -2/9, 7/9)$, the point on the plane that is also on the line through P. Now $\overline{RQ} = R - Q = (-10/9, -2/9, 7/9)$. To complete the computation we need to find

$$|\overline{PQ}| \sin(\theta) = \frac{|\overline{PQ} \times \overline{RQ}|}{|\overline{RQ}|} = \frac{|(20/9, -30/9, 20/9)|}{|(-10/9, -2/9.7/9)|} = \frac{\frac{10}{9}\sqrt{17}}{\frac{1}{3}\sqrt{17}} = \frac{10}{3}.$$

■

■ **Example 2.26** Find the distance from the point $P=(1,2,2)$ to the line joining $Q=(1,0,0)$ and $R=(-1,1,0)$.

Observe here that R is not necessarily the closest point to P so we will use the formula from the last part of (8.111). First, define $\mathbf{v}=R-Q=(-2,1,0)$. Then $|\mathbf{v}|=\sqrt{(-2)^2+1^2+0^2}=\sqrt{5}$ and $\mathbf{v}/|\mathbf{v}|=(-2/\sqrt{5},1/\sqrt{5},0)$. Also, $\overline{PQ}=P-Q=(0,2,2)$. Then from (8.111)

$$\text{dist}=\left|\overline{PQ}\times\frac{\mathbf{v}}{|\mathbf{v}|}\right|=|(0,2,2)\times(-2/\sqrt{5},1/\sqrt{5},0)|=\frac{1}{\sqrt{5}}|(0,2,2)\times(-2,1,0)| \tag{2.26}$$

$$=\frac{1}{\sqrt{5}}|(-2,-4,4)|=\frac{\sqrt{(-2)^2+(-4)^2+4^2}}{\sqrt{5}}=\frac{\sqrt{36}}{\sqrt{5}}=\frac{6}{\sqrt{5}}=\frac{6\sqrt{5}}{5}. \tag{2.27}$$

■

■ **Example 2.27** Find the distance between the lines $\mathbf{x}_1=-\hat{\mathbf{i}}+2\hat{\mathbf{j}}+(\hat{\mathbf{i}}-\hat{\mathbf{k}})t$ and $\mathbf{x}_2=\hat{\mathbf{j}}-2\hat{\mathbf{k}}+(\hat{\mathbf{j}}-\hat{\mathbf{i}})t$.

Step 1: Write each line in the parametric form $\mathbf{x}=\mathbf{x}_0+\mathbf{v}t$

$$\mathbf{x}_1=\mathbf{x}_{0,1}+\mathbf{v}_1t=\begin{cases}x_1=-1+t\\ y_1=2\\ z_1=-1t\end{cases}\qquad \mathbf{x}_2=\mathbf{x}_{0,2}+\mathbf{v}_2t=\begin{cases}x_2=-t\\ y_2=1+t\\ z_2=-2\end{cases} \tag{2.28}$$

To use (2.23), we need to identify P, Q and $\mathbf{n}$. First, P and Q are just $\mathbf{x}_{0,1}$ and $\mathbf{x}_{0,2}$ respectively. Thus, $P=(-1,2,0)$ and $Q=(0,1,-2)$ and $\overline{PQ}=P-Q=(1,-1,-2)$. To find the normal vector $\mathbf{n}$ we use the only tool we have for finding a vector orthogonal to another ... the cross product

$$\mathbf{n}=\mathbf{v}_1\times\mathbf{v}_2=\begin{vmatrix}\hat{\mathbf{i}}&\hat{\mathbf{j}}&\hat{\mathbf{k}}\\ 1&0&-1\\ -1&1&0\end{vmatrix}=-\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}=(-1,1,1),\quad |\mathbf{n}|=\sqrt{3}. \tag{2.29}$$

Therefore, using (2.23), we find

$$\text{dist}=\left|\overline{PQ}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|=\left|(1,-1,-2)\cdot(-1/\sqrt{3},1/\sqrt{3},1/\sqrt{3})\right|=|-1/\sqrt{3}-1/\sqrt{3}-2/\sqrt{3}|=|4/\sqrt{3}|=\frac{4}{\sqrt{3}}. \tag{2.30}$$

■

■ **Example 2.28** i) Find the direction of the line of intersection of the two planes $-x+2y+z=3$ and $x+y-3z=1$.

Since the intersection of the two planes must lie in both planes (by definition), then the line we are looking for must be orthogonal/perpendicular to the normal vector for each plane

$$\mathbf{n}_1=(-1,2,1)\qquad \mathbf{n}_2=(1,1,-3).$$

So the direction of the line is

$$\mathbf{n}_1\times\mathbf{n}_2=\begin{vmatrix}\hat{\mathbf{i}}&\hat{\mathbf{j}}&\hat{\mathbf{k}}\\ -1&2&1\\ 1&1&-3\end{vmatrix}=-7\hat{\mathbf{i}}+2\hat{\mathbf{j}}-3\hat{\mathbf{k}}=(-7,-2,-3). \tag{2.31}$$

ii) Find the cosine of the angle between the two planes.

This is equivalent to finding the cosine of the angle between the normal vectors. Using the definition of the scalar product

$$\begin{aligned}
\mathbf{n}_1 \cdot \mathbf{n}_2 &= |\mathbf{n}_1||\mathbf{n}_2|\cos(\theta) \\
(-1,2,1)\cdot(1,1,-3) &= \sqrt{6}\sqrt{11}\cos(\theta) \\
\frac{-2}{\sqrt{66}} &= \cos(\theta).
\end{aligned}$$

■

2.6 Matrix Operations

In this section we consider different ways of manipulating matrices to solve problems:

- Matrix Equations, $A\mathbf{x} = \mathbf{b}$
- Matrix Algebra $(+, -, \times, \div)$
- Inverse of a Matrix, $\mathbf{x} = A^{-1}\mathbf{b}$
- Functions of Matrices (e.g., e^A)

Definition 2.6.1 *(Equal Matrices)* Two matrices A and B are *equal* if and only if every single element is equal. In particular they must be the same size (e.g., both are $m \times n$).

■ **Example 2.29** Let $A = \begin{bmatrix} x^2-3 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find the values of x so that $A = B$.

Solution: By definition the two matrices are equal if all the elements are equal. So $x^2 - 3 = 1 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$.

Can a 2×3 matrix equal a 3×2 matrix? **No!** the dimensions must be the same! ■

2.6.1 Scalar Multiplication of Matrices

For any real number c, the matrix $cA = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ \vdots & & \vdots & \\ ca_{n1} & ca_{n2} & \cdots & ca_{nn} \end{bmatrix}$. It is important to notice that the scalar number c multiplies every entry.

Recall Section 3.3, and consider how the determinant is effected by scalar multiplication.

■ **Example 2.30** Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, which has determinant $\det(A) = 1(4) - 2(3) = 4 - 6 = -2$. Let $c = 2$ and find the determinant of cA.

Solution: First, compute $cA = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$, then $\det(cA) = 2(8) - 4(6) = 16 - 24 = -8$. It turns out in general $\det(cA) = c^n \det(A)$ where n is the size of the square matrix, A is $n \times n$. ■

2.6.2 Addition and Subtraction of Matrices

Before considering any algebraic operations on matrices one must make sure they are the same size! Addition and Subtraction of two matrices are similar just add/subtract the matching components.

$$A + B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$$

■ **Example 2.31** Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and let $B = \begin{bmatrix} 0 & -1 \\ -2 & 4 \end{bmatrix}$. Find $5A$, $A+B$, and $A-B$.

$$5A = \begin{bmatrix} 5(1) & 5(2) \\ 5(3) & 5(4) \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix}$$
$$A+B = \begin{bmatrix} 1+0 & 2+(-1) \\ 3+(-2) & 4+4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 8 \end{bmatrix}$$
$$A-B = \begin{bmatrix} 1-0 & 2-(-1) \\ 3-(-2) & 4-4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 5 & 0 \end{bmatrix}$$

■

Also, observe that $2A = A+A$. In order for addition and subtraction to make sense the dimensions of the matrices must be the same.

2.6.3 Multiplication and Division of Matrices

First, we must define what we mean by $AB = C$ for matrices A, B, and C. Given two matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, then the product

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+dh \\ ce+dg & cf+dh \end{bmatrix}. \tag{2.32}$$

The (i, j)th entry of the resulting matrix C is the scalar product of the ith row of A with the jth column of B.

■ **Example 2.32** Given two matrices $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}$, then the product

$$AB = \begin{bmatrix} 1(0)+0(2) & 1(-1)+0(1) \\ 2(0)+3(2) & 2(-1)+3(1) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 6 & 1 \end{bmatrix}. \tag{2.33}$$

■

R **Key Observation:** In order for matrix multiplication to work, the number of columns of matrix A must be equal to the number of rows in matrix B (for the scalar products to be well-defined). Thus, if A is $m \times n$, then B must be $n \times d$ for multiplication to work! In addition, the resulting matrix will have dimension $m \times d$.

So the inner dimensions of AB must be equal (e.g., n) and the outer dimensions give the size of the resulting matrix ($m \times d$).

■ **Example 2.33** Multiply

$$A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -3 \\ 6 & -7 \end{bmatrix} \tag{2.34}$$

Step 1: Check that the dimensions are valid for matrix multiplication.

Matrix A is 3×2 and matrix B is 2×2. The inner dimensions agree (e.g., 2) so we can multiply and the resulting matrix should be 3×2.

Thinking Question: What if we wanted to multiply BA? No! We cannot because the inner dimensions do not agree in this case (e.g., $3 \neq 2$).

Step 2: Carry Out the Multiplication

$$AB = \begin{bmatrix} 4(2)-2(6) & 4(-3)-2(-7) \\ 3(2)-5(6) & 3(-3)-5(-7) \\ 0(2)+1(6) & 0(-3)+1(-7) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ -24 & 26 \\ 6 & -7 \end{bmatrix}$$

■

What is the determinant of a product of matrices?

$$\det(AB) = \det(A)\det(B) = \det(AB).$$

This is only valid for square ($n \times n$ matrices).

2.6.4 Matrix Equation

A fundamental problem in Linear Algebra is finding a vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$. Consider

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}.$$

This matrix equation represents the following system of linear equations.

$$\begin{cases} x + 2y + 3z = 2 \\ x - y = 3 \\ z = -1 \end{cases}$$

How can we solve this system using Matrix Operations.

2.6.5 Solution Sets of Linear Systems

The most basic type of matrix equation to solve is that of a *homogeneous linear system*

$$A\mathbf{x} = \mathbf{0}.$$

■ **Example 2.34** Solve the following linear system with row reduction

$$\begin{aligned} x_1 + 10x_2 &= 0 \\ 2x_1 + 20x_2 &= 0 \end{aligned}$$

corresponding to the matrix equation $A\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} 1 & 10 \\ 2 & 20 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A homogeneous equation always has the zero or *trivial solution* $\mathbf{x} = [0, 0]$. A non-zero solution is called a *non-trivial solution*.

Do any non-trivial solutions exist for this problem?

After row reduction we see that

$$\begin{bmatrix} 1 & 10 & 0 \\ 2 & 20 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Using back substitution we find that $\mathbf{x} = x_2 \begin{bmatrix} -10 \\ 1 \end{bmatrix}$ where x_2 is the free variable. Thus, there are infinitely many solutions of this form. ■

R A homogeneous equation $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions if and only if the system of equations has at least one free variable.

■ **Example 2.35** Determine if the following homogeneous system has nontrivial solutions and then describe the solution set

$$\begin{aligned} 2x_1 + 4x_2 - 6x_3 &= 0 \\ 4x_1 + 8x_2 - 10x_3 &= 0 \end{aligned}$$

corresponding to the matrix equation $A\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} 2 & 4 & -6 \\ 4 & 8 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Notice first that there must be at least one free variable. Why? There are more columns than rows in the matrix A. Carry out the row reduction to find

$$\begin{bmatrix} 2 & 4 & -6 & 0 \\ 4 & 8 & -10 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Using back substitution we find that $\mathbf{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ where x_2 is the free variable. Thus, there are infinitely many solutions of this form. ■

We can also use the determinant in an alternate method for determining the existence of nontrivial solutions.

Theorem 2.6.1 A system of n homogeneous linear equations in n unknowns has a nontrivial solution if and only if the determinant of the coefficient matrix, $\det(A) = 0$.

We see from the first example (which is 2×2) that $\det(A) = 1(20) - 2(10) = 0$ and it had non-trivial solutions.

■ **Example 2.36** For what values of λ do we have non-trivial solutions for the following linear system

$$\begin{cases} (1-\lambda)x - 3y = 0 \\ 3x + (1-\lambda)y = 0 \end{cases}$$

To solve this problem consider the coefficient matrix $A = \begin{bmatrix} 1-\lambda & -3 \\ 31-\lambda & \end{bmatrix}$ and compute its determinant

$$0 = \det(A) = (1-\lambda)(1-\lambda) - 3(3) = \lambda^2 - 2\lambda - 8 = (\lambda + 2)(\lambda - 4).$$

So the system has non-trivial solutions when $\lambda = -2, 4$. We will see problems like this again in future sections when talking about eigenvalues, eigenvectors, and diagonalization of matrices! ■

Now that we understand how to solve homogeneous linear system what about *non-homogeneous* or *inhomogeneous* matrix equations?

■ **Example 2.37** Determine the solution set of

$$\begin{aligned} 2x_1 + 4x_2 - 6x_3 &= 0 \\ 4x_1 + 8x_2 - 10x_3 &= 4 \end{aligned}$$

Carry out the row reduction to find

$$\begin{bmatrix} 2 & 4 & -6 & 0 \\ 4 & 8 & -10 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Using back substitution we find

$$\mathbf{x} = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

■

So the solution set of the inhomogeneous equation is the homogeneous solution $\mathbf{x}_c = x_2[-2,1,0]$ added to a particular solution $\mathbf{x}_p = [6,0,2]$. If we change the righthand side in the original problem the only thing that will change is the particular solution $\mathbf{x}_p$.

Summary: The solution to a matrix equation $A\mathbf{x} = \mathbf{b}$ is the sum of the homogeneous solution $\mathbf{x}_c$ (to $A\mathbf{x} = \mathbf{0}$) and a particular solution $\mathbf{x}_p$. One can think of the solution to the inhomogeneous equation as a translation of the solution set to the homogeneous equation by $\mathbf{x}_p$.

Theorem 2.6.2 Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{x} = \mathbf{p} + \mathbf{x}_c$ where $\mathbf{x}_c$ is a solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

■ **Example 2.38** Determine the solution set of

$$2x_1 + -4x_2 - 4x_3 = 0$$

and compare it to the solution set of

$$2x_1 + -4x_2 - 4x_3 = 6.$$

This is just one equation in three unknowns so there is no row reduction to be done. In both cases we can solve for x_1 in terms of the free variables x_2, x_3. For the first equation

$$\mathbf{x} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix},$$

and for the first equation

$$\mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix},$$

So the inhomogeneous equation has a solution set composed of the homogeneous solution added to a particular solution $\mathbf{x}_p = [3,0,0]$. Solving another inhomogeneous system

$$2x_1 + -4x_2 - 4x_3 = 4$$

we find

$$\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix},$$

which has the same homogeneous part, but the particular solution is now $\mathbf{x}_p = [2,0,0]$. ■

2.6.6 Inverse Matrix

First, recall what we do for one equation and one unknown $ax = b$. We divide by a and find $x = b/a$. The matrix equivalent of this operation is to multiply both sides by the inverse $A\mathbf{x} = \mathbf{b} \Rightarrow \mathbf{x} = A^{-1}\mathbf{b}$. So if we can develop an algorithm for finding the inverse of a matrix A, then we can easily solve a system of linear equations by just multiplying both sides by the inverse A^{-1}.

Definition 2.6.2 *(Matrix Inverse)* The inverse of a matrix only makes sense for a square $n \times n$ matrix A. The inverse, denoted A^{-1}, is the unique matrix such that $AA^{-1} = I_n = A^{-1}A$. (For comparison to real numbers $aa^{-1} = a/a = 1$).

If a matrix A has an inverse, then it is said to be *invertible* or *non-singular*.

R Observe that $1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1})$. Thus, $\det(A^{-1}) = \frac{1}{\det(A)}$. Also, if $\det(A) = 0$, then the matrix is *singular* and has no inverse.

2.6.7 Ways to Compute A^{-1}

Case I: 2×2 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If the $\det(A) \neq 0$ or $ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $\det(A) = ad - bc = 0$, the the matrix is not invertible.

■ **Example 2.39** Let $A = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$. Find A^{-1}.

Step 1: Find $\det(A) = -7(-2) - 3(5) = -1$.

Step 2: Use the formula for A^{-1}

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} -2 & -3 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}.$$

Step 3: You can check your work, $AA^{-1} = I_2$. ■

This formula works great for 2×2 matrices, but as soon as the dimension becomes $n \times n$ where $n \geq 3$ the corresponding formula becomes unmanageable. Thus, we need to come up with a general method for computing the inverse of a matrix.

Case II: $n \times n$ Given a matrix A we can setup the augmented matrix

$$[A|I_n] \qquad \text{Row reduce } A \text{ to } I \qquad \left[I_n|A^{-1}\right].$$

As we apply the row operations to I_n the righthand side will transform into the inverse.

■ **Example 2.40** Let $A = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$ and check that we get the same answer as the formula.

Solution: In class! ■

■ **Example 2.41** Solve the matrix equation

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ -2 \end{bmatrix}. \tag{2.35}$$

by finding the inverse, A^{-1}.

Solution: In class! ■

2.6.8 Rotation Matrices

Given a vector in two-dimensions $\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}$, we can rotate it about the origin by an angle θ using a *rotation matrix*

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{2.36}$$

Then the new coordinates are

$$\mathbf{r}' = R\mathbf{r} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{2.37}$$

■ **Example 2.42** If $\mathbf{r} = [1, 0]$ and we want to rotate it by $\theta = \pi/2$, then

$$\mathbf{r}' = R\mathbf{r} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2.38}$$

This is a common example of a linear transformation. More on the definition and use of linear transformations soon! ■

2.6.9 Functions of Matrices

We can take the power of a matrix. For example raising a square $n \times n$ matrix A to the kth power is equivalent to multiplying A to itself k times

$$A^k = AA...A \neq \begin{bmatrix} a_{11}^k & \cdots & a1n^k \\ \vdots & & \vdots \\ a_{n1}^k & \cdots a_{nn}^k & \end{bmatrix}. \tag{2.39}$$

We cannot just raise each element to the kth power, the matrix multiplication must be performed.

■ **Example 2.43** Let $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$. Find A^2.

$$A^2 = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 4 \end{bmatrix} \neq \begin{bmatrix} 1^2 & 0^2 \\ (-1)^2 & 2^2 \end{bmatrix} \tag{2.40}$$

■

What about an arbitrary function of a matrix, $f(A)$. To get an idea of how these work we can expand the function in a Taylor series.

$$e^{kA} = 1 + kA + \frac{k^2A^2}{2!} + \ldots \tag{2.41}$$

Observe that both sides of this equation are matrices. So the exponential function of a matrix has a meaning.

Two other quick observations:

1. $(A+B)^2 = A^2 + AB + BA + B^2$. The middle terms may not be the same since matrix multiplications is not commutative! (In general $AB \neq BA$).

2. $e^{A+B} \neq e^A e^B$.

2.7 Linear Combinations, Functions, and Operators

In this section we will study a more general class of operations that matrices are a particular example of, *Linear Transformations*. However, we must first understand the basic definition of linearity and different operations associated with this definition.

Definition 2.7.1 *(Linear Combination)* Given two vectors $\mathbf{u}$ and $\mathbf{v}$ as well as two scalars a, b then a *linear combination* of $\mathbf{u}$ and $\mathbf{v}$ is any sum of the form

$$a\mathbf{u} + b\mathbf{v}.$$

■ **Example 2.44** Find $3\mathbf{u} - \mathbf{v}$ for $\mathbf{u} = [2,1]$ and $\mathbf{v} = [-2,2]$.

Solution: Using scalar multiplication of vectors and subtraction we find

$$3\mathbf{u} - \mathbf{v} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}.$$

■

R Any position vector $\mathbf{r} = (x,y,z) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ is a linear combination of the three unit basis vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$.

■ **Example 2.45** Determine if $\mathbf{w} = [-4,1]$ is a linear combination of $\mathbf{u} = [2,1]$ and $\mathbf{v} = [-2,2]$.

Need: To find scalars a, b so that $a\mathbf{u} + b\mathbf{v} = \mathbf{w}$. This can be rewritten as

$$\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}. \tag{2.42}$$

So determining if some vector is a linear combination of other vectors is equivalent to solving (6.202). For our particular problem we can setup the augmented matrix and row reduce to solve

$$\begin{bmatrix} 2 & -2 & -4 \\ 1 & 2 & 1 \end{bmatrix}.$$

After row reduction we find

$$\begin{bmatrix} 2 & -2 & -4 \\ 0 & 3 & 3 \end{bmatrix}.$$

Using back substitution we find that $b = 1$ and then $2a - 2b = -4 \to a = -1$. Therefore, $[-4, 1] = -\mathbf{u} + \mathbf{v}$. ■

Definition 2.7.2 *(Span)* Given a set of vectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ in $\mathbb{R}^n$, then the span denoted, $\text{span}\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is the set of all linear combinations of the vectors or equivalently it is all vectors that can be written as

$$\mathbf{u} = a_1\mathbf{v}_1 + \cdots + a_p\mathbf{v}_p.$$

■ **Example 2.46** i) What is the span of the vector $\mathbf{v} = [1, 2]$.

Solution: The span is any scalar multiple of this vector, $\mathbf{u} = c[1, 2]$. Visually this is a line in two dimensions through the point $(1, 2)$ with slope $m = y/x = 2/1 = 2$.

ii) Find the span of $\mathbf{v}_1 = [1, 2]$ and $\mathbf{v}_2 = [-1, 0]$.

The span is all vectors of the form $a\mathbf{v}_1 + b\mathbf{v}_2$. In two-dimensions this is the entire plane. Two vectors will always span the two-dimensional plane if they are not scalar multiples of each other, $\mathbf{v}_1 = c\mathbf{v}_2$. If this is the case, then we return to the i) where we have only a line. ■

2.7.1 Linear Functions

Definition 2.7.3 A function of a vector, $f(\mathbf{v})$, is called *linear* if

$$f(\mathbf{v}_1 + \mathbf{v}_2) = f(\mathbf{v}_1) + f(\mathbf{v}_2) \qquad \text{and} \qquad f(a\mathbf{v}_1) = af(\mathbf{v}_1)$$

for any scalar a.

■ **Example 2.47** a. Let $\mathbf{u} = (1, 2, 3)$ and $\mathbf{v} = (x, y, z)$. Is $f(\mathbf{v}) = \mathbf{u} \cdot \mathbf{v} = x + 2y + 3z$ linear?

Check the properties:

i) $f(\mathbf{v}_1 + \mathbf{v}_2) = (x_1 + x_2) + 2(y_1 + y_2) + 3(z_1 + z_2) = x_1 + 2y_1 + 3z_1 + x_2 + 2y_2 + 3z_2 = f(\mathbf{v}_1) + f(\mathbf{v}_2)$.

ii) $f(a\mathbf{v}) = ax + 2(ay) + 3(az) = a(x + 2y + 3z) = af(\mathbf{v})$. So yes it is linear!

b. Is $f(\mathbf{v}) = \mathbf{v} \cdot \mathbf{v} = x^2 + y^2 + z^2$ linear?

Check the properties:

i) $f(\mathbf{v}_1 + \mathbf{v}_2) = (x_1 + x_2)^2 + (y_1 + y_2)^2 + (z_1 + z_2)^3 = x_1^2 + y_1^2 + z_1^2 + x_2^2 + y_2^2 + z_2^2 + 2(x_1x_2 + y_1y_2 + z_1z_2) = f(\mathbf{v}_1) + f(\mathbf{v}_2) + 2(x_1x_2 + y_1y_2 + z_1z_2)$ So no it is not linear. ■

The key to determine if a function is linear is to look for typical nonlinearities:
i) Powers of variables, $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$.
ii) Trig functions, $\sin(\mathbf{v})$.
iii) Multiplication of different components, $f(\mathbf{v}) = [xy, y]$.

Question: What if the function is a vector (e.g., magnetic field $\mathbf{B}(\mathbf{x})$).

Definition 2.7.4 $\mathbf{F}(\mathbf{x})$ is a linear vector function if:
i) $\mathbf{F}(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{F}(\mathbf{x}_1) + \mathbf{F}(\mathbf{x}_2)$.

ii) $\mathbf{F}(a\mathbf{x}) = a\mathbf{F}(\mathbf{x})$ for any scalar a.

Example 2.48 a. Is $\mathbf{F}(\mathbf{x}) = 3\mathbf{x} - \mathbf{v}$ where $\mathbf{v} = [1,1,1]$ linear?

Check the properties:

i) $\mathbf{F}(\mathbf{x}_1 + \mathbf{x}_2) = 3(\mathbf{x}_1 + \mathbf{x}_2) - \mathbf{v} = 3\mathbf{x}_1 - \mathbf{v} + 3\mathbf{x}_2 = \mathbf{F}(\mathbf{x}_1) + \mathbf{x}_2$. No! Not linear.

b. Is $\mathbf{F}(\mathbf{x}) = 3\mathbf{x}$ linear?

Check the properties:

i) $\mathbf{F}(\mathbf{x}_1 + \mathbf{x}_2) = 3(\mathbf{x}_1 + \mathbf{x}_2) = 3\mathbf{x}_1 + 3\mathbf{x}_2 = \mathbf{F}(\mathbf{x}_1) + \mathbf{F}(\mathbf{x}_2)$.

ii) $\mathbf{F}(a\mathbf{x}) = 3(a\mathbf{x}) = a(3\mathbf{x}) = a\mathbf{F}(\mathbf{x})$. Yes, linear! ■

Example 2.49 Consider a rotation by $\theta = \pi/2$ in two dimensions. Recall the rotation matrix $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Is $\mathbf{F}(\mathbf{x}) = R_{\pi/2}\mathbf{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$ linear?

Check the properties:

i) $\mathbf{F}(\mathbf{x}_1 + \mathbf{x}_2) = R_{\pi/2}(\mathbf{x}_1 + \mathbf{x}_2) = R_{\pi/2}\mathbf{x}_1 + R_{\pi/2}\mathbf{x}_2 = \mathbf{F}(\mathbf{x}_1) + \mathbf{F}(\mathbf{x}_2)$.

ii) $\mathbf{F}(a\mathbf{x}) = R_{\pi/2}(a\mathbf{x}) = aR_{\pi/2}(\mathbf{x}) = a\mathbf{F}(\mathbf{x})$. Yes, linear! ■

A matrix applied to a vector is referred to as a *linear operator*.

2.7.2 Linear Operators

Definition 2.7.5 An operator is a rule or instruction for how to act on some scalar or vector. An operator L is *linear* if:
i) $L(\mathbf{u} + \mathbf{v}) = L\mathbf{u} + L\mathbf{v}$.
ii) $L(c\mathbf{v}) = cL(\mathbf{v})$ for any real scalar c.
Here $\mathbf{u}$ and $\mathbf{v}$ can be scalars, vectors, matrices, functions, etc.

R Every example above that was shown to be linear is a linear operator.

Example 2.50 Important for Math Methods II, is differentiation d/dx a linear operator?

Check the properties:

i) $\frac{d}{dx}[f(x) + g(x)] = \frac{df}{dx} + \frac{dg}{dx}$.

ii) $\frac{d}{dx}[cf(x)] = c\frac{df}{dx}$. Yes, derivatives are linear operators! ■

Example 2.51 Is $f(x) = x^{1/n}$ a linear operator? If so, when (for what values of n)?

Solution: In general, $\sqrt[n]{x+y} \neq \sqrt[n]{x} + \sqrt[n]{y}$ unless $n = 1$. In this case $f(x) = x$ for all other powers this function is NOT a linear operator. ■

2.8 Matrix Operations and Linear Transformations

Recall the equivalence of a system of linear equations to a matrix equation, $A\mathbf{x} = \mathbf{b}$:

$$\begin{cases} ax+by=e \\ cx+dy=f \end{cases} \quad \text{or} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}.$$

Every point $\mathbf{x} = [x,y]$ is moved to a new point $[e,f]$ (mapping/transformation). All the necessary information is built into the matrix A. Observe that:
i) $A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2$
ii) $A(c\mathbf{x}) = cA\mathbf{x}$.
Both hold by general matrix properties, so all matrices are linear transformations.

■ **Example 2.52** Let $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $\mathbf{x} = [1,0]$. Then $A\mathbf{x} = \mathbf{x}' = [1,1]$. So $|\mathbf{x}| = 1$, but $|\mathbf{x}'| = \sqrt{2}$. So lengths and distances are not preserved by this matrix, but it is still a linear transformation. ■

R If a linear transformation does preserve lengths and distance then it is called *orthogonal*. A classic example of this is a rotation matrix, which clearly does not change the length of a vector. A matrix of an orthogonal transformation is an *orthogonal matrix* and has the property that $M^{-1} = M^T$.

■ **Example 2.53** i) Let $R_{\pi/2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ then one can check that $R_{\pi/2}^{-1} = R_{\pi/2}^T$.

ii) This also holds for reflections $S = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. ■

Observe also that if a matrix M is orthogonal, then $I = M^{-1}M = M^TM$ and thus, $1 = \det(I) = \det(M^TM) = \det(M)^2$. This implies that $\det(M) = \pm 1$. If the determinant is positive it is a pure rotation and if it is negative then the transformation involves some form of reflection.

R In 3D if you want to rotate an object you have to pick an axis to rotate about. The rotation matrices about the x-axis, y-axis, and z-axis respectively are

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & cos\theta \end{bmatrix} \quad R_y = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & cos\theta \end{bmatrix} \quad R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Are the $3D$ rotation matrices orthogonal? Yes! you can check by finding $R_i^T = R_i^{-1}$.

Question: Are all matrices that have $\det(M) = 1$ orthogonal?

■ **Example 2.54**

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad A^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \neq A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

■

Theorem 2.8.1 Every linear transformation can be expressed with a matrix.

One of the most useful techniques in linear algebra is to take a mapping or transformation and express it as a matrix in order to analyze it. A mapping $T(\mathbf{x}) : \mathbb{R}^n \mapsto \mathbb{R}^m$ can be written as an $m \times n$ matrix A where each column of A is the output of the set of unit basis vectors (e.g., $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$)

$$A = \left[T(\hat{\mathbf{i}})T(\hat{\mathbf{j}})T(\hat{\mathbf{k}})\right]. \tag{2.43}$$

■ **Example 2.55** Find the matrix associated to the transformation $T(\mathbf{x}) = \begin{bmatrix} x+y \\ 3y \\ z \end{bmatrix}$.

Step 1: First check that the transformation is indeed linear!

Step 2: Since we are in three dimensions, compute how it acts on the three basis vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$:
i) $T(\hat{\mathbf{i}}) = [1,0,0]$
ii) $T(\hat{\mathbf{j}}) = [1,3,0]$
iii) $T(\hat{\mathbf{k}}) = [0,0,1]$.

Step 3: Each of these solutions is a column of A

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Observe that if we want to know what point maps to $\mathbf{b} = [2,3,1]$ is equivalent to finding a point $\mathbf{x} = A^{-1}\mathbf{b}$. We can check that $\mathbf{x} = [1,1,1]$. ■

In a future section we can define one-to-one and onto linear transformations by looking at their corresponding matrices.

2.9 Linear Dependence and Independence

A homogeneous system such as

$$\begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 9 \\ 5 & 9 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

can be viewed as a linear combination of vectors

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This equation has the trivial solution ($x_1 = 0, x_2 = 0, x_3 = 0$), but is this the *only* solution?

Definition 2.9.1 A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{0}, \tag{2.44}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$ is said to be **linearly dependent** if there exists weights $c_1, ..., c_p$ not all zero such that

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}. \tag{2.45}$$

■ **Example 2.56** Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix}$.

a. Determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

b. If possible, find a linear dependence relation.

Step 1: Set up an augmented matrix with columns $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and righthand side $\mathbf{0}$.

Step 2: Row reduce the matrix to echelon form:

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 3 & 5 & 9 & 0 \\ 5 & 9 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & -1 & 18 & 0 \end{bmatrix}.$$

Step 3: See if there are any free variables. If there are then the vectors are linearly dependent and the dependence relation has coefficients $c_1 = x_1 = -33x_3, c_2 = x_2 = 18x_3, c_3 = x_3$ for any real number x_3. One possible dependence relation is $-33\mathbf{v}_1 + 18\mathbf{v}_2 + \mathbf{v}_3 = 0$ by choosing $x_3 = 1$. ■

R The linear dependence relation among the columns of a matrix A corresponds to a nontrivial solution to $A\mathbf{x} = \mathbf{0}$.

2.9.1 Special Cases

1. Consider the set containing one nonzero vector $\{\mathbf{v}_1\}$. The only solution to $x_1\mathbf{v}_1 = \mathbf{0}$ is $x_1 = 0$. Thus, a set of only one vector is automatically linearly independent as long as it is not the zero vector $\mathbf{v}_1 \neq \mathbf{0}$.

2. Consider a set of two vectors

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Notice that $\mathbf{u}_2 = 2\mathbf{u}_1$ and therefore

$$2\mathbf{u}_1 + (-1)\mathbf{u}_2 = \mathbf{0}.$$

Thus, the set $\{\mathbf{u}_1, \mathbf{u}_2\}$ is linearly dependent.

Since $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$ a similar relationship cannot be found and the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ must be linearly independent.

R A set of two vectors is **linearly dependent** if one of the vectors is a scalar multiple of the other.

A set of two vectors is **linearly independent** if and only if neither of the vectors is a multiple of the other.

3. Consider a set of vectors containing the zero vector, $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{p-1}, \mathbf{0}\}$. Then

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + ... + 0\mathbf{v}_{p-1} + 1\mathbf{0} = \mathbf{0}.$$

Thus, any set of vectors containing the zero vector, $\mathbf{0}$, must be linearly dependent.

4. “A set containing too many vectors".

Theorem 2.9.1 If a set contains more vectors than there are entries in each vector, then the set is linearly dependent (e.g.,any set $\{\mathbf{v}_1,...,\mathbf{v}_p\}$ in $\mathbb{R}^n$ where $p > n$).

2.9.2 Linear Independence of Functions

Consider a set of functions $\{f_1(x), f_2(x),..., f_n(x)\}$. This set is linearly dependent if there exists constants $k_1,...,k_n$ such that

$$k_1 f_1 + ... + k_n f_n = 0.$$

Definition 2.9.2 If functions $f_1,..., f_n$ have derivatives up to order $n-1$ and

$$W(f_1,..., f_n) = \begin{vmatrix} f_1 & \cdots & f_n \\ \vdots & & \vdots \\ f_1^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix} \neq 0,$$

then the functions are linearly independent. This determinant, W, is called the *Wronskian* of the functions (we will see this again when solving differential equations!).

■ **Example 2.57** a) Is the set of functions $\{\cos(x), \sin(x)\}$ linearly independent?

Solution: The Wronskian $W = \cos^2(x) + \sin^2(x) = 1 \neq 0$. Yes!

b) Is the set of functions $\{1, x, 2+5x\}$ linearly independent?

Solution: The Wronskian $W = 0$. No!

c) Is the set of functions $\{1, \cos(x), \cos(2x)\}$ linearly independent?

Solution: The Wronskian $W = 4\sin(x)\sin(2x) - 2\cos(x)\sin(2x) \neq 0$. Yes!

d) Is the set of functions $\{1, x^2, \cos(2x)\}$ linearly independent?
Solution: $W = -8x\cos(2x) + 4\sin(2x) \neq 0$. ■

2.9.3 Basis Functions

Recall the definition of the span of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2,..., \mathbf{v}_p\}$ is the set of all linear combinations. While to be linearly independent, the set of vectors must not contain any vector that can be made as a linear combination of the others.

Definition 2.9.3 A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2,..., \mathbf{v}_p\}$ is called a **basis** for $\mathbb{R}^n$ if the set is *linearly independent* and the linear combinations of the vectors *span* all of $\mathbf{R}^n$.

■ **Example 2.58** The standard basis in $\mathbf{R}^3$, $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$. ■

2.10 Special Matrices

Matrix	Symbol	Condition
Transpose Matrix	A^T	$a_{ij} = a_{ji}$
Conjugate Matrix	$\overline{A}$ or $A^\star$	$a_{ij} \to \overline{a}_{ij}$
Hermitian (Self-adjoint) Matrix	$A^\dagger = \overline{A}^T$	$a_{ij} = \overline{a}_{ji}$
Anti-Hermitian Matrix	$A = -\overline{A}^T$	$a_{ij} = -\overline{a}_{ji}$
Inverse Matrix	A^{-1}	$[A\|I] \to [I\|A^{-1}]$
Real Matrix	$A = \overline{A}$	$a_{ij} = \bar{a}_{ij}$
Imaginary Matrix	$A = -\overline{A}$	$a_{ij} = -\overline{a}_{ij}$
Symmetric Matrix	$A = A^T$	$a_{ij} = a_{ji}$, A real
Skew Symmetric	$A = -A^T$	$a_{ij} = -a_{ji}$, A real
Orthogonal	$A^{-1} = A^T$	
Unitary Matrix	$A^{-1} = \overline{A}^T$	
Normal Matrix	$A\overline{A}^T = \overline{A}^T A$	

2.11 Eigenvalues and Eigenvectors

The basic concepts here – eigenvalues and eigenvectors – are used through mathematics, physics and chemistry.

■ **Example 2.59** Let $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Examine the images of $\mathbf{u}$ and $\mathbf{v}$ under multiplication by A.

$$A\mathbf{u} = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -2\mathbf{u}$$

$$A\mathbf{v} = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix} \neq \lambda \mathbf{v}.$$

Here $\mathbf{u}$ is called an *eigenvector* of A. $\mathbf{v}$ is not an eigenvector of A since $A\mathbf{v}$ is not a multiple of $\mathbf{v}$. ■

Definition 2.11.1 An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda \mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** if there is a nontrivial solution $\mathbf{x}$ for $A\mathbf{x} = \lambda \mathbf{x}$, such an $\mathbf{x}$ is called the *eigenvector corresponding to* λ.

■ **Example 2.60** Show that 4 is an eigenvalue of $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$ and find the corresponding eigenvectors.

Solution: The scalar 4 is an eigenvalue of A if and only if $A\mathbf{x} = 4\mathbf{x}$ has a nontrivial solution. This is equivalent to $(A - 4I)\mathbf{x} = \mathbf{0}$ having a nontrivial solution. To solve this problem, we first find $A - 4I$:

$$A - 4I = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ -4 & -2 \end{bmatrix}. \tag{2.46}$$

Now solve $(A - 4I)\mathbf{x} = \mathbf{0}$ using row reduction

$$\begin{bmatrix} -4 & -2 & 0 \\ -4 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.47}$$

Thus, using back substitution we find $x_1 = -\frac{1}{2}x_2$ or in vector form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}. \tag{2.48}$$

Thus, each vector of the form $x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$ is an eigenvector for the eigenvalue $\lambda = 4$. ■

R The method just used to find eigenvectors cannot be used to find eigenvalues. We must come up with a procedure for finding each eigenvalue and then implement the above strategy for finding the corresponding eigenvalues.

■ **Example 2.61** Suppose that λ is an eigenvalue of A. Determine an eigenvalue of A^2 and A^3. In general, what is the eigenvalue of A^n.

Solution: Since λ is an eigenvalue of A, there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. Apply A to both sides:

$$A^2\mathbf{x} = A(\lambda\mathbf{x})$$
$$A^2\mathbf{x} = \lambda A\mathbf{x}$$
$$A^2\mathbf{x} = \lambda^2\mathbf{x}.$$

In general, λ^n is an eigenvalue of A^n. ■

Theorem 2.11.1 The eigenvalues of a triangular matrix are the entries on the diagonal.

Theorem 2.11.2 If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are linearly independent.

2.11.1 The Characteristic Equation: Finding Eigenvalues

To find eigenvectors we need to solve $(A - \lambda I)\mathbf{x} = \mathbf{0}$ and find nontrivial solutions, but how do we find the eigenvalues, λ? There are two ways to think about it:

1. $(A - \lambda I)\mathbf{x} = \mathbf{0}$ must have nontrivial solutions. Then $(A - \lambda I)$ is not invertible. Thus $\det(A - \lambda I) = 0$.

2. For there to be nontrivial solutions of $(A - \lambda I)\mathbf{x} = \mathbf{0}$, Cramer's rule must fail. It fails when the $\det(A - \lambda I) = 0$.

Definition 2.11.2 *(Characteristic Equations)* To find the eigenvalues of a matrix one must solve the **characteristic equation**

$$\det(A - \lambda I) = 0. \tag{2.49}$$

■ **Example 2.62** Find the eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$.

Solution: Since

$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ -6 & 5-\lambda \end{bmatrix},$$

the characteristic equation becomes

$$\begin{aligned} -\lambda(5-\lambda)+6&=0\\ \lambda^2-5\lambda+6&=0\\ (\lambda-2)(\lambda-3)&=0. \end{aligned}$$

Thus, the eigenvalues are $\lambda=2$ and $\lambda=3$. ■

R For a 3×3 matrix or larger, recall that the determinant can be computed by cofactor expansion.

■ **Example 2.63** Find the eigenvalues of $A=\begin{bmatrix}1&2&1\\0&-5&0\\1&8&1\end{bmatrix}$.

Solution: Since

$$A-\lambda I=\begin{bmatrix}1-\lambda&2&1\\0&-5-\lambda&0\\1&8&1-\lambda\end{bmatrix}.$$

Then

$$\det(A-\lambda I)=(-5-\lambda)\begin{vmatrix}1-\lambda&1\\1&1-\lambda\end{vmatrix}.$$

the characteristic equation becomes

$$\begin{aligned} (-5-\lambda)[(1-\lambda^2)-1]&=0\\ (-5-\lambda)[\lambda^2-2\lambda]&=0\\ (-5-\lambda)\lambda(\lambda-2)&=0. \end{aligned}$$

Thus, the eigenvalues are $\lambda=-5$, $\lambda=0$, and $\lambda=2$. ■

■ **Example 2.64** Find the eigenvalues of $A=\begin{bmatrix}3&2&3\\0&6&10\\0&0&2\end{bmatrix}$.

Solution: Since

$$A-\lambda I=\begin{bmatrix}3-\lambda&2&3\\0&6-\lambda&10\\0&0&2-\lambda\end{bmatrix}.$$

Then

$$\det(A-\lambda I)=(3-\lambda)(6-\lambda)(2-\lambda)=0.$$

Thus, the eigenvalues are $\lambda=3$, $\lambda=6$, and $\lambda=2$. ■

2.11.2 Similarity

Definition 2.11.3 For $n\times n$ matrices A and B, we say A *is similar to* B if there is an invertible matrix P such that

$$P^{-1}AP=B \qquad \text{or} \qquad A=PBP^{-1}.$$

Theorem 2.11.3 If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues!

Definition 2.11.4 A square matrix A is **diagonalizable** f A is similar to a diagonal matrix, i.e. if $A = PDP^{-1}$ where P is invertible and D is diagonal.

R If we consider the matrix P a linear transformation, then diagonalizable matrices can be transformed from one coordinate system to another. In particular, working with D is much simpler than working with A.

2.12 Diagonalization

One of the goals of the section is to develop a useful factorization $A = PDP^{-1}$, when A is $n \times n$. We can use this to quickly find A^k quickly for large k. The matrix D is diagonal, and D^K is trivial to compute (each element is raised to the kth power). Thus, $A^k = (PDP^{-1})^k = PD^kP^{-1}$, which is easier to compute. But how do we find the matrix P.

Theorem 2.12.1 *(Diagonalization Theorem)* An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In fact, $A = PDP^{-1}$ with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

■ **Example 2.65** Diagonalize the following matrix (if possible). $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

Step 1: **Find the eigenvalues of** A. Use the characteristic equation!

$$0 = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{vmatrix} = (2-\lambda)^2(1-\lambda).$$

Thus, the eigenvalues are $\lambda = 1$ and $\lambda = 2$. Even though we do not have three unique eigenvalues, we hopefully will find three linearly independent eigenvectors.

Step 2: **Find three linearly independent eigenvectors of** A. To find the eigenvectors we must solve $(A - \lambda I)\mathbf{x} = \mathbf{0}$, for each value of λ.

Case 1 ($\lambda = 1$): Solve $(A - I)\mathbf{x} = \mathbf{0}$, by writing the augmented matrix a row-reducing

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, from back substitution, we see $x_2 + x_3 = 0$ or $x_2 = -x_3$ with x_3 free. Also from the first row, $x_1 = 0$. So the eigenvector corresponding to $\lambda = 1$ has the form $\mathbf{v}_1 = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

Case 2 ($\lambda = 2$): Solve $(A - 2I)\mathbf{x} = \mathbf{0}$, by writing the augmented matrix a row-reducing

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, from back substitution, we see $x_1 + x_3 = 0$ or $x_1 = -x_3$ with x_2, x_3 free. So the eigenvectors corresponding to $\lambda = 2$ are $\mathbf{v}_2 = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and we must pick another linearly independent eigenvector, since x_2 is free just change its value and pick $x_3 = 0$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Step 3: **Construct P from the vectors in Step 2.**

$$P = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Step 4: **Construct D from the corresponding eigenvalues.**

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Note that the eigenvector in the first column of P must be associated to the eigenvalue in the first column of D.

Step 5: **Check your work by verifying** $AP = PD$. It is easier to check this then computing P^{-1}. ■

■ **Example 2.66** Diagonalize the following matrix (if possible). $A = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$

Step 1: **Find the eigenvalues of** A. Use the characteristic equation!

$$0 = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 4 & 6 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 4-\lambda \end{vmatrix} = (2-\lambda)^2(4-\lambda).$$

Thus, the eigenvalues are $\lambda = 4$ and $\lambda = 2$. Even though we do not have three unique eigenvalues, we hopefully will find three linearly independent eigenvectors.

Step 2: **Find three linearly independent eigenvectors of** A. To find the eigenvectors we must solve $(A - \lambda I)\mathbf{x} = \mathbf{0}$, for each value of λ.

Case 1 ($\lambda = 4$): Solve $(A - 4I)\mathbf{x} = \mathbf{0}$, by writing the augmented matrix a row-reducing

$$\begin{bmatrix} -2 & 4 & 6 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, from back substitution, we see $-2x_2+2x_3=0$ or $x_2=x_3$ with x_3 free. Also from the first row, $-2x_1+4x_2+6x_3=0$ or $x_1=2x_2+3x_3=5x_3$. So the eigenvector corresponding to $\lambda=4$ has the form $\mathbf{v}_1=x_3\begin{bmatrix}5\\1\\1\end{bmatrix}$.

Case 2 ($\lambda=2$): Solve $(A-2I)\mathbf{x}=\mathbf{0}$, by writing the augmented matrix a row-reducing

$$\begin{bmatrix}0&4&6&0\\0&0&2&0\\0&0&2&0\end{bmatrix}\rightarrow\begin{bmatrix}0&4&6&0\\0&0&2&0\\0&0&0&0\end{bmatrix}.$$

Thus, from back substitution, we see $x_3=0$. From the first equation $4x_2+6x_3=0$ or $x_2=0$ with x_1 free. So the eigenvector corresponding to $\lambda=2$ is $\mathbf{v}_2=x_1\begin{bmatrix}1\\0\\0\end{bmatrix}$ and we cannot find another linearly independent eigenvector. Thus, A is not diagonalizable. ■

■ **Example 2.67** Why is $A=\begin{bmatrix}2&0&0\\2&6&0\\3&2&1\end{bmatrix}$ diagonalizable?

First, find the eigenvalues with the characteristic equation

$$0=\det(A-\lambda I)=\begin{vmatrix}2-\lambda&0&0\\2&6-\lambda&0\\3&2&1-\lambda\end{vmatrix}=(2-\lambda)(6-\lambda)(4-\lambda).$$

Thus, the eigenvalues are $\lambda=4$, $\lambda=6$, and $\lambda=2$. Since the eigenvalues are distinct, then they will each have at least one linearly independent eigenvector. Thus, we are guaranteed to have enough eigenvectors to build the matrices P and D. ■

R In the special case that the matrix A is *real and symmetric*, then it can always be diagonalized and the eigenvectors have an additional property. They are no longer just linearly independent, they are also orthogonal (e.g., $\mathbf{v}_1\cdot\mathbf{v}_2=0$).

■ **Example 2.68** Diagonalize the following matrix (if possible). $A=\begin{bmatrix}5&0&2\\0&3&0\\2&0&5\end{bmatrix}$

Step 1: **Find the eigenvalues of** A. Use the characteristic equation!

$$0=\det(A-\lambda I)=\begin{vmatrix}5-\lambda&0&2\\0&3-\lambda&0\\2&0&5-\lambda\end{vmatrix}=(3-\lambda)\begin{vmatrix}5-\lambda&2\\2&5-\lambda\end{vmatrix}=(3-\lambda)(3-\lambda)(7-\lambda).$$

Thus, the eigenvalues are $\lambda=3$ and $\lambda=7$. Even though we do not have three unique eigenvalues, we hopefully will find three linearly independent eigenvectors.

Step 2: **Find three linearly independent eigenvectors of** A. To find the eigenvectors we must solve $(A-\lambda I)\mathbf{x}=\mathbf{0}$, for each value of λ.

Case 1 ($\lambda=7$): Solve $(A-7I)\mathbf{x}=\mathbf{0}$, by writing the augmented matrix a row-reducing

$$\begin{bmatrix}-2&0&2&0\\0&-4&0&0\\2&0&-2&0\end{bmatrix}\rightarrow\begin{bmatrix}-2&0&2&0\\0&-4&0&0\\0&0&0&0\end{bmatrix}.$$

Thus, from back substitution, we see $-4x_2 = 0$ or $x_2 = 0$. Also from the first row, $-2x_1 + 2x_3 = 0$ or $x_1 = x_3$ with x_3 free. So the eigenvector corresponding to $\lambda = 7$ has the form $\mathbf{v}_1 = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Case 2 ($\lambda = 3$): Solve $(A - 3I)\mathbf{x} = \mathbf{0}$, by writing the augmented matrix a row-reducing

$$\begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, from back substitution, we see $2x_1 + 2x_3 = 0$ or $x_1 = -x_3$ with x_2, x_3 free. So the eigenvectors corresponding to $\lambda = 3$ are $\mathbf{v}_2 = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and we must pick another linearly independent eigenvector, since x_2 is free just change its value and pick $x_3 = 0$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Step 3: **Construct** P **from the vectors in Step 2**.

$$P = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Step 4: **Construct** D **from the corresponding eigenvalues**.

$$D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Note that the eigenvector in the first column of P must be associated to the eigenvalue in the first column of D.

Observe that since the matrix was real an symmetric the eigenvalues are all orthogonal! ■

■ **Example 2.69** Diagonalize the following matrix (if possible). $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Step 1: **Find the eigenvalues of** A. Use the characteristic equation!

$$0 = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = \cdots = (1-\lambda)(1-\lambda)(4-\lambda).$$

Thus, the eigenvalues are $\lambda = 1$ and $\lambda = 4$. Even though we do not have three unique eigenvalues, we hopefully will find three linearly independent eigenvectors.

Step 2: **Find three linearly independent eigenvectors of** A. To find the eigenvectors we must solve $(A - \lambda I)\mathbf{x} = \mathbf{0}$, for each value of λ.

Case 1 ($\lambda = 4$): Solve $(A - 4I)\mathbf{x} = \mathbf{0}$, by writing the augmented matrix a row-reducing

$$\begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, from back substitution, we see $-3x_2+3x_3=0$ or $x_2=x_3$ with x_3 free. Also from the first row, $x_1-2x_2+x_3=0$ or $x_1=x_3$. So the eigenvector corresponding to $\lambda=4$ has the form $\mathbf{v}_1=x_3\begin{bmatrix}1\\1\\1\end{bmatrix}$.

Case 2 ($\lambda=1$): Solve $(A-I)\mathbf{x}=\mathbf{0}$, by writing the augmented matrix a row-reducing

$$\begin{bmatrix}1&1&1&0\\1&1&1&0\\1&1&1&0\end{bmatrix}\rightarrow\begin{bmatrix}1&1&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}.$$

Thus, from back substitution, we see $x_1+x_2+x_3=0$ or $x_1=-x_2-x_3$ with x_2,x_3 free. So $\mathbf{x}=x_2\begin{bmatrix}-1\\1\\0\end{bmatrix}+x_3\begin{bmatrix}-1\\0\\1\end{bmatrix}$. So the eigenvectors corresponding to $\lambda=3$ are $\mathbf{v}_2=\begin{bmatrix}-1\\1\\0\end{bmatrix}$ and $\mathbf{v}_3=\begin{bmatrix}-1\\0\\1\end{bmatrix}$.

Step 3: **Construct P from the vectors in Step 2**.

$$P=\begin{bmatrix}1&-1&-1\\1&1&0\\1&0&1\end{bmatrix}$$

Step 4: **Construct D from the corresponding eigenvalues**.

$$D=\begin{bmatrix}4&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Note that the eigenvector in the first column of P must be associated to the eigenvalue in the first column of D.

Observe that since the matrix was real an symmetric the eigenvalues are all orthogonal! ■

2.12.1 Physical Interpretation of Eigenvalues, Eigenvectors, and Diagonalization

In physics it is important to track deformation of a material. To do so, we consider an initial position at the point (x,y), then the system is stretched, rotated, reflected, etc. until the original point is at a new position (X,Y). This deformation can be described by a M.

The first natural question is if there exists such deformations that a vector just gets stretched or shrinks along the same direction. In other words

$$\begin{bmatrix}X\\Y\end{bmatrix}=\lambda\begin{bmatrix}x\\y\end{bmatrix}.$$

Such vectors with this property are the *eigenvectors* of the transformation and the special values λ are the *eigenvalues* of the transformation. If there exists a matrix P such that $P^{-1}MP=D$, then we say that we have diagonalized M by a similarity transformation. Physically, this amounts to a simplification of the problem using a better choice of variables.

Now consider the physical meaning of P and D. Consider a set of two axes, the traditional (x,y) and a rotated set of axes (x',y') (by angle θ). The relation of one coordinate system to another can be expressed as a system of linear equations:

$$\begin{cases}x=x'\cos(\theta)+y'\sin(\theta)\\y=y'\sin(\theta)+y'\cos(\theta).\end{cases}$$

or in matrix notation

$$\mathbf{r} = P\mathbf{r}', \qquad \text{where } P = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Recall that M is the matrix that described the deformation in the (x,y)-plane. Then $\mathbf{R} = M\mathbf{r}$ shows that the vector $\mathbf{r}$ becomes the vector $\mathbf{R}$ after the deformation.

Thinking Question: How can we describe the deformation in the (x',y') system? In other words, what matrix takes $\mathbf{r}'$ to $\mathbf{R}'$?

By using the above relations we find:

$$\begin{aligned} \mathbf{R} &= M\mathbf{r} \\ P\mathbf{R}' &= MP\mathbf{r}' \\ \mathbf{R}' &= P^{-1}MP\mathbf{r}'. \end{aligned}$$

Thus, $D = P^{-1}MP$ is the matrix which describes in the (x',y') system the same deformation that M describes in the (x,y) system.

Thinking Question: What happens in the case that P is chosen to make D a diagonal matrix?

If this is the case then then new axes (x',y') are along the directions of the eigenvectors of M. If the eigenvectors are orthogonal, then the new axes will be orthogonal as well. In principle if P is not an orthogonal matrix (composed of orthogonal eigenvectors), then the new axes will not be orthogonal. The only case where we are guaranteed orthogonal eigenvectors is if the original deformation M is real and symmetric.

When D is diagonal, in the (x',y') coordinate system the system is either stretched or shrunk along the axes no matter how complicated the original deformation M was.

Definition 2.12.1 If two or more eigenvalues are the same, then this eigenvalue is called *degenerate*. Degeneracy means that two independent eigenvectors correspond the to same eigenvalue.

Part Three: Multivariable Calculus

3. Partial Differentiation

3.1 Introduction and Notation

In Calculus I-II we focus on functions of one variable $y = f(x)$. In reality we need to be able to take derivatives in many dimensions (e.g., velocity, acceleration, etc.). Other applications of derivatives we have seen before include:

- Power Series Expansions (e.g., $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \ldots$
- Max/Min Problems, Extrema

We start by considering functions of several real variables $z = f(x,y)$ (we will stay in real space for now so we can consider both two- and three-dimensions.

Example 3.1 The volume of a cylinder $V(r,h) = \pi r^2 h$ depends on the radius and the height. ■

Example 3.2 Suppose $z = f(x,y)$, but $x = 5$, then $z = f(x,y) = f(y)$ is a 2D curve in a 3D area made up of the intersection of the plane $x = 5$ with the function $f(x,y)$. ■

Definition 3.1.1 *(Level Curves)* A level curve is made up of points (x,y) where $f(x,y) = const$.

Example 3.3 Let $z = f(x,y) = x^2 + y^2$, then the level curves have the form $x^2 + y^2 = const$ and are circles! ■

In two dimensions the derivatives are simple. If $y = f(x) = x^2$, then $\frac{dy}{dx} = y' = 2x$ and $\frac{d^2y}{dx^2} = y'' = 2$. Is there an analogue in three dimensions? $z = f(x,y) = x^2 y$

We have several options for a first derivative:

- Take the derivative with respect to x holding y constant (fixed), denoted $\frac{\partial f}{\partial x} = 2xy$
- Take the derivative with respect to y holding x constant, denoted $\frac{\partial f}{\partial y} = x^2$.
- Take the gradient, a vector in the direction of the maximal change in both x and y composed of the previous two examples, $\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xy \\ x^2 \end{bmatrix}$.

R Notice that the partial derivative of f with respect to x is not the same as the total derivative, $\frac{\partial f}{\partial x} \neq \frac{df}{dx}$. More on this later!

There are also many ways to take a second derivative:

- Take the derivative with respect to x a second time holding y constant (fixed), denoted $\frac{\partial^2 f}{\partial x^2} = 2y$
- Take the derivative with respect to y a second time holding x constant, denoted $\frac{\partial^2 f}{\partial y^2} = 0$.
- Take one derivative of each (while holding the other fixed), denoted $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 2x$. This is called the "Mixed Derivative". Observe that we can switch the order and get the same result for "nice functions" (e.g., continuous, differentiable, etc. *Clairaut's Theorem*)

R There are additional notation for partial derivatives. The partial derivative of $z = f(x,y)$ with respect to x can be denoted, $z_x, f_x, \frac{\partial f}{\partial x}, f_1$ among others.

3.1.1 Review of Product, Quotient, and Chain Rule

Definition 3.1.2 *(Product Rule)* Given two differentiable functions $f(x)$ and $g(x)$, the derivative of the product

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + g'(x)f(x).$$

Definition 3.1.3 *(Quotient Rule)* Given two differentiable functions $f(x)$ and $g(x)$, the derivative of the quotient

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

Definition 3.1.4 *(Chain Rule)* Given two differentiable functions $f(x)$ and $g(x)$, the derivative of the composition $g \circ f$ is

$$\frac{d}{dx}[g(f(x))] = g'(f(x))f'(x).$$

Definition 3.1.5 *(Derivatives with Log or Exp)* Given a differentiable function $f(x)$,

$$\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$$

$$\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$$

$$\frac{d}{dx}x^y = x^y\ln(x)$$

■ **Example 3.4** Let $z = f(x,y) = -x^2y^3 + e^{-x^2y}$. Find $f_x, f_y, f_{xx}, f_{yy}, f_{xy} = f_{yx}$.

Solution:

$$\begin{aligned}
f_x &= -2xy^3 - 2xye^{-x^2y} \\
f_y &= -3x^2y^2 - x^2e^{-x^2y} \\
f_{xx} &= -2y^3 - 2ye^{-x^2y} + 4x^2y^2e^{-x^2y} \\
f_{yy} &= -6x^2y + x^4e^{-x^2y} \\
f_{xy} &= -6xy^2 - 2xe^{-x^2y} + 2x^3ye^{-x^2y}
\end{aligned}$$

■

■ **Example 3.5** Let $z = f(x,y) = \frac{2x^2y}{3x+1}$ and find f_x, f_y, f_{xy}.

Solution:

$$\begin{aligned}
f_x &= \frac{4xy(3x+1) - 6x^2y}{(3x+1)^2} \\
f_y &= \frac{2x^2}{3x+1} \\
f_{xy} &= \frac{4x(3x+1) - 6x^2}{(3x+1)^2} = \frac{6x^2+4x}{(3x+1)^2}.
\end{aligned}$$

■

■ **Example 3.6** Let $z = f(x,y) = \ln(3x+1)$ and find f_x.

Solution:

$$f_x = \frac{3}{3x+1}.$$

■

■ **Example 3.7** If $f(x,y) = \sin\left(\frac{x}{1+y}\right)$, find f_x and f_y.

Solution: Compute

$$\begin{aligned}
f_x &= \frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right)\frac{1}{1+y} \\
f_y &= \frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right)\frac{-x}{(1+y)^2}.
\end{aligned}$$

■

■ **Example 3.8** If $f(x,y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2,1)$ and $f_y(2,1)$.

Solution: Compute

$$\begin{aligned}
f_x &= \left.\frac{\partial f}{\partial x}\right|_{(x,y)=(2,1)} = 3x^2 + 2xy^3\Big|_{(x,y)=(2,1)} = 3(4) + 2(2)(1) = 16 \\
f_y &= \left.\frac{\partial f}{\partial y}\right|_{(x,y)=(2,1)} = 3x^2y^2 - 4y\Big|_{(x,y)=(2,1)} = 3(4)(1) + 4(1) = 8.
\end{aligned}$$

■

■ **Example 3.9** If $f(x,y) = 4 - x^2 - y^2$, find $f_x(1,1)$ and $f_y(1,1)$.

Solution: Compute

$$f_x = \left.\frac{\partial f}{\partial x}\right|_{(x,y)=(1,1)} = -2x\Big|_{(x,y)=(1,1)} = -2$$
$$f_y = \left.\frac{\partial f}{\partial y}\right|_{(x,y)=(1,1)} = -4y\Big|_{(x,y)=(1,1)} = -4.$$

■

■ **Example 3.10** If $f(x,y,z) = e^{xy}\ln(z)$, find f_x, f_y, and f_z.

Solution: Compute

$$f_x = \frac{\partial f}{\partial x} = ye^{xy}\ln(z)$$
$$f_y = \frac{\partial f}{\partial y} = xe^{xy}\ln(z)$$
$$f_z = \frac{\partial f}{\partial z} = \frac{e^{xy}}{z}.$$

■

We can even work in other coordinate systems such as polar (r,θ).

■ **Example 3.11** If $f(x,y) = 3x^2 - y^2 = 3r^2\cos^2(\theta) - r^2\sin^2(\theta) = g(r,\theta)$, find f_r and f_θ.

Solution: Compute

$$f_r = 2r\left[3\cos^2(\theta) - \sin^2(\theta)\right]$$
$$f_\theta = r^2\left[-6\cos(\theta)\sin(\theta) - 2\sin(\theta)\cos(\theta)\right] - 8r^2\sin\theta\cos\theta.$$

■

There are several physical scalar quantities that are functions of more then one variables. For example the temperature in a material depends on space and time $T = T(x,y,z,t)$. In Physics, these quantities have physical meaning. One way physicists denote taking derivative while other parameters are left constant is $\left(\frac{\partial T}{\partial p}\right)_V$. This indicates that we take the derivative of the temperature, T, with respect to the pressure, p, leaving the volume V a fixed constant.

3.2 Power Series in Two Variables

Recall the power series for a function of one real variable

$$f(x-a) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \ldots$$

Apply a similar idea to functions of multiple variables.

Case I: Separable A function of two variables, $f(x,y)$, is separable if it can be written as a product of a function of x and a function of y, $f(x,y) = g(x)h(y)$. In this case you can expand each function in a power series in one variable and multiple to get the power series of f.

Step 1: Expand each function in a 1D Taylor series.

Step 2: Multiply the terms and group by increasing total power $\alpha+\beta$, $x^\alpha y^\beta$.

■ **Example 3.12** *Step 1:*

$$f(x,y) = e^x \sin(y) = \left(1+x+\frac{x^2}{2}+\ldots\right)\left(y-\frac{y^3}{3!}+\ldots\right).$$

Step 2:

$$f(x,y) = e^x \sin(y) = y+xy+\frac{x^2y}{2}-\frac{y^3}{3!}+\frac{xy^3}{3!}+\frac{x^3y}{6}+\ldots$$

■

■ **Example 3.13** *Step 1:*

$$f(x,y) = \cos(x)\cos(y) = \left(1-\frac{x^2}{2!}+\ldots\right)\left(1-\frac{y^2}{2!}+\ldots\right).$$

Step 2:

$$f(x,y) = \cos(x)\cos(y) = 1-\frac{x^2}{2}-\frac{y^2}{2}+\frac{x^2y^2}{4}+\ldots$$

■

Definition 3.2.1 The Taylor Series Expansion of $f(x,y)$ about the point (a,b) uses powers of $(x-a)$ and $(y-b)$ in the form

$$f(x,y) = a_{00}+a_{10}(x-a)+a_{01}(y-b)+a_{20}(x-a)^2+a_{11}(x-a)(y-b)+a_{02}(y-b)^2+\ldots$$

where $f(a,b) = a_{00}$ and

$$\begin{aligned}
f_x &= a_{10}+2a_{20}(x-a)+a_{11}(y-b)+\ldots & &\text{if } x=a \text{ and } y=b, \text{ then } f_x(a,b)=a_{10}\\
f_y &= a_{01}+a_{11}(x-a)+2a_{02}(y-b)+\ldots & &\text{if } x=a \text{ and } y=b, \text{ then } f_y(a,b)=a_{01}\\
f_{xx} &= 2a_{20}+\ldots & &\text{if } x=a \text{ and } y=b, \text{ then } f_{xx}(a,b)=2a_{20}\\
f_{yy} &= 2a_{02}+\ldots & &\text{if } x=a \text{ and } y=b, \text{ then } f_{yy}(a,b)=2a_{02}\\
f_{xy} = f_{yx} &= a_{11}+\ldots & &\text{if } x=a \text{ and } y=b, \text{ then } f_{xy}(a,b)=a_{11}
\end{aligned}$$

Therefore.

$$f(x,y) = f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)+\frac{f_{xx}(a,b)}{2!}f(x-a)^2+f_{xy}(a,b)(x-a)(y-b)+\frac{f_{yy}(a,b)}{2!}(y-b)^2+$$

If we let $h=(x-a)$ and $k=(y-b)$, then the second order terms have the form

$$\frac{1}{2!}\left[f_xx(a,b)h^2+2f_{xy}(a,b)hk+f_{yy}(a,b)k^2\right] = \frac{1}{2!}\left[h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right]^2 f(a,b).$$

The Taylor Series Expansion can then be written in the general form

$$f(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right]^n f(a,b), \tag{3.1}$$

where $\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right]^n$ are the binomial coefficients (see Pascal's Triangle!).

R This is a Maclaurin Series if $a = b = 0$. This procedure works on all functions (not just separable functions).

R Compare the series representation for the 2D series with that of the 1D series for a function of one real variable

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[h\frac{\partial}{\partial x} \right]^n f(a). \tag{3.2}$$

■ **Example 3.14** Find the Taylor Series Expansion of $f(x,y) = e^{x+y}$:

$$e^{x+y} = 1 + x + y + \frac{1}{2}\left[x^2 + 2xy + y^2\right] + \dots$$

■

■ **Example 3.15** Find the Taylor Series Expansion of $f(x,y) = e^x \sin(y)$:

$e^x \sin(y) = y + xy + \dots$ matchs a previous example

■

■ **Example 3.16** Find the Taylor Series Expansion of $f(x,y) = \sin(x-y)$:

$\sin(x-y) = x - y + \dots$

■

3.3 Total Differentials

Definition 3.3.1 The *differential* dx of the independent variable x is: $dx = \Delta x$, but $dy \neq \Delta y$. The change in y, Δy, is the actual change in the y value whereas dy is the change in y along a tangent at $x + \Delta x$.

R Of course, if dx is small, then $\Delta y \approx dy$ and $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. This follows from the fact that $y' = \frac{dy}{dx} \Rightarrow dy = y'dx \Rightarrow dy = \frac{\partial y}{\partial x}dx$.

In the case of a multi-variable function $z = f(x,y)$ we see that

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

This is called the total differential of the function. Also, $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are the slops of the tangent lines in each direction. The total derivative is different from the partial derivative in that we longer assume the one of the variables is constant (fixed). Observe that if y is held constant then the $dy = 0$ and the total differential reduces to $dz = \frac{\partial z}{\partial x}dx \Rightarrow \frac{dx}{dx} = \frac{\partial z}{\partial x}$.

The total differential can always be taken no matter how many variables are present. Let $u = f(x_1, x_2, x_3, x_4, ..., x_n)$. Then the total differential is

$$du = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 + \ldots \frac{\partial f}{\partial x_n}dx_n.$$

■ **Example 3.17** Recall the concept of *Implicit Differentiation* from Calculus I. If we are given a function $f(x,y) = x^4 + 2y^2 = 8$ it is hard to solve for y', so we use the idea of implicit differentiation

$$4x^3 + 4y\frac{dy}{dx} = 0 \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{-4x^3}{4y} = \frac{-x^3}{y}. \tag{3.3}$$

The left hand side can be written as

$$\frac{1}{dx}\left[\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dx\right] = 0.$$

The term inside the square brackets is the total differential of f. So even back in Calculus 1 we were using this concept without knowing it. ■

3.4 Approximations Using Differentials

■ **Example 3.18** Find the total differential of $f(x,y) = 10x^3 - 8x^2y + 4y^3$

Solution:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}d = \left[30x^2 - 16xy\right]dx + \left[-8x^2 + 12y^2\right]dy.$$

■

■ **Example 3.19** Find the approximate value of $\sqrt{.5 + 10^{-19}} = \sqrt{.5}$ using total differentials.

Solution: Let $f(x) = x^{1/2}$, the $\Delta f = f(.5 + 10^{-19}) - f(.5) \approx df$. Here $x = .5$ and $dx = 10^{-19}$. Thus,

$$df = \frac{\partial f}{\partial x}dx = \frac{1}{2}(.5)^{-1/2}(10^{-19}) = \frac{1}{2}(1.41)(10^{-19}) = 7 \times 10^{-20}.$$

■

■ **Example 3.20** Find the approximate value of $\frac{1}{(n+1)^2} - \frac{1}{(n-1)^2}$ using total differentials.

Solution: Let $f(x) = \frac{1}{(x+1)^2}$, the $\Delta f = f(n) - f(n-2) \approx df$. Here $x = n$ and $dx = -2$. Thus,

$$df = \frac{\partial f}{\partial x}dx = \frac{-2}{(x+1)^3}(-2) = \frac{4}{(n+1)^3}.$$

This has physical meaning. Two forces with decay n^{-2} can sum to produce something with extra decay n^{-3}. ■

■ **Example 3.21** Let $z = f(x,y) = 2\sqrt{x^2+y^2}$.
a) Use the total differential to approximate Δz when x changes from $3 \to 2.98$ and y changes from $4 \to 4.01$.
b) Calculate the actual change Δz.

Solution: First, using the total differential

$$\begin{aligned}\Delta z \approx dz &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \\ &= \left[\frac{2x}{(x^2+y^2)^{-1/2}}\right]dx + \left[\frac{2y}{(x^2+y^2)^{-1/2}}\right]dy \\ &= \frac{2(3)}{(3^2+4^2)^{-1/2}}(-.02) + \frac{2(4)}{(3^2+4^2)^{-1/2}}(.01) = -.008\end{aligned}$$

Next, compute the actual change

$$\begin{aligned}\Delta z &= f(x+\Delta x, y+\Delta y) - f(x,y) \\ &= f(2.98, 4.01) - f(3,4) = 2\sqrt{2.98^2+4.01^2} - 2\sqrt{3^2+4^2} = -.007903.\end{aligned}$$

■

R Observe that

$$\begin{aligned}f(x+\Delta x, y+\Delta y) - f(x,y) &= f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y) + f(x+\Delta x, y) - f(x,y) \\ &\approx \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial x}dx.\end{aligned}$$

■ **Example 3.22** Find the approximate value of $(1.92^2+2.1^2)^{1/3}$ using total differentials.

Solution: Let $z = f(x,y) = \sqrt[3]{x^2+y^2} = (x^2+y^2)^{1/3}$. Here $x = 2$, $y = 2$, $dx = -.08$, and $dy = .1$. Thus,

$$\begin{aligned}df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \\ &= \left[\frac{2x}{3(x^2+y^2)^{2/3}}\right]dx + \left[\frac{2y}{3(x^2+y^2)^{2/3}}\right]dy \\ &= \frac{2(2)}{3(2^2+2^2)^{2/3}}(-.08) + \frac{2(2)}{3(2^2+2^2)^{2/3}}(.1) = .0067\end{aligned}$$

So $f(1.92, 2.1) \approx f(2,2) + dz = 2 + .0067 = 2.0067$, actual value: 2.008. ■

■ **Example 3.23** Approximate the change in volume of a beverage can in the shape of a right circular cylinder as the radius changes from 3 to 2.5 and the height changes from 14 to 14.2.

Solution: Let $V = f(r,h) = \pi r^2 h$. Here $r = 3$, $h = 14$, $dr = -.5$, and $dh = .2$. Thus,

$$\begin{aligned}dV &= \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial h}dh \\ &= [2\pi rh]\,dr + \left[\pi r^2\right]dh \\ &= 2\pi(3)(14)(-.5) + \pi(3^2)(.2) = -126.2920\end{aligned}$$

Thus, decreasing the radius by .5 and increasing the height by .2 results in a decrease in the volume by 126.29units^2. ■

■ **Example 3.24** Making a profit depends on the level of inventory x and the floor space y in the following way (in thousands)

$$P(x,y) = -.02x^2 - 15y^2 + xy + 39x + 25y - 20000.$$

Currently, we have $4,000,000$ in inventory and $150,000$ sq. feet. So $x = 4000$ and $y = 150000$. Find the expected change in profit if management decides to increase the inventory by $500,000$ and decrease the floor space by 10000 sq. feet.

Solution: Here $x = 4000$, $y = -150$, $dx = 500$, and $dy = -10$. Thus,

$$\begin{aligned} dP &= \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy \\ &= [-.04x + y + 39]\,dx + [-30y + x + 25]\,dy \\ &= [-.04(4000) + 150 + 39]\,(500) + [-30(150) + 4000 + 25]\,(-10) = \$19250 \end{aligned}$$

Thus, management will have made a good decision that results in an increase in the profits! ■

■ **Example 3.25** The reduced mass μ of a system of two bodies is $\mu^{-1} = \frac{1}{m_1} + \frac{1}{m_2} \Rightarrow \mu = \frac{m_1 m_2}{m_1 + m_2}$.

From Newton's 2nd Law: $F_{21} = m_1 a_1$ and $F_{12} = m_2 a_2$. From Newton's 3rd Law: $F_{21} = -F_{12} \Rightarrow m_1 a_1 = -m_2 a_2 \Rightarrow a_2 = \frac{-m_1}{m_2} a_1$.

The relative acceleration $a_{rel} = a_1 - a_2 = \left(1 + \frac{m_1}{m_2}\right) a_1 = \frac{m_2 + m_1}{m_1 m_2} m_1 a_1 = \frac{F_{12}}{m_{rel}}$. Thus, $m_{rel} a_{rel} = F_{12}$.

If m_1 increases by 2% then what is the % change on m_2 so that the relative mass μ remains unchanged.

Solution: Using total derivatives with $dm_1 = .02m_1$ we see

$$0 = -m_1^{-2} dm_1 - m_2^{-2} dm_2 \qquad \Rightarrow \qquad \frac{dm_2}{m_2^2} = -\frac{dm_1}{m1^2} = \frac{-.02m_1}{m_1^2}.$$

Thus,

$$\frac{dm_2}{m_2} = -\frac{.02dm_s}{m1} \qquad \Rightarrow \qquad dm_2 = -.02\frac{m_2}{m_1} m_2.$$

Therefore, the relative mass remains unchanged if m_2 decreases by $.2\frac{m_2}{m_1}$%. ■

3.5 Chain Rule or Differentiating a Function of a Function

Recall the Chain Rule for functions of a single variable.

■ **Example 3.26** If $y = f(x)$ and $x = g(t)$, then $y(t) = f(g(t))$. We then can take the derivative of y with respect to t

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = f'(g(t))g'(t).$$

■

Now, recall the total differential for a multi-variable function $z = f(x,y)$

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

If x and y are functions of t, then divide by dt to find

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \qquad \text{(Chain Rule)} \tag{3.4}$$

■ **Example 3.27** Let $z = x^2y + 3xy^4$ where $x = \sin(2t)$ and $y = \cos(t)$. Find $\frac{dz}{dt}$ when $t = 0$.

Solution: By the Chain Rule

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \left[2xy + 3y^4\right]\left[2\cos(2t)\right] + \left[x^2 + 12xy^3\right]\left[-\sin(t)\right].$$

Note that when $t = 0$, then $x = 0, y = 1$. Thus,

$$\left.\frac{dz}{dt}\right|_{t=0} = [0+3][2] + [0+0][0] = 6.$$

This is interpreted as the rate of change of z as (x, y) moves along a curve $\mathscr{C}$. ■

■ **Example 3.28** Let $z = x^2 + y^2 + xy$ where $x = \sin(t)$ and $y = e^t$. Find $\frac{dz}{dt}$.

Solution: By the Chain Rule

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = [2x+y]\left[\cos(t)\right] + [2y+x]\left[e^t\right] = 2\sin(t)\cos(t) + e^t\cos(t) + 2e^{2t} + e^t\sin(t).$$

■

■ **Example 3.29** Let $w = \ln\sqrt{x^2+y^2+z^2} = \frac{1}{2}\ln(x^2+y^2+z^2)$ where $x = \sin(t)$, $y = \cos(t)$, and $z = \tan(t)$. Find $\frac{dw}{dt}$.

Solution: By the Chain Rule

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\
&= \left[\frac{2x}{x^2+y^2+z^2}\right]\left[\cos(t)\right] + \left[\frac{y}{x^2+y^2+z^2}\right]\left[-\sin(t)\right] + \left[\frac{z}{x^2+y^2+z^2}\right]\left[\sec^2(t)\right] \\
&= \frac{\cos(t)\sin(t) - \cos(t)\sin(t) + \tan(t)\sec^2(t)}{1+\tan^2(t)} = \frac{\tan(t)\sec^2(t)}{\sec^2(t)} = \tan(t).
\end{aligned}$$

■

■ **Example 3.30** (Application) The pressure (kPa), volume V (L), and temperature (K) of a mole of ideal gas are related by the equation $PV = 8.31T$ (Ideal Gas Law). Find the rate at which the pressure is changing when the temperature is $300K$ increasing at a rate of $.1K/s$ and $V = 100L$ increasing at a rate of $.2L/s$.

Solution: Here $T = 300$, $\frac{dT}{dt} = .1$, $V = 100$, $\frac{dV}{dt} = .2$, and $P = 8.31\frac{T}{V}$. By the Chain Rule:

$$\begin{aligned}
\frac{dP}{dt} &= \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt} = \frac{8.31}{V}\frac{dT}{dt} - \frac{8.31T}{V^2}\frac{dV}{dt} \\
&= \frac{8.31}{100}(.1) - \frac{8.31(300)}{100^2}(.2) = -0.04155
\end{aligned}$$

■

Question: What if $x(t)$ and $y(t)$ were functions of two variables $x = x(t,s)$ and $y = y(t,s)$?

Then $z = f(x(t,s), y(t,s))$ and by the Chain Rule:

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

■ **Example 3.31** Let $z = e^x \sin(y)$ where $x = st^2$ and $y = s^2 t$. Find $\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}$.

Solution: Thus,

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = [e^x \sin y][t^2] + [e^x \cos y][2st] = t^2 e^{st^2} \sin(s^2 t) + 2ste^{st^2}\cos(s^2 t)$$
$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = [e^x \sin y][2st] + [e^x \cos y][s^2] = 2ste^{st^2} \sin(s^2 t) + s^2 e^{st^2}\cos(s^2 t).$$

■

■ **Example 3.32** Let $z = e^{x+2y}$ where $x = \frac{s}{t}$ and $y = \frac{t}{s}$. Find $\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}$.

Solution: Thus,

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = [e^{x+2y}][\frac{1}{t}] + [2e^{x+2y}][-\frac{-t}{s^2}] = e^{\frac{s}{t}+\frac{2t}{s}}\left[\frac{1}{t} - \frac{2t}{s^2}\right]$$
$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = [e^{x+2y}][\frac{-s}{t^2}] + [2e^{x+2y}][\frac{1}{s}] = e^{\frac{s}{t}-\frac{2t}{s}}\left[-\frac{s}{t^2} - \frac{2}{s}\right].$$

■

3.6 Implicit Differentiation

Recall from Calculus that the method of *Implicit Differentiation* is a special case of the Chain Rule. The method is “implicit" due to the fact that often y cannot be solved as a function of x.

■ **Example 3.33** Let $x^2 + y^2 = 25 \Rightarrow y = \pm\sqrt{25 - x^2}$. Thus,

$$y' = \frac{x}{\sqrt{25-x^2}} \qquad \text{or} \qquad y' = -\frac{x}{\sqrt{25-x^2}}.$$

Implicit Differentiation lets one find the derivative of y WITHOUT writing y explicitly as a function of x. Take the derivative of each side with respect to x

$$2x + 2y\frac{dy}{dx} = 0$$
$$2y\frac{dy}{dx} = -2x$$
$$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y} = \frac{-x}{\pm\sqrt{25-x^2}}.$$

■

This method is usually used when we want to know the value of the derivative at a point (x_0, y_0).

■ **Example 3.34** $x^2+\sin(x)=t$. Find $\frac{dx}{dt}$.

Solution: This can be solved using total differentiation or implicit differentiation. First with TD we rewrite $0=x^2+\sin(x)-t=f(x,y)$.

$$0=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial t}\frac{dt}{dt}=[2x+\cos(x)]\frac{dx}{dt}-1\Rightarrow\frac{dx}{dt}=\frac{1}{2x+\cos(x)}.$$

To use implicit differentiation we think of x as a function of t, then by Chain Rule

$$2x\frac{dx}{dt}+\cos(x)\frac{dx}{dt}=1 \qquad \Rightarrow \qquad \frac{dx}{dt}=\frac{1}{2x+\cos(x)}.$$

■

For higher derivatives we do not use differentials! We only use implicit differentiation.

■ **Example 3.35** $x^2+\sin(x)=t$. Find $\frac{d^2x}{dt^2}$.

Solution: To use implicit differentiation we think of x as a function of t, then by Chain Rule on the result of the prior example

$$2\left(\frac{dx}{dt}\right)^2+2x\frac{d^2x}{dt^2}+\cos(x)\frac{d^2}{dt^2}-\sin(x)\left(\frac{dx}{dt}\right)^2=0$$

$$\frac{d^2x}{dt^2}[2x+\cos(x)]+\left(\frac{dx}{dt}\right)^2[2-\sin(x)]=0$$

$$\frac{d^2x}{dt^2}=\frac{-\left(\frac{dx}{dt}\right)[2-\sin(x)]}{2x+\cos(x)}$$

$$\frac{d^2x}{dt^2}=\frac{-(2-\sin(x)}{(2x+\cos(x))^3}.$$

■

Suppose we want the value of the derivative at a point. In the previous two examples, if we want th values at $x=\pi$ and $t=0$, then from the implicit differentiation we find $\frac{dx}{dt}=\frac{1}{2\pi-1}$ and $\frac{d^2x}{dt^2}=\frac{-2}{(2\pi-1)^3}$.

The most prevalent application for Implicit Differentiation is to find the equation of a tangent lien to a curve, which also gives the slope at a point.

■ **Example 3.36** Find the equation of the tangent line to the curve $x^2y^3+x^3y^2-y=0$ at the point $(1,1)$.

Solution: Using Implicit Differentiation we find

$$2xy^3+3x^2y^2\frac{dy}{dx}+3xy^2+2x^3y\frac{dy}{dx}-\frac{dy}{dx}=0$$

$$\frac{dy}{dx}\left[3x^2y^2+2x^3y-1\right]=-2xy^3+3xy^2$$

$$\frac{dy}{dx}=\frac{-2xy^3+3xy^2}{3x^2y^2+2x^3y-1}$$

$$\text{Plug in }(1,1)\qquad \frac{dy}{dx}=\frac{1}{4}.$$

Thus, the equation for the tangent line has the form $y=mx+b$ where $m=\frac{y-y_0}{x-x_0}=\frac{y-1}{x-1}=\frac{1}{4}$. Thus, $y=\frac{1}{4}(x-1)+1$. ■

■ **Example 3.37** Find the equation of the tangent line to the curve $3-x+\sqrt{x^2+y^2}=0$ at the point $(0,2)$.

Solution: Using Implicit Differentiation we find

$$\begin{aligned} -1+\frac{1}{2}(x^2+y^2)(2x+2y\frac{dy}{dx}) &= 0 \\ x(x^2+y^2)+y(x^2+y^2)\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1-x(x^2+y^2)}{y(x^2+y^2)} \\ \text{Plug in } (0,2) \qquad \frac{dy}{dx} &= \frac{1}{8}. \end{aligned}$$

Thus, the equation for the tangent line has the form $y=mx+b$ where $m=\frac{y-y_0}{x-x_0}=\frac{y-2}{x-0}=\frac{1}{8}$. Thus, $y=\frac{1}{8}x+2$. ■

3.7 More Chain Rule

Continuing from the last section, we consider functions $z=f(x,y)$ where $x=x(s,t)$ and $y=y(s,t)$.

■ **Example 3.38** Let $z=xy$ where $x=\sin(s+t)$ and $y=s-t$. Find $\frac{\partial z}{\partial t},\frac{\partial z}{\partial s}$.

Solution: Using the Chain Rule we find

$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}=[y][\cos(s+t)]+[x](-1)=y\cos(s+t)-x \\ \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}=[y][\cos(s+t)]+[x](1)=y\cos(s+t)+x \end{aligned}$$

■

■ **Example 3.39** Let $u=x^2+2xy-y\ln(z)$ where $x=s+t^2$, $y=s-t^2$, and $z=2t$. Find $\frac{\partial u}{\partial s},\frac{\partial u}{\partial t}$.

Solution: Using the Chain Rule we find

$$\begin{aligned} \frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial s}=[2x+2y](1)+[2x-\ln(z)](1)+[-y/z](0)=4x+2y-\ln(z) \\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial t}=[2x+2y][2t]+[2x-\ln(z)](-2t)+[-y/z](2)=4yt+2t\ln(z)-\frac{2y}{t}. \end{aligned}$$

■

Notation: Sometime it is useful to write the Chain Rule formulas in matrix form. If $u=f(x,y,z)$ where $x=x(s,t)$, $y=y(s,t)$, and $z=z(s,t)$. Recall from linear algebra

$$\begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{bmatrix}. \tag{3.5}$$

One recovers the Chain Rule through matrix multiplication.

3.7.1 Using Cramer's Rule

One can use Cramer's Rule to solve for dx, dy, etc.

■ **Example 3.40** Find $\frac{dz}{dt}$ given that $z = x - y$ where x and y are defined implicitly $x^2 + y^2 = t^2$ and $x\sin(t) = ye^y$.

Solution: Here we cannot solve for x and y in terms of t explicitly! Instead find dx and dy first, then use the Chain Rule to get the desired derivative. From Total Differentiation we have

$$\begin{aligned} 2xdx + 2ydy &= 2tdt \\ \sin(t)dx + x\cos(t)dt &= (ye^y + e^y)dy. \end{aligned}$$

Simplify and Rearrange:

$$\begin{aligned} xdx + ydy &= tdt \\ \sin(t)dx - (y+1)e^y dy &= -x\cos(t)dt. \end{aligned}$$

This can be written as a product of matrices:

$$\begin{bmatrix} x & y \\ \sin(t) & -(y+1)e^y \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} tdt \\ -x\cos(t)dt \end{bmatrix}.$$

This is just a linear system of the form $A\mathbf{x} = \mathbf{b}$. Recall from Cramer's Rule that as long as the determinant of A is not zero, $\det(A) \neq 0$, the the solution to $A\mathbf{x} = \mathbf{b}$ is $x = \frac{\det(A_x)}{\det(A)}$ and $y = \frac{\det(A_y)}{\det(A)}$ where A_x is A with the first column replaced by the righthand side $\mathbf{b}$. Thus,

$$\begin{aligned} \det(A) &= \begin{vmatrix} x & y \\ \sin(t) & -(y+1)e^y \end{vmatrix} = -x(y+1)e^y - y\sin(t) \\ \det(A_x) &= \begin{vmatrix} tdt & y \\ -x\cos(t)dt & -(y+1)e^y \end{vmatrix} = -tdt(y+1)e^y + xy\cos(t)dt \\ \det(A_y) &= \begin{vmatrix} x & tdt \\ \sin(t) & -x\cos(t)dt \end{vmatrix} = -x^2\cos(t)dt - tdt\sin(t). \end{aligned}$$

Therefore, Cramer's Rule gives:

$$\begin{aligned} dx &= \frac{-t(y+1)e^y + xy\cos(t)}{x(y+1)e^y - y\sin(t)} dt \\ dy &= \frac{-x^2\cos(t) - t\sin(t)}{x(y+1)e^y - y\sin(t)} dt \end{aligned}$$

Thus, we can now compute the differential dz

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = 1\left[\frac{-t(y+1)e^y + xy\cos(t)}{x(y+1)e^y - y\sin(t)}\right]dt + (-1)\left[\frac{-x\cos(t) - t\sin(t)}{x(y+1)e^y - y\cos(t)}\right]dt.$$

Now to find the derivative $\frac{dz}{dt}$ just divide the above expression by dt on both sides to find

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = 1\left[\frac{-t(y+1)e^y + xy\cos(t)}{x(y+1)e^y - y\sin(t)}\right] + (-1)\left[\frac{-x\cos(t) - t\sin(t)}{x(y+1)e^y - y\cos(t)}\right].$$

■

3.8 Maximum and Minimum Problems with Constraints

Finding maximum and minimum values of a function have wide-ranging physical applications. For example, a fundamental law of physics says that a system wants to be in the state that minimizes energy. Maxima also have many applications such as a system wanting to be in a maximum state of entropy (disorder). If we can view the function (through graphing) we observe local extrema (maxima and minima) in the form of peaks and valleys in the graph. But how can we determine which points (x,y) give rise to the maximum and minimum values without plotting them? The goal of this section is to develop a method to answer this question.

Definition 3.8.1 *(Local Extrema)* A function of two variables, $z = f(x,y)$, has a local maximum at (a,b) if $f(x,y) \le f(a,b)$ when (x,y) is near (a,b). If $f(x,y) \ge f(a,b)$ for all (x,y) near (a,b), then $f(a,b)$ is a local minimum.

Definition 3.8.2 *(Global Extrema)* A function of two variables, $z = f(x,y)$, has a global maximum at (a,b) if $f(x,y) \le f(a,b)$ for every point (x,y). If $f(x,y) \ge f(a,b)$ for all (x,y), then $f(a,b)$ is a global minimum.

Theorem 3.8.1 If a function $f(x,y)$ has a local minimum/maximum at a point (a,b), then the first order partial derivatives $\frac{\partial f}{\partial x}(a,b) = f_x(a,b) = 0$ and $\frac{\partial f}{\partial y}(a,b) = f_y(a,b) = 0$. This is analogous to the 1D case where $f'(x) = 0$ at all extrema.

Definition 3.8.3 Any location (x,y) where $f_x(x,y) = f_y(x,y) = 0$ is called a *critical point*.

■ **Example 3.41** Let $f(x,y) = x^2 + y^2 - 2x - 6y + 14$. Find the critical points.

Solution: All that must be done is take the first order partial derivatives f_x, f_y, set them equal to zero and solve for x, y.

$$
\begin{aligned}
0 = f_x = 2x - 2 \qquad &\Rightarrow \qquad x = 1 \\
0 = f_y = 2y - 6 \qquad &\Rightarrow \qquad y = 3.
\end{aligned}
$$

Thus, the only critical point is $(1,3)$. ■

■ **Example 3.42** Let $f(x,y) = y^2 - x^2$. Find the critical points.

Solution: All that must be done is take the first order partial derivatives f_x, f_y, set them equal to zero and solve for x, y.

$$
\begin{aligned}
0 = f_x = -2x \qquad &\Rightarrow \qquad x = 0 \\
0 = f_y = 2y \qquad &\Rightarrow \qquad y = 0.
\end{aligned}
$$

Thus, the only critical point is $(0,0)$. ■

Finding the critical points only gives us candidates for extrema. Being a critical point is *necessary* for being a maximum or minimum, but it is not *sufficient*. Recall the three relevant cases from 1D and how we determine if we have a maximum, minimum, or point of inflection (all of which are critical points with $f'(x) = 0$).

Theorem 3.8.2 *(Second Derivative Test)* Suppose that the first order partial derivatives are zero, $\frac{\partial f}{\partial x}(a,b) = 0$ and $\frac{\partial f}{\partial y}(a,b) = 0$ (critical point at (a,b)). In addition, f_{xx}, f_{xy}, f_{yy} exist. Define

$$D = D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2 . \tag{3.6}$$

Type of Critical Point	Conditions Needed
Maximum	$f'(x) = 0$ and $f''(x) < 0$
Minimum	$f'(x) = 0$ and $f''(x) > 0$
Point of Inflection	$f'(x) = 0$ and $f''(x) = 0$

Then:
a) If $D > 0$ and $f_{xx} > 0$, then $f(a,b)$ is a local minimum.
b) If $D > 0$ and $f_{xx} < 0$, then $f(a,b)$ is a local maximum.
c) If $D < 0$, then $f(a,b)$ is a saddle point.
d) If $D = 0$, then the second derivative test is inconclusive.

Is there a nice way to remember the formula for D? Yes! With determinants

$$D = \begin{vmatrix} f_x x & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2.$$

■ **Example 3.43** Find the local maximum, minimum, and saddle points of $f(x,y) = x^4 + y^4 - 4xy + 1$.

Solution:
Step 1: Find all possible critical points.

$$\begin{aligned} 0 = f_x &= 4x^3 - 4y \qquad \Rightarrow \qquad y = x^3 \\ 0 = f_y &= 4y^3 - 4x \end{aligned}$$

By substituting the first expression into the second we find

$$0 = 4x^9 - 4x = 4x(x^8 - 1) = 4x(x^4 + 1)(x^4 - 1) = 4x(x^4 + 1)(x^2 - 1)(x^2 + 1).$$

Thus, the real roots are $x = 0, 1, -1$. Using the relation that $y = x^3$, then the three critical points are $(0,0), (1,1), (-1,-1)$.

Step 2: Find the 2nd order partial derivatives

$$\begin{aligned} f_{xx} &= 12x^2 \\ f_{yy} &= 12y^2 \\ f_{xy} = f_{yx} &= -4. \end{aligned}$$

Step 3: Compute $D(a,b)$ for each critical point.

$$D(x,y) = (12x^2)(12y^2) - (-4)^2 = 144x^2y^2 - 16$$

For the critical point $(0,0)$, $D(0,0) = -16 < 0$, so it is a saddle point. For the critical point $(1,1)$, $D(1,1) = 144 - 16 > 0$ and $f_{xx}(1,1) = 12 > 0$, so it is a local minimum. Finally, for the critical point $(-1,-1)$, $D(-1,-1) = 144 - 16 > 0$ and $F_{yy}(-1,-1) = 12 > 0$, so it is also a local minimum. ■

■ **Example 3.44** Find the shortest distance from the point $(1,0,-2)$ to the plane $x + 2y + z = 4$.

Solution: Recall that the distance between a point (x,y,z) and $(1,0,-2)$ is

$$d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}.$$

Using the equation for the plane $z = 4 - x - 2y$ and

$$d^2 = (x-1)^2 + y^2 + (6 - x - 2y)^2.$$

Observe that if we minimize d^2 we will also minimize d.

Step 1: Find all possible critical points.

$$0 = f_x = 2(x-1) + 2(6-x-2y)(-1) = -14 + 4x + 4y$$
$$0 = f_y = 2y + 2(6-x-2y)(-2) = -24 + 4x + 10y$$

By solving this system we find that $x = 11/6$ and $y = 5/3$. Thus, the critical point as $(11/6, 5/3)$.

Step 2: Find the 2nd order partial derivatives

$$f_{xx} = 4$$
$$f_{yy} = 10$$
$$f_{xy} = f_{yx} = 4.$$

Step 3: Compute $D(a,b)$ for each critical point.

$$D(x,y) = 4(10) - (4)^2 = 24 > 0.$$

Since $D > 0$ and $f_{xx} > 0$, then $(11/6, 5/3)$ is a local minimum. Plugging this point back into the distance formula gives $d = \frac{5}{6}\sqrt{6}$. ■

■ **Example 3.45** A cardboard box without a lid is to be made from $12m^2$ of cardboard. Find the maximum volume of such a box.

Solution: Recall the volume of a box

$$V = xyz$$

for a box with side lengths x, y, z.

Step 0: Express the desired function as a function of only two variables using another relation such as the surface area $A = xy + 2xz + 2yz = 12$. Solving for z

$$z = \frac{12 - xy}{2x + 2y}, \qquad V = xyz = \frac{12xy - x^2y^2}{2x + 2y}.$$

Step 1: Find all possible critical points.

$$0 = V_x = \frac{y^2(12 - 2xy - x^2)}{2(x+y)^2} \qquad \Rightarrow 0 = y^2(12 - 2xy - x^2)$$
$$0 = V_y = \frac{x^2(12 - 2xy - y^2)}{2(x+y)^2} \qquad \Rightarrow 0 = x^2(12 - 2xy - y^2)$$

Observe that if either x or y is zero, then we get the minimum volume, $V = 0$. Ignoring these cases we find that either $x = y$ or $x = -y$. In the real world, side lengths cannot be negative, so $x = y$. By substitution we find $x = 2$ and $y = 2$. Thus, the critical point as $(2, 2)$.

Step 2: Find the 2nd order partial derivatives. In this case there is only one critical point left which is not the minimum, so it must be a maximum. Therefore, $V_{max} = xyz = 2(2)(1) = 4$. ■

■ **Example 3.46** A cardboard box without a lid has a volume of $5m^3$. Find the minimum surface area of such a box.

Solution: Recall the volume of a box

$$V = xyz = 5$$

for a box with side lengths x, y, z. The surface area is $A = xy + 2xz + 2yz$

Step 0: Express the desired function as a function of only two variables using another relation. Solving for z

$$z = \frac{5}{xy}, \qquad A = xy + 2xz + 2yz = xy + \frac{10}{y} + \frac{10}{z}.$$

Step 1: Find all possible critical points.

$$0 = A_x = y - \frac{10}{x^2} \qquad \Rightarrow \qquad y = \frac{10}{x^2}$$
$$0 = A_y = x - \frac{10}{y^2}$$

By substituting the first equation into the second we find $0 = x - \frac{10x^4}{100} = x\left(1 - \frac{1}{10}x^3\right)$. Thus, $x = 10^{1/3}$ and then $y = 10^{1/3}$.

Step 2: Find the 2nd order partial derivatives. Recall $z = \frac{5}{xy} = \frac{5}{10^{2/3}} = \frac{1}{2}10^{1/3}$. Thus the surface area is minimized when the height is half the length x and width y. ■

■ **Example 3.47** A trapezoidal gutter has an opening of $24cm$. Find the angle of the sides θ so that the cross-sectional area is maximized.

Solution: Recall the area of a trapezoid (if x is the base and y is the side

$$A = \frac{x + x + 2y\cos(\theta)}{2} y\cos(\theta) = (x + y\cos(\theta))y\sin(\theta).$$

We also know that the width is $24 = x + 2y$.

Step 0: Express the desired function as a function of only two variables using another relation. Solving for x

$$x = 24 - 2y, \qquad A = (24y - 2y^2 + y^2\cos(\theta))\sin(\theta).$$

Step 1: Find all possible critical points.

$$\begin{aligned} 0 = A_x &= -y^2\sin^2(\theta) + 24y\cos(\theta) - 2y^2\cos(\theta) + y^2\cos^2(\theta) \\ &= -y^2\sin^2(\theta) + 24y\cos(\theta) - 2y^2\cos(\theta) + 2y^2\cos^2(\theta) - y^2 \\ 0 = A_y &= (24 - 4y + 2y\cos(\theta))\sin(\theta) \end{aligned}$$

From the first equation $\theta = 0$ or $24 - 4y + 2y\cos(\theta) = 0 \Rightarrow \cos(\theta) = \frac{-12+2y}{y}$. If $\theta = 0$ we have a minimum cross-sectional area, so disregard this case. Thus,

$$0 = a_x = -y^2\left(1 - \frac{144 - 48y + 4y}{y^2}\right) + (24y^2 - 2y^2)(\frac{-12 + 2y}{y}) + y^2\left(\frac{144 - 48y + 4y^2}{y^2}\right)$$
$$0 = 3y^2 - 24y = 3y(y - 8).$$

Thus, $y = 8$. Then $x = 24 - 2y = 8$. Also, $\cos(\theta) = \frac{-12+2y}{y} = \frac{4}{8} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$. Thus, the maximal area

$$A = (24y - 2y^2 + y^2\cos(\theta))\sin(\theta) = \left((24(8) - 2(64) + (64)\cos(\frac{\pi}{3})\right)\sin(\frac{\pi}{3}) = (192 - 128 + 32)\frac{\sqrt{3}}{2} = 48\sqrt{3}.$$

■

3.9 Lagrange Multipliers

In the last section we studied a problem of maximizing the volume $V = xyz$ of a rectangular box subject to the constraint that the area of the surfaces $A = xy + 2xz + 2yz = 12$.

Before: We had to use substitution and solve multiple equations.

Now: Solve a max/min problem with a constraint simultaneously using *Lagrange Multipliers*.

This method was developed long ago to solve a classical problem the so-called *Milkmaid Problem*. Imagine you are on a farm and it is time to get milk. The maid has to get the day's milk from the cow. The sun is setting and she has a date with a handsome shepherd and wants to complete the task as quickly as possible. Before she can get the milk she must rinse her bucket in the nearby river. She wants to take the shortest possible path from her location, M, to the river to the cow C.

Thinking Question: What point P along the river should she rinse her bucket?

This question can be restated as finding a point P for which the distance from M to P and from P to C is minimum. If she only needed to go to the cow the obvious solution is a straight line, but the problem is not as simple with 3 points. We need to satisfy the constraint that P is on the river bank.

Suppose the shape of the river is described as a curve, $g(x,y) = 0$ where $g(x,y) = y - x^2$ (parabola) or $g(x,y) = x^2 + y^2 - r^2$ (circle). We want to minimize the function

$$F(P) = \text{dist}(M,P) + \text{dist}(P,C) \qquad \text{subject to} \qquad g(P) = 0.$$

Graphically: For every point P on an ellipse the distance from a focus to P to the other focus is constant. Take M, C as the foci of an ellipse, any point on increasing ellipses has the same distance from both. To find the point P, find the smallest ellipse that intersects the curve defining the river. This occurs when the smallest ellipse and river are tangent.

Algebraically: Usually to find a maximum or minimum we need to set derivatives equal to zero, $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ or $\nabla f = 0$, but we must pair this with our constraint. How to do this? Add a new variable and define a new function!

$$F(P,\lambda) := f(P) - \lambda g(P).$$

To find the critical points we set all first derivatives equal to zero, $\nabla F = 0 \Leftrightarrow \nabla f = \lambda \nabla g$

$$0 = \frac{\partial f}{\partial x}(P) - \lambda \frac{\partial g}{\partial x}$$
$$0 = \frac{\partial f}{\partial y}(P) - \lambda \frac{\partial g}{\partial y}$$
$$0 = g(P).$$

The first two equations are used to find the critical point and the last equation enforces the constraint.

The variable λ is a dummy variable used to get a system of equations, we really only care about x,y,z. Once you have found all the critical points where f you plug them into f to see which are maxima and which are minima. Solving this system of equation can be hard! Some tricks:

1. Since we do not care what λ is, you can solve first for λ in terms of x,y,z to remove λ from the equations.

2. Try first solving for one variable in terms of the others.

3. Remember when taking the square root to consider both the positive and negative root.

4. Remember when dividing an equation by some expression, you must be sure that the expression is not zero. Often it is helpful to consider two cases: first solve the equations assuming that a variable is 0, and then solve the equations assuming that it is not zero.

In physics, the Lagrange multiplier is te relative weight of the constraint on the problem. In economics, it represents the fact that the maximum profit is subject to limited resources where λ is the marginal value. This can also represent the rate at which the optimal value of $f(P)$ changes if you change the constraint.

■ **Example 3.48** *Parabolic River.* Assume the maid is at $(0,5)$, the cow is at $(8,0)$ and the river curve is $g(x,y) = (y-2)-(x-4)^2$.

Solution: We want to minimize the distance functional

$$F(x,y,\lambda) := \sqrt{x^2+(y-5)^2} + \sqrt{(x-8)^2+y^2} - \lambda\left[-(x-4)^2+(y-2)^2\right] \tag{3.7}$$

We can make the problem easier by replacing each distance (square root) with its square (if the square of the distance is minimized, then so is the distance).

$$\tilde{F}(x,y,\lambda) := x^2+(y-5)^2+(x-8)^2+y^2-\lambda\left[-(x-4)^2+(y-2)^2\right]$$

We now take the gradient and set it equal to zero, $\nabla F = 0$

$$0 = \frac{\partial F}{\partial x} = 2x+2(x-8)+2\lambda(x-4)$$
$$0 = \frac{\partial F}{\partial y} = 2(y-5)+2y-\lambda$$
$$0 = \frac{\partial F}{\partial \lambda} = (x-4)^2-(y-2)$$

Step 1: Solve for λ in terms of x,y. Thus, $\lambda = 4y-10$.

Step 2: Plug this value into $\frac{\partial F}{\partial x} = 0$

$$\begin{aligned} 0 = \frac{\partial F}{\partial x} &= 4x-16+2(4y-10)(x-4) \\ &= 4(x-4)(8y-16) \\ &= 32(x-4)(y-2) \text{Using the constraint:} \qquad &= 32(x-4)^3 \end{aligned}$$

Thus, the critical point is $x = 4$. This implies (from the constraint) that $y = 2$ and $\lambda = -2$. Plugging this back into the original distance equation (6.202) gives $F(4,2,-2) = 5+\sqrt{8} = 7.828$. Just to check that this is the minimal path try another point on the river (e.g., $(2,6) \Rightarrow \lambda = 14$). $F(2,6,14) = 6+\sqrt{5} = 8.2366$ is a larger distance to travel. ■

Definition 3.9.1 *(Method of Lagrange Multipliers)* to find the maximum or minimum values of a function $f(x,y,z)$ subject to the constraint $g(x,y,z) = k$.
a) Find all values of x,y,z,λ such that

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z), \qquad g(x,y,z) = k.$$

b) Evaluate f at all critical points from (a). The largest value is the maximum of f and the smallest value is the minimum of f.

Let's recall some examples solved in the last section.

■ **Example 3.49** Maximize the volume $V = xyz$ of a cardboard box subject to the constraint that we only have $12m^2$ of cardboard.

Solution: Solve using Lagrange Multipliers.
Step 1: Define $F(x,y,z,\lambda) := f(x,y,z) - \lambda g(x,y,z)$

$$F(x,y,z,\lambda) = xyz - \lambda\,[xy + 2xz + 2yz - 12]\,.$$

Step 2: Find the partial derivatives

$$\begin{aligned}
0 &= \frac{\partial F}{\partial x} = yz - \lambda\,[y + 2z] \\
0 &= \frac{\partial F}{\partial y} = xz - \lambda\,[x + 2z] \\
0 &= \frac{\partial F}{\partial z} = xy - \lambda\,[2x + 2y] \\
0 &= \frac{\partial F}{\partial \lambda} = xy + 2xz + 2yz - 12.
\end{aligned}$$

We have to be a little clever to solve this problem! Multiply the first equation by x, the second equation by y, and the third equation by z:

$$\begin{aligned}
xyz &= \lambda\,[xy + 2xz] \\
xyz &= \lambda\,[xy + 2yz] \\
xyz &= \lambda\,[2xz + 2yz]\,.
\end{aligned}$$

Observe that $\lambda \neq 0$ or every equation would be false (since none of the three dimensions can be zero! Combining the equations we find

$$\begin{aligned}
0 &= \lambda\,[xy + 2xz] - \lambda\,[xy + 2yz] \quad \Rightarrow \quad xz = yz \quad \Rightarrow \quad x = y \\
0 &= \lambda\,[xy + 2yz] - \lambda\,[2xz + 2yz] \quad \Rightarrow \quad xy = 2xz \quad \Rightarrow \quad y = 2z.
\end{aligned}$$

Thus, $x = y = 2z$. Plugging these into the constraint we find

$$0 = 4z^2 + 4z^2 + 4z^2 - 12 \quad \Rightarrow \quad z^2 = 1 \quad \Rightarrow \quad z = 1.$$

Therefore, $x = y = 2$. Matching the previous answer! ■

■ **Example 3.50** Find the extremal values of the function $f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

Solution: Solve using Lagrange Multipliers.
Step 1: Define $F(x,y,z,\lambda) := f(x,y,z) - \lambda g(x,y,z)$

$$F(x,y,z,\lambda) = x^2 + 2y^2 - \lambda\left[x^2 + y^2 - 1\right].$$

Step 2: Find the partial derivatives

$$0 = \frac{\partial F}{\partial x} = 2x - \lambda\,[2x]$$
$$0 = \frac{\partial F}{\partial y} = 4y - \lambda\,[2y]$$
$$0 = \frac{\partial F}{\partial \lambda} = x^2 + y^2 - 1.$$

From the first equation we see that either $x = 0$ or $\lambda = 1$. If $x = 0$, then $y = \pm 1$. If $\lambda = 1$, then $y = 0$ and $x = \pm 1$. So there are four critical points $(0,1), (0,-1), (1,0), (-1,0)$. To find the extrema (maximum or minimum) plug in the critical points to $f(x,y)$:

$$f(0,1) = 2$$
$$f(0,-1) = 2$$
$$f(1,0) = 1$$
$$f(-1,0) = 1.$$

Therefore, the max value is 2 occurring at $(0,1)$ and $(0,-1)$. The minimum value is 1 occurring at $(1,0)$ and $(-1,0)$. ■

■ **Example 3.51** Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are the closest and furthest from $(3,1,-1)$.

Solution: Solve using Lagrange Multipliers. If the distance is minimized so is the distance squared.
Step 1: Define $F(x,y,z,\lambda) := f(x,y,z) - \lambda g(x,y,z)$

$$F(x,y,z,\lambda) = (x-3)^2 + (y-1)^2 = (z+1)^2 - \lambda\left[x^2 + y^2 + z^2 - 4\right].$$

Step 2: Find the partial derivatives

$$0 = \frac{\partial F}{\partial x} = 2(x-3) - \lambda\,[2x] \quad \Rightarrow \quad x = \frac{3}{1-\lambda}$$
$$0 = \frac{\partial F}{\partial y} = 2(y-1) - \lambda\,[2y] \quad \Rightarrow \quad y = \frac{1}{1-\lambda}$$
$$0 = \frac{\partial F}{\partial z} = 2(z+1) - \lambda\,[2z] \quad \Rightarrow \quad z = \frac{-1}{1-\lambda}$$
$$0 = \frac{\partial F}{\partial \lambda} = x^2 + y^2 + z^2 - 4.$$

Plugging the values of x, y, z into the constraint gives $\lambda = 1 \pm \frac{\sqrt{11}}{2}$. So there are two critical points $(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, \frac{-2}{\sqrt{11}})$ (max distance) and $(\frac{-6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}, \frac{2}{\sqrt{11}})$ (min distance). ■

What happens if we have more than one constraint? Then we add all constraints on as different Lagrange Multipliers.

Definition 3.9.2 To maximize or minimize a function $f(x,y,z)$ with two constraints $g(x,y,z) = C_1$ and $h(x,y,z) = C_2$ we set up the functional to minimize as

$$F(x,y,z,\lambda,\mu) := f(x,y,x) - \lambda\,[g(x,y,z) - C_1] - \mu\,[h(x,y,z) - C_2].$$

Then solve for unknowns (x,y,z,λ,μ).

■ **Example 3.52** Find the maximum value of the function $f(x,y,z) = x+2y+3z$ on the curve of intersection of the plane $x-y+z=1$ and the cylinder $x^2+y^2=1$.

Solution: Solve using Lagrange Multipliers.
Step 1: Define $F(x,y,z,\lambda,\mu) := f(x,y,x) - \lambda\,[g(x,y,z)-C_1] - \mu\,[h(x,y,z)-C_2]$

$$F(x,y,z,\lambda,\mu) := x+2y+3z-\lambda\,[x-y+z-1]-\mu\,\left[x^2+y^2-1\right].$$

Step 2: Find the partial derivatives

$$\begin{aligned}
0 &= \frac{\partial F}{\partial x} = 1-\lambda-2\mu x \\
0 &= \frac{\partial F}{\partial y} = 2+\lambda-2\mu y \\
0 &= \frac{\partial F}{\partial z} = 3-\lambda \qquad \Rightarrow \qquad \lambda = 3 \\
0 &= \frac{\partial F}{\partial \lambda} = x-y+z-1 \\
0 &= \frac{\partial F}{\partial \mu} = x^2+y^2-1.
\end{aligned}$$

Since $\lambda = 3$, the first equation gives that $x = \frac{-1}{\mu}$ and the second equation gives that $y = \frac{5}{2\mu}$. Using these relations and the μ constraint we see that $\mu = \pm\frac{\sqrt{29}}{2}$ resulting in $x = \mp\frac{2}{\sqrt{29}}, y = \pm\frac{5}{\sqrt{29}}$. From the constraint g we find that $z = 1\pm\frac{7}{\sqrt{29}}$.

Step 3: Plug the critical points into f to determine the maximum and minimum, $\mp\frac{2}{\sqrt{29}}+2\left(\pm\frac{5}{\sqrt{29}}\right)+3\left(1\pm\frac{7}{\sqrt{29}}\right) = 3\pm\sqrt{29}$ (max with $+$). ■

4. Multivariable Integration and Applications

4.1 Introduction

Recall how integration works in one dimension. Given a function $f(x)$ defined in the interval $a \leq x \leq b$, we can approximate the integral value via an Riemann sum. Divide the interval $[a,b]$ into n sub-intervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \frac{b-a}{n}$. Then we can multiply the width of the interval by the height, $f(x^*)$ to form the Riemann sum:

$$\sum_{i=1}^{n} f(x_i^*)\Delta x \to \int_a^b f(x)dx \text{ as } \Delta x \to 0,$$

where $x_{i-1} \leq x_i^* \leq x_i$ is a point in the ith interval. The integral is obtained by taking the limit as $n \to \infty$ ($\Delta x \to 0$). Thus, the integral represents the “area under the curve" (see Fig. 4.1).

We can use a similar procedure to define integration in two dimensions. Instead of intervals $[x_{i-1}, x_i]$ we have little rectangles, $R = [a,b] \times [c,d]$, with area $\Delta A = \Delta x \Delta y$. Approximate the volume under the surface by a sum of these boxes.

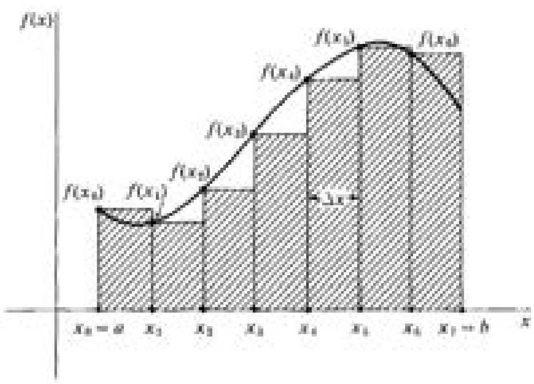


Figure 4.1: The typical way to envision integration is the sum of a bunch of tiny rectangles under the desired curve.

Definition 4.1.1 *(Double Integral)* The volume under the curve $f(x,y)$ is defined as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A \to \iint_R f(x,y)dA \text{ as } \Delta A \to 0.$$

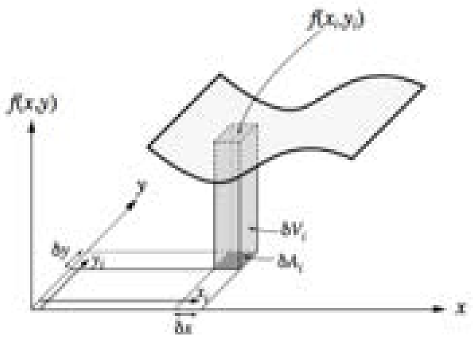


Figure 4.2: As in 1D we can approximate a curve by rectangles, only here. in 2D, we use rectangular prisms.

Properties of Double Integrals:

1. Additive: $\iint [f(x,y)+g(x,y)]\,dA = \iint_R f(x,y)dA + \iint_R g(x,y)dA$.

2. Scalar Multiple: $\iint_R cf(x,y)dA = c\iint_R f(x,y)dA$.

3. Bounds: If $f(x,y) \geq g(x,y)$ in a region R, then $\iint_R f(x,y)dA \geq \iint_R g(x,y)dA$.

Recall that in 1D we do not use the Riemann sums in practice, rather we use the *Fundamental Theorem of Calculus*

$$\int_a^b f(x)dx = F(b) - F(a), \tag{4.1}$$

where F is the indefinite integral of f or $F'(x) = f(x)$.

In two dimensions, suppose $f(x,y)$ is a function of two variables and is integrable on the rectangle $R = [a,b]\times[c,d]$. First, integrate in one of the variables (for example y first):

$$A(x) = \int_c^d f(x,y)dy.$$

Then integrate in the remaining variable:

$$V = \int_a^b A(x)dx = \int_a^b \left[\int_c^d f(x,y)dy\right]dx = \int_c^d \left[\int_a^b f(x,y)dy\right]dx.$$

V is referred to as an **iterated integral**. The iterated integral implies that we integrate with respect to y treating x as a constant, then integrate the resulting function with respect to x.

R The fact that we can integrate in either variable first is the result of *Fubini's Theorem*. Once can find many references online if interested.

■ **Example 4.1** Evaluate: $\int_0^3 \int_1^2 x^2 y dy dx$.

Solution: First, integrate in y, then x:

$$\int_0^3 \int_1^2 x^2 y dy dx = \int_0^3 \left[\frac{1}{2}x^2 y^2 \Big|_1^2 \right] dx = \int_0^3 2x^2 - \frac{1}{2}x^2 dx$$
$$= \int_0^3 \frac{3}{2}x^2 dx = \frac{1}{2}x^3 \Big|_0^3 = \frac{1}{2}[27-0] = \frac{27}{2}.$$

As a check recompute by first integrating in x, then y:

$$\int_1^2 \int_0^3 x^2 y dy dx = \int_1^2 \left[\frac{1}{3}x^3 y \Big|_0^3 \right] dy = \int_1^2 9y - 0 dy$$
$$= \int_1^2 9y dy = \frac{9}{2}y^2 \Big|_1^2 = \frac{36}{2} - \frac{9}{2} = \frac{27}{2}.$$

■

■ **Example 4.2** Evaluate: $\iint_R (x - 3y^2) dA$ where $R = [0,2] \times [1,2]$.

Solution: First, integrate in y, then x:

$$\int_0^2 \int_1^2 x - 3y^2 dy dx = \int_0^2 \left[xy - y^3 \Big|_1^2 \right] dx = \int_0^2 2x - 8 - x + 1 dx$$
$$= \int_0^2 x - 7 dx = \frac{1}{2}x^2 - 7x \Big|_0^2 = 2 - 14 = -12.$$

■

■ **Example 4.3** Evaluate: $\iint_R y \sin(xy) dA$ where $R = [1,2] \times [0,\pi]$.

Solution: First, integrate in x, then y:

$$\int_0^\pi \int_1^2 y \sin(xy) dy dx = \int_0^\pi \left[-\cos(xy) \Big|_1^2 \right] dy = \int_0^\pi -\cos(2y) + \cos(y) dy$$
$$= -\frac{1}{2}\sin(2y) + \sin(y) \Big|_0^\pi = 0.$$

■

■ **Example 4.4** Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$ and the planes $x = 2, y = 2$ as well as $x = 0, y = 0, z = 0$.

Solution: Set up the volume integral by finding the appropriate bounds. Also, we need to integrate z as a function of x and y, $z = 16 - x^2 - y^2$:

$$V = \int_0^2 \int_0^2 (16 - x^2 - y^2) dx dy = \int_0^2 \left[16x - \frac{1}{3}x^3 - 2y^2 x \Big|_0^2 \right] dy = \int_0^2 32 - \frac{8}{3} - 4y^2 dy$$
$$= \int_0^2 \frac{88}{3} - 4y^2 dy = \frac{88}{3} - \frac{4}{3}y^3 \Big|_0^2 = \frac{144}{3} = 48.$$

■

The double integral simplifies in the special case that the function $z = f(x,y)$ is separable (e.g., $f(x,y) = g(x)h(y)$).

$$\int_a^b \int_c^d f(x,y)dydx = \int_a^b \int_c^d g(x)h(y)dydx = \int_a^b g(x)\left[\int_c^d h(y)dy\right]dx = \int_a^b g(x)dx \int_c^d h(y)dy. \tag{4.2}$$

■ **Example 4.5** Evaluate: $\iint_R \sin(x)\cos(y)dA$ where $R = [0,\pi/2]\times[0,\pi/2]$.

Solution: The function is separable, so split the integral:

$$\int_0^{\pi/2}\int_0^{\pi/2} \sin(x)\cos(y)dydx = \left[\int_0^{\pi/2}\sin(x)dx\right]\left[\int_0^{\pi/2}\cos(y)dy\right]$$
$$= \left[-\cos(x)\Big|_0^{\pi/2}\right]\left[\sin(y)\Big|_0^{\pi/2}\right] = [0+1][1-0] = 1.$$

■

■ **Example 4.6** Evaluate: $\int_0^1\int_0^1 \sqrt{s+t}dsdt$.

Solution: First, integrate in s, then t:

$$\int_0^1\int_0^1 \sqrt{s+t}dsdt = \int_0^1\int_0^1 (s+t)^{1/2}dsdt = \int_0^1 \left[\frac{2}{3}(s+t)^{3/2}\Big|_0^1\right]dt = \int_0^1 \frac{2}{3}(1+t)^{3/2} - \frac{2}{3}t^{3/2}dt$$
$$= \frac{4}{15}(1+t)^{5/2} - \frac{4}{15}t^{5/2}\Big|_0^1 = \frac{4}{15}2^{5/2} = \frac{8\sqrt{2}}{15}.$$

■

■ **Example 4.7** Evaluate: $\int_0^1\int_0^3 e^{x+3y}dxdy$.

Solution: First, integrate in x, then y:

$$\int_0^1\int_0^3 e^{x+3y}dxdy = \int_0^1 \left[e^{x+3y}\Big|_0^3\right]dy = \int_0^1 e^{3+3y} - e^{3y}dy$$
$$= \int_0^1 e^{3y}\left[e^3-1\right]dy = \frac{1}{3}e^{3y}\left[e^3-1\right]\Big|_0^1 = \frac{1}{3}\left[e^3-1\right]^2.$$

■

4.2 Double Integrals Over General Regions

There are two basic cases to consider: (i) area is between two functions of x or (ii) the area is between two functions of y.

Case I: A region D in between the graphs of two continuous functions of x:

$$D := \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}.$$

Thus, the integral over this region can be computed as

$$\iint_D f(x,y)dydx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx.$$

Case II: A region D in between the graphs of two continuous functions of y:

$$D := \{(x,y) \mid h_1(y) \le x \le h_2(y), c \le y \le d\}.$$

Thus, the integral over this region can be computed as

$$\iint_D f(x,y)dydx = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy.$$

■ **Example 4.8** Evaluate: $\iint_D (x+2y)dA$ where D is bounded by $y = 2x^2$ and $y = 1+x^2$.

Solution:
Step 1: Determine the bounds of D by graphing the two given curves and finding the points of intersection.

To find the points of intersection set the two curves equal to each other and solve for x

$$2x^2 = 1+x^2 \qquad \Rightarrow \qquad x^2 - 1 = 0 \qquad \Rightarrow \qquad x = \pm 1.$$

If $x = 1$, then $y = 2x^2 = 2$ and if $x = -1$, then $y = 1+x^2 = 2$.

Step 2: Set up the integral, then solve. Observe that it is very important to figure out which curve is on top!!

$$\begin{aligned}\iint_D (x+2y)dA &= \int_{-1}^{1}\int_{2x^2}^{1+x^2}(x+2y)dydx = \int_{-1}^{1}\left[xy+y^2\Big|_{2x^2}^{1+x^2}\right]dx\\ &= \int_{-1}^{1} x(1+x^2)+(1+x^2)^2-2x^3-4x^4dx = \int_{-1}^{1} x+x^3+1+2x^2+x^4-2x^3-4x^4dx\\ &= \int_{-1}^{1} -3x^4-x^3+2x^2+x+1dx = -\frac{3}{5}x^5-\frac{1}{4}x^4+\frac{2}{3}x^3+\frac{1}{2}x^2+x\Big|_{-1}^{1} = \frac{32}{15}.\end{aligned}$$

■

R You must draw a diagram to find the bound correctly! **Question:** What if you accidentally switch $g_1(x)$ and $g_2(x)$? The magnitude of the answer will be the same, but with the wrong sign, $(-)$ correct answer.

■ **Example 4.9** Find the volume of the solid S that is bounded by the paraboloid $z = x^2+y^2$ and above the region D bounded by $y = 2x$ and $y = x^2$.

Solution:
Step 1: Determine the bounds of D by graphing the two given curves and finding the points of intersection.

To find the points of intersection set the two curves equal to each other and solve for x

$$2x = x^2 \qquad \Rightarrow \qquad x^2 - 2x = 0 \qquad \Rightarrow \qquad x = 0, x = 2.$$

Step 2: Set up the integral, then solve. Observe that it is very important to figure out which curve is on top!!

$$\iint_D x^2+y^2 dA = \int_0^2 \int_{x^2}^{2x} x^2+y^2 dydx = \int_0^2 \left[x^2y+\frac{1}{3}y^3 \Big|_{x^2}^{2x} \right] dx$$
$$= \int_0^2 2x^3+\frac{8}{3}x^3-x^4-\frac{1}{3}x^6 dx = \int_0^2 \frac{14}{3}x^3-x^4-\frac{1}{3}x^6 dx$$
$$= \frac{14}{12}x^4-\frac{1}{5}x^5-\frac{1}{21}x^7\Big|_0^2 = \frac{216}{35}.$$

■

R In the previous example, we could also write the domain D in terms of functions of y, $x = y/2$ and $x = \sqrt{y}$. Then the points of intersection are $(0,0)$ and $(2,4)$ with integral

$$\int_0^4 \int_{\sqrt{y}}^{y/2} x^2+y^2 dxdy = \frac{216}{35}.$$

■ **Example 4.10** Evaluate: $\iint_D xydA$ where D is bounded by $y = x-1$ and $y^2 = 2x+6$.

Solution:
Step 1: Determine the bounds of D by graphing the two given curves and finding the points of intersection.

To find the points of intersection set the two curves equal to each other and solve for x

$$x = y+1 = \frac{1}{2}y^2-3 \quad \Rightarrow \quad y^2-2y-8 = (y-4)(y+2) = 0 \quad \Rightarrow \quad y = 4, y = -2.$$

Step 2: Set up the integral, then solve. Observe that it is very important to figure out which curve is on top!!

$$\iint_D xydA = \int_{-2}^4 \int_{\frac{1}{2}y^2-3}^{y+1} xydxdy = \int_{-2}^4 \left[\frac{1}{2}x^2y \Big|_{\frac{1}{2}y^2-3}^{y+1} \right] dy$$
$$= \int_{-2}^4 -\frac{1}{2}(\frac{1}{2}y^2-3)^2y+\frac{1}{2}(y+1)^2ydy = \int_{-2}^4 -\frac{1}{8}y^5+2y^3+y^2-4ydy$$
$$= -\frac{1}{48}y^6+\frac{1}{2}y^4+\frac{1}{3}y^3-2y^2\Big|_{-2}^4 = 36.$$

■

R If we wanted to integrate the previous example in y and define the domain between two functions of x, first we would need to break the integral into two parts:

$$\iint_D xydA = \int_{-3}^{-1} \int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xydxdy + \int_{-1}^5 \int_{x-1}^{\sqrt{2x+6}} xydxdy.$$

■ **Example 4.11** Find the volume of a tetrahedron that is bounded by the planes $x+2y+z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

Solution:
Step 1: Determine the bounds of D by graphing the two given curves and finding the points of intersection.

To find the points of intersection set the two curves equal to each other and solve for x

$$y = 1 - \frac{x}{2} = \frac{x}{2} \qquad \Rightarrow \qquad x = 1.$$

Step 2: Set up the integral, then solve. Observe that it is very important to figure out which curve is on top!!

$$\begin{aligned}\iint_D z dA &= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y) dy dx = \int_0^1 \left[2y - xy - y^2 \Big|_{x/2}^{1-x/2} \right] dx \\ &= \int_0^1 x^2 - 2x + 1 dx \\ &= \frac{1}{3}x^3 - x^2 + x \Big|_0^1 = \frac{1}{3} - 1 + 1 = \frac{1}{3}.\end{aligned}$$

■

■ **Example 4.12** Evaluate: $\int_0^1 \int_x^1 \sin(y^2) dy dx$.

Solution: This would be too hard to evaluate as written. Instead, it may be easier to integrate as a function of x first.

Step 1: Determine the bounds of D by graphing the two given curves and finding the points of intersection.

To find the points of intersection set the two curves equal to each other and solve for x

$$x = y = 1 \qquad \Rightarrow \qquad x = 1.$$

Step 2: Set up the integral, then solve. Observe that it is very important to figure out which curve is on top!!

$$\begin{aligned}\int_0^1 \int_x^1 \sin(y^2) dy dx &= \int_0^1 \int_0^y \sin(y^2) dx dy = \int_0^1 \left[\sin(y^2) x \Big|_0^y \right] dy \\ &= \int_0^1 \sin(y^2) y dy \qquad = -\frac{1}{2}\cos(y^2)\Big|_0^1 = \frac{1}{2}\left[\cos(1) + 1\right].\end{aligned}$$

■

4.2.1 Integrals Over Subregions

A double integral can always be broken into subregions. If D is broken into two pieces D_1 and D_2 such that $D = D_1 \cup D_2$ and $D_1 \cap D_2 = 0$, then

$$\iint_D f(x,y) dA = \iint_{D_1} f(x,y) dA + \iint_{D_2} f(x,y) dA. \tag{4.3}$$

4.2.2 Area Between Curves

We can also calculate the area between two curves using a double integral

$$\iint_D 1dA = \text{Area}(D). \tag{4.4}$$

■ **Example 4.13** Calculate the area of the circle $x^2+y^2=1$.

$$
\begin{aligned}
\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1dydx &= \int_{-1}^{1}\sqrt{1-x^2}+\sqrt{1-x^2} \\
&= 2\int_{-1}^{1}\sqrt{1-x^2}dx \\
\text{Substitute } x=\sin(\theta),\ dx=\cos(\theta)d\theta: \quad &= 2\int_{-\pi/2}^{\pi/2}\sqrt{1-\sin^2(\theta)}(\cos(\theta))d\theta \\
&= 2\int_{-\pi/2}^{\pi/2}\cos^2(\theta)d\theta \\
&= 2\int_{-\pi/2}^{\pi/2}\frac{1}{2}+\frac{1}{2}\cos(2\theta)d\theta \\
&= 2\left[\frac{1}{2}\theta+\frac{1}{4}\sin(2\theta)\Big|_{-\pi/2}^{\pi/2}\right] \\
&= 2\left[\frac{\pi}{4}+0+\frac{\pi}{4}+0\right]=\pi.
\end{aligned}
$$

■

4.3 Triple Integrals

Now we consider integrating a function of three variables over three-dimensional space. This requires defining the triple integral. Given a box

$$B := \{(x,y,z) \mid a\le x\le b, c\le y\le d, r\le z\le s\},$$

the triple integral can be defined as

$$\iiiint_B f(x,y,z)dV = \int_r^s\int_c^d\int_a^b f(x,y,z)dxdydz. \tag{4.5}$$

■ **Example 4.14** Evaluate the triple integral $\iiiint_B xyz^2dV$ where $B=\{0\le x\le 1, -1\le y\le 2, 0\le z\le 3\}$.

Solution: In 3D there are now 6 possible orders of integration! So it is important to choose the one that seems to result int he easiest integrals to compute.

$$
\begin{aligned}
\int_0^3\int_{-1}^2\int_0^1 xyz^2dxdydz &= \int_0^3\int_{-1}^2\frac{1}{2}x^2yz^2\Big|_0^1 dydz = \int_0^3\int_{-1}^2\frac{1}{2}yz^2dydz \\
&= \int_0^3\frac{1}{4}y^2z^2\Big|_{-1}^2 dz = \int_0^3 z^2-\frac{1}{4}z^2dz = \int_0^3\frac{3}{4}z^2dz = \frac{1}{4}z^3\Big|_0^3 = \frac{27}{4}.
\end{aligned}
$$

■

(R) The Fubini Theorem still holds so we can take the three integrals in any order we see fit!

Definition 4.3.1 *(Iterated Integrals)* Suppose a region is bounded between two surfaces. Then we can compute the three-dimensional iterated integrals

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dzdydx \quad \text{or} \quad \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dzdxdy. \tag{4.6}$$

■ **Example 4.15** Evaluate: $\iiint_E zdV$ where E is a solid tetrahedron bounded by $x=0$, $y=0$, $z=0$, $x+y+z=1$.

Solution:

Step 1: Draw Two Diagrams! One for the 3D surfaces and one for the two-dimensional area D we will integrate over. These will be helpful in finding the curves and points of intersection. The points of intersection in the plane $z=0$ are the lines $y=0$ and $y=1-x$.

Step 2: Setup the Iterated Integrals:

$$\begin{aligned}
\int_0^1 \int_0^{1-x} \int_0^{1-x-y} zdzdydx &= \int_0^1 \int_0^{1-x} \frac{1}{2}z^2 \Big|_0^{1-x-y} dydx \\
&= \int_0^1 \int_0^{1-x} \frac{1}{2}(1-x-y)^2 dydx \\
&= \int_0^1 -\frac{1}{6}(1-x-y)^3 \Big|_0^{1-x} dx \\
&= \int_0^1 \frac{1}{6}(1-x)^3 dx \\
&= -\frac{1}{24}(1-x)^4 \Big|_0^1 = \frac{1}{24}.
\end{aligned}$$

■

■ **Example 4.16** Evaluate: $\iiint_E 6xydV$ where E is a solid tetrahedron bounded by the plane $z=1+x+y$, the curve $y=\sqrt{x}$, $y=0$, $x=1$.

Solution:

Step 1: Draw Two Diagrams! One for the 3D surfaces and one for the two-dimensional area D we will integrate over. These will be helpful in finding the curves and points of intersection. The points of intersection in the plane $z=0$ are the lines $y=0$ and $y=\sqrt{x}$ where x is from 0 to 1.

Step 2: Setup the Iterated Integrals:

$$\begin{aligned}
\int_0^1\int_0^{\sqrt{x}}\int_0^{1+x+y} 6xydzdydx &= \int_0^1\int_0^{\sqrt{x}} 6xy(1+x+y)dydx \\
&= \int_0^1\int_0^{\sqrt{x}} 6xy+6x^2y+6xy^2dydx \\
&= \int_0^1 3xy^2+3x^2y^2+2xy^3\Big|_0^{\sqrt{x}} dx \\
&= \int_0^1 3x^2+3x^3+2x^{5/2}dx \\
&= x^3+\frac{3}{4}x^4+\frac{4}{7}x^{7/2}\Big|_0^1 = \frac{65}{28}.
\end{aligned}$$

■

4.3.1 Volume Between Surfaces

We can also calculate the area between two curves using a double integral

$$\iiint_B 1dV = \text{Volume}(B). \tag{4.7}$$

4.4 Applications of Integration

There are many real world physical applications that can found using double and triple integrals. In the previous section, we observed that these integrals can be used to compute the area $A = \iint 1dxdy$ and volume $V = \iiint 1dxdydz$. Typical questions one may ask have the form: Given the curve $y = x^2 - x$ from $x = 0$ to $x = 1$, find

a) Area under the curve
b) Mass of a sheet of material cut in the shape of this area with a given density $\rho(x,y)$
c) Arc Length of the curve
d) Centroid of the area
e) Centroid of the arc
f) Moments of Inertia

We must first define these quantities.

4.4.1 Mass

Suppose a plate occupies a region D in the xy-plane with variable density $\rho(x,y)$.

Definition 4.4.1 *(Mass)* In physics, the density is defined as the mass per unit of volume, $\rho = \frac{m}{V}$. We can define the pass even when the density is non-uniform $\rho = \rho(x,y)$,

$$m = \iint_D \rho(x,y)dxdy. \tag{4.8}$$

Similarly, if an electric charge is distributed over a region D with a charge density (charge/area) given by $\sigma(x,y)$, then the total charge is

$$Q := \iint_D \sigma(x,y)dA.$$

■ **Example 4.17** Suppose the charge is distributed over a triangular region D between $x=1$, $y=1$, $y=1-x$ so that the charge density at (x,y) is $\sigma(x,y)=xy$ (C/m^2). Find the charge Q.

Solution: As in the last section we need to find the points of intersection before defining the bounds of the integral. Here, the lines intersect at $(1,0)$, $(0,1)$, and $(1,1)$.

Then we need to setup the appropriate integral

$$\begin{aligned} Q=\iint_D xydA &= \int_0^1\int_{1-x}^1 xydydx = \int_0^1 \frac{1}{2}xy^2\Big|_{1-x}^1 dx \\ &= \int_0^1 \frac{1}{2}x-\frac{1}{2}x(1-x)^2dx = \int_0^1 -\frac{1}{2}x^3+x^2dx \\ &= -\frac{1}{8}x^4+\frac{1}{3}x^3\Big|_0^1 = -\frac{1}{8}+\frac{1}{3}=\frac{5}{24}. \end{aligned}$$

■

4.4.2 Moments and Center of Mass

In physics, the moment of force (often referred to as just *moment*) is a measure of its tendency to cause a body to rotate about a specific point or axis.

Definition 4.4.2 *(Moments)* The *moment* of an object about an axis is the product of the mass and the directed distance from the axis

$$M_x := \iint_D y\rho(x,y)dxdy, \qquad M_y := \iint_D x\rho(x,y)dxdy. \tag{4.9}$$

In physics, the *center of mass* for a distribution of mass in space is the unique point where the weighted relative position of the distributed mass sums to zero. In other words, it is the point where if a force is applied the object will move in direction of force without rotation.

Definition 4.4.3 *(Centers of Mass)* The *center of mass* of an object

$$\bar{x} := \frac{1}{m}\iint_D x\rho(x,y)dxdy = \frac{M_y}{m}, \qquad \bar{y} := \frac{1}{m}\iint_D y\rho(x,y)dxdy = \frac{M_x}{m}, \tag{4.10}$$

where the mass is $m=\iint_D \rho(x,y)dydx$.

In mathematics and physics, the *centroid* or geometric center of a two-dimensional region (area) is the arithmetic mean ("average") position of all the points in the shape.

Definition 4.4.4 *(Centroid)* The *centroid* of an object is the point where it would balance on the end of the rod if the density were uniform.

$$x_{cent} := \frac{\iint_D x\rho(x,y)dxdy}{\iint_D \rho(x,y)dydx} =_{\rho=const.} \frac{1}{A}\iint_D xdA, \qquad y_{cent} := \frac{\iint_D y\rho(x,y)dxdy}{\iint_D \rho(x,y)dydx} =_{\rho=const.} \frac{1}{A}\iint_D ydA. \tag{4.11}$$

■ **Example 4.18** Find the mass and the center of mass of a triangular plate with vertices $(0,0),(1,0),(0,2)$ and density $\rho(x,y)=1+3x+y$.

Solution: First we need to find the boundary curves, in particular the line L forming the hypotenuse of the triangular plate. Using point slope form: $(y-2)=\frac{2-0}{0-1}(x-0)\Rightarrow y=-2x+2$.

Now compute the total mass using the formula:

$$m = \iint_D \rho(x,y)dA = \int_0^1 \int_0^{-2x+2} 1+3x+ydydx = \int_0^1 y+3xy+\frac{1}{2}y^2\Big|_0^{-2x+2} dx$$
$$= \int_0^1 -4x^2+4dx = -\frac{4}{3}x^3+4x\Big|_0^1 = \frac{8}{3}.$$

Now we must compute the centers of mass $\bar{x}$, $\bar{y}$.

$$\bar{x} = \frac{1}{m}\int_0^1 \int_0^{-2x+2} x(1+3x+y)dydx = \frac{3}{8}\int_0^1 xy+3x^2y+\frac{1}{2}xy^2\Big|_0^{-2x+2} dx$$
$$= \frac{3}{8}\int_0^1 -4x^3+4xdx = \frac{3}{8}\left[-x^4+2x^2\Big|_0^1\right] = \frac{3}{8},$$

and

$$\bar{y} = \frac{1}{m}\int_0^1 \int_0^{-2x+2} y(1+3x+y)dydx = \frac{3}{8}\int_0^1 \frac{1}{2}y^2+\frac{3}{2}xy^2+\frac{1}{3}y^3\Big|_0^{-2x+2} dx$$
$$= \frac{3}{8}\int_0^1 -6x^3-10x^2+2x+2+\frac{1}{3}(-2x+2)^3dx = \frac{3}{8}\left[\frac{3}{2}x^4-\frac{10}{3}x^3+x^2+2x-\frac{1}{24}(-2x+2)^4\Big|_0^1\right]$$
$$= \frac{11}{16}.$$

In addition, let's compute the centroid to show how it differs from the center of mass. First we must compute the area

$$A := \int_0^1 \int_0^{-2x+2} 1dydx = \int_0^1 -2x+2dx = -x^2+2x\Big|_0^1 = 1.$$

Now, compute x_{cent}, y_{cent}:

$$x_{cent} = \int_0^1 \int_0^{-2x+2} xdydx = \int_0^1 xy\Big|_0^{-2x+2} dx = \int_0^1 -2x^2+2xdx = -\frac{2}{3}x^3+x^2\Big|_0^1 = \frac{1}{3},$$
$$y_{cent} = \int_0^1 \int_0^{-2x+2} ydydx = \int_0^1 \frac{1}{2}y^2\Big|_0^{-2x+2} dx = \int_0^1 2x^2+-4x+2dx = \frac{2}{3}x^3-2x^2+2x\Big|_0^1 = \frac{2}{3}.$$

■

4.4.3 Moment of Inertia

The *moment of inertia*, otherwise known as the angular mass or rotational inertia, of a rigid body determines the torque needed for a desired angular acceleration about a rotational axis. It depends on the body's mass distribution and the axis chosen, with larger moments requiring more torque to change the body's rotation. Typically the moment of inertia requires a mass and radius of rotation, $I = mr^2$.

Definition 4.4.5 *(Moment of Inertia)*

$$I_x := \iint_D y^2\rho(x,y)dA, \qquad I_y := \iint_D x^2\rho(x,y)dA, \qquad \text{In 3D} \quad I_z := \iint_D (x^2+y^2)\rho(x,y)dA. \tag{4.12}$$

■ **Example 4.19** Find the moments of inertia I_x, I_y, I_z of a homogeneous rectangular plate with corners $(0,0),(0,1),(2,1),(2,0)$ and constant density $\rho(x,y)=\rho$.

$$I_x = \int_0^2 \int_0^1 x^2 \rho dydx = \int_0^2 x^2 \rho dx = \frac{\rho}{3}x^3\Big|_0^2 = \frac{8}{3}\rho,$$
$$I_y = \int_0^2 \int_0^1 y^2 \rho dydx = \int_0^2 \frac{\rho}{3} dx = \frac{\rho}{3}x\Big|_0^2 = \frac{2}{3}\rho,$$
$$I_z = \int_0^2 \int_0^1 (x^2+y^2)\rho dydx = \int_0^2 (x^2y+\frac{1}{3}y^3)\rho\Big|_0^1 dx = \int_0^2 \rho x^2 + \frac{\rho}{3}dx = \frac{\rho}{3}x^3 + \frac{\rho}{3}x\Big|_0^2 = \frac{10}{3}\rho.$$

Notice along the way we proved the *Perpendicular Axis Theorem* giving a relation between the moments, $I_x + I_y + I_z$. ■

4.4.4 Generalization of Physical Quantities to 3D

Mass: $m := \iiint_E \rho(x,y,z)dV$

Moments: $M_{yz} := \iiint_E x\rho(x,y,z)dV, \quad M_{xz} := \iiint_E y\rho(x,y,z)dV, \quad M_{xy} := \iiint_E z\rho(x,y,z)dV$

Center of Mass: $\bar{x} := \frac{M_{yz}}{m}, \quad \bar{y} := \frac{M_{xz}}{m}, \quad \bar{z} := \frac{M_{xy}}{m}$

Moments of Inertia: $I_x := \iiint_E (y^2+z^2)\rho(x,y,z)dV,$

$$I_y := \iiint_E (x^2+z^2)\rho(x,y,z)dV, \quad I_z := \iiint_E (x^2+y^2)\rho(x,y,z)dV.$$

4.4.5 Applications to Probability

Definition 4.4.6 *(Prob. Density)* A function $f(x)$ can be taken as a probability density if it satisfies

$$f(x) \geq 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} f(x)dx = 1. \tag{4.13}$$

In one dimension, the probability of finding a value in the interval $[a,b]$ given a probability density f is

$$P(a \leq x \leq b) = \int_a^b f(x)dx.$$

The analogous calculation in two-dimension refers to a joint probability density, $f(x,y)$, in two variables

$$P(a \leq x \leq b, c \leq y \leq d) = \int_a^b \int_c^d f(x,y)dydx, \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx = 1.$$

■ **Example 4.20** The joint density function for X and Y is given by:

$$f(x,y) := \begin{cases} C(x+2y), & \text{if } 0 \leq x \leq 10, 0 \leq yleq10 \\ 0m & \text{otherwise} \end{cases}.$$

a) Find the constant C such that $f(x,y)$ is a probability density.

b) Find the probability that $x \leq 7$ and $y \geq 2$.

Solution: First, we must find C by integrating f in both variables from $-\infty$ to ∞

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx = \int_0^{10}\int_0^{10} C(x+2y)dyd = \int_0^{10} Cxy + Cy^2\Big|_0^{10} dx$$
$$= \int_0^{10} 10Cx + 100Cdx = 5Cx^2 + 100Cx\Big|_0^{10} = 1500C$$

Thus, $C = \frac{1}{1500}$.

Next, the probability that $x \leq 7$ and $y \geq 2$ can be computed:

$$P(x \leq 7, y \geq 2) = \int_{-\infty}^{7}\int_{2}^{\infty} f(x,y)dydx = \int_0^7\int_2^{10} \frac{1}{1500}(x+2y)dydx = \frac{1}{1500}\int_0^2 xy + y^2\Big|_2^{10} dx$$
$$= \frac{1}{1500}\int_0^7 8x + 96dx = \frac{1}{1500}\left[4x^2 + 96x\Big|_0^7\right] = \frac{868}{1500} = .5787$$

■

Another widely used concept from probability is the concept of an *expected value*. This is the value of x and y one should expect to see on average if many trials are run.

Definition 4.4.7 *(Expected Value)* Given a joint probability density $f(x,y)$, the expected values, μ_x, μ_y, can be computed as

$$\mu_x := \iint xf(x,y)dA = m\bar{x}, \qquad \mu_y := \iint yf(x,y)dA = m\bar{y}. \tag{4.14}$$

Observe the relationship between the expected values and the centers of mass defined earlier.

■ **Example 4.21** Given the curve $y = x - x^2$ from $x = 0$ to $x = 1$.
a) Find the area under the curve.
b) Find the mass of the plane sheet cut to fit this area with density $\rho(x,y) = xy$.
c) Find the Center of Mass of the sheet.
d) Find the Volume of Revolution.
e) Find an expression for the Surface Area of Revolution.

Solution: a) First, compute the area under the curve:

$$A := \int_0^1\int_0^{x-x^2} 1dydx = \int_0^1 y\Big|_0^{x-x^2} dx = \int_0^1 x - x^2dx$$
$$= \frac{1}{2}x^2 - \frac{1}{3}x^3\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

b) Next, find the total mass of the sheet

$$m = \iint_D \rho(x,y)dydx = \int_0^1\int_0^{x-x^2} xydydx = \int_0^1 \frac{1}{2}xy^2\Big|_0^{x-x^2} dx$$
$$= \int_0^1 \frac{1}{2}x^5 - x^4 + \frac{1}{2}x^3dx = \frac{1}{12}x^6 - \frac{1}{5}x^5 + \frac{1}{8}x^4\Big|_0^1 = \frac{1}{120}.$$

c) Next, find the center of mass

$$\bar{x} = \frac{1}{m}\iint_D x\rho(x,y)dA = 120\int_0^1\int_0^{x-x^2} x^2 y\,dydx = 120\int_0^1 \frac{1}{2}x^2y^2\Big|_0^{x-x^2} dx$$
$$= 120\int_0^1 \frac{1}{2}x^6 - x^5 + \frac{1}{2}x^4 dx = 120\left[\frac{1}{14}x^7 - \frac{1}{6}x^6 + \frac{1}{10}x^5\Big|_0^1\right] = \frac{4}{7}.$$

$$\bar{y} = \frac{1}{m}\iint_D y\rho(x,y)dA = 120\int_0^1\int_0^{x-x^2} xy^2\,dydx = 120\int_0^1 \frac{1}{3}xy^3\Big|_0^{x-x^2} dx$$
$$= 120\int_0^1 -\frac{1}{3}x^7 + x^6 - x^5 + \frac{1}{3}x^4 dx = 120\left[-\frac{1}{24}x^8 + \frac{1}{7}x^7 - \frac{1}{6}x^6 + \frac{1}{15}x^5\Big|_0^1\right] = \frac{1}{7}.$$

d) Find the volume of revolution. Recall volume of a cylinder is πr^2, but here the curve $y(x)$ is the radius and the x interval $[0,1]$ is the height

$$V = \int_0^1 \pi y^2 dx = \int_0^1 \pi(x-x^2)^2 dx = \pi\int_0^1 x^4 - 2x^3 + x^2 dx$$
$$= \pi\left[\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3\Big|_0^1\right] = \frac{\pi}{30}.$$

e) The expression for the Surface Area of Revolution is:

$$A := \int_0^1 2\pi y ds = \int_0^1 2\pi(x-x^2)\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = \int_0^1 2\pi(x-x^2)\sqrt{1+(1-2x)^2}dx.$$

■

■ **Example 4.22** Find the moments of inertia for the same density, but under the curve $f(x) = x^2$ (see Book).

Solution: Compute the moments:

$$I_x = \int_0^1\int_0^{x^2} xy^3 dydx = \int_0^1 \frac{1}{4}xy^4\Big|_0^{x^2} = \int_0^1 \frac{1}{4}x^9 dx = \frac{1}{40}x^{10}\Big|_0^1 = \frac{1}{40},$$
$$I_y = \int_0^1\int_0^{x^2} x^3 y dydx = \int_0^1 \frac{1}{2}x^3y^2\Big|_0^{x^2} = \int_0^1 \frac{1}{2}x^2 dx = \frac{1}{16}x^8\Big|_0^1 = \frac{1}{16},$$
$$I_z = \int_0^1\int_0^{x^2} (x^2+y^2)xy\,dydx = \int_0^1\int_0^{x^2} xy^3 dydx + \int_0^1\int_0^{x^2} x^3 y dydx = \frac{1}{40} + \frac{1}{16} = \frac{7}{80}.$$

■

4.5 Change of Variables in Integrals

The fundamental question in this section is can we study multiple integrals in other coordinate systems. The three basic systems to consider are:

- (2D) Polar Coordinates (r,θ)
- (3D) Cylindrical Coordinates (r,θ,z)
- (3D) Spherical Coordinates (ρ,θ,ϕ)

A typical problem arises when we want to compute $\iint_R f(x,y)dA$ where R is a disk or a ring. This is much easier in polar coordinates! Recall the definition of polar coordinates.

Definition 4.5.1 *(Polar Coordinates)* Recall the relationship between (x,y) and (r,θ):

$$r^2 = x^2 + y^2, \qquad \theta = \tan^{-1}(y/x)$$
$$x = r\cos(\theta), \qquad y = r\sin(\theta).$$

For normal integration in Cartesian coordinates, we divide the region into rectangles of area $A = dxdy$, not so in polar coordinates! Instead we have a "polar rectangle" whose ares is not $drd\theta$, but $A = rdrd\theta$ since

$$\begin{aligned} \text{Area: } &= \frac{1}{2}r_2^2\Delta\theta - \frac{1}{2}r_1^2\Delta\theta = \frac{1}{2}(r_1^2 - r_2^2)\Delta\theta \\ &= \frac{1}{2}(r_2 - r_1)(r_2 + r_1)\Delta\theta = r^*\Delta r\Delta\theta \to rdrd\theta \end{aligned}$$

as $\Delta r \to 0, \Delta\theta \to 0$.

4.5.1 Changing to Polar Coordinates in a Double Integral

Given a region r and a function $f(x,y)$ we can convert the double integral to polar coordinates

$$\iint_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta))rdrd\theta. \tag{4.15}$$

■ **Example 4.23** Evaluate $\iint_{R_2}(3x+4y^2)dA$ where $R_2 = \{1 \le r \le 2, 0 \le \theta \le \pi\}$.

Solution: Compute

$$\begin{aligned} \iint_{R_2}(3x+4y^2)dA &= \int_0^\pi \int_1^2 [3r\cos(\theta) + 4r^2\sin^2(\theta)]rdrd\theta = \int_0^\pi \int_1^2 3r^2\cos(\theta) + 4r^3\sin^2(\theta)drd\theta \\ &= \int_0^\pi r^3\cos(\theta) + r^4\sin^2(\theta)\Big|_1^2 d\theta = \int_0^\pi 7\cos(\theta) + 15\sin^2(\theta)d\theta \\ &= \int_0^\pi 7\cos(\theta) + \frac{15}{2} - \frac{15}{2}\cos(2\theta)d\theta = 7\sin(\theta) + \frac{15}{2}\theta - \frac{15}{4}\sin(2\theta)\Big|_0^\pi = \frac{15\pi}{2}. \end{aligned}$$

■

■ **Example 4.24** Find the volume of the solid bounded by the plane $z = 0$ and the parabloid $z = 1 - x^2 - y^2$.

Solution: Compute

$$\begin{aligned} V &= \iint_D (1 - x^2 - y^2)dydx = \int_0^{2\pi} \int_0^1 (1 - r^2)rdrd\theta = \int_0^{2\pi} \frac{1}{2}r^2 - \frac{1}{4}r^4\Big|_0^1 d\theta \\ &= \int_0^{2\pi} \frac{1}{4}d\theta = \frac{\theta}{4}\Big|_0^{2\pi} = \frac{\pi}{2}. \end{aligned}$$

■

■ **Example 4.25** Given a semicircular sheet of material of radius a, $\theta \in [-\pi/2.\pi/2]$, and constant density ρ find (a) the center of mass, (b) Moments.

Solution: (a) By symmetry $\bar{y}=0$ and we must compute $\bar{x}$:

$$\begin{aligned}\bar{x} &= \iint x\rho dA = \int_{-\pi/2}^{\pi/2}\int_0^a r\cos(\theta)\rho r dr d\theta = \int_{-\pi/2}^{\pi/2}\int_0^a r^2\cos(\theta)\rho dr d\theta \\ &= \int_{-\pi/2}^{\pi/2} \frac{1}{3}\rho r^3\cos(\theta)\Big|_0^a d\theta = \int_{-\pi/2}^{\pi/2}\rho\frac{a^3}{3}\cos(\theta)d\theta \\ &= \rho\frac{a^3}{3}\sin(\theta)\Big|_{-\pi/2}^{\pi/2} = \frac{2a^3}{3}\rho.\end{aligned}$$

(b) By symmetry $I_x=0$ and the mass

$$m=\iint \rho dA = \int_{-\pi/2}^{\pi/2}\int_0 \rho r dr d\theta = \int_{-\pi/2}^{\pi/2}\frac{1}{2}r^2\Big|_0^a d\theta = \int_{-\pi/2}^{\pi/2}\frac{a^2}{2}d\theta = \frac{\pi a^2}{2}.$$

Then

$$\begin{aligned}I_y &= \frac{\rho}{m}\iint x^2 dA = \frac{\rho}{m}\int_{-\pi/2}^{\pi/2}\int_0^a r^2\cos^2(\theta) r dr d\theta = \frac{\rho}{m}\int_{-\pi/2}^{\pi/2}\int_0^a r^3\cos^2(\theta)dr d\theta \\ &= \frac{\rho}{m}\int_{-\pi/2}^{\pi/2}\frac{1}{4}r^4\cos^2(\theta)\Big|_0^a dr d\theta = \frac{\rho}{m}\int_{-\pi/2}^{\pi/2}\frac{a^4}{8}(1-\sin(2\theta))d\theta \\ &= \frac{\rho}{m}(\frac{a^4}{8}(\theta+\frac{1}{2}\cos(2\theta)))\Big|_{-\pi/2}^{\pi/2} = \frac{\rho}{m}\frac{\pi a^4}{8} = \rho\frac{a^2}{4}.\end{aligned}$$

■

We can also compute double integrals for arbitrarily polar regions bounded by continuous curves

$$\iint_R f(x,y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos(\theta), r\sin(\theta)) r dr d\theta. \tag{4.16}$$

■ **Example 4.26** Use the double integral to find the area of 1 loop of the 4 leaved rose $r=\cos(2\theta)$.

Solution: The area can be found as

$$\begin{aligned}A(D) &= \iint_D dA = \int_{-\pi/4}^{\pi/4}\int_0^{\cos(2\theta)} r dr d\theta = \int_{-\pi/4}^{\pi/4}\frac{1}{2}r^2\Big|_0^{\cos(2\theta)} d\theta \\ &= \int_{-\pi/4}^{\pi/4}\frac{1}{2}\cos^2(2\theta)d\theta = \frac{1}{4}\int_{-\pi/4}^{\pi/4} 1+\cos(4\theta)d\theta = \frac{1}{4}\left[\theta+\frac{1}{4}\sin(4\theta)\Big|_{-\pi/4}^{\pi/4}\right] = \frac{\pi}{8}.\end{aligned}$$

■

■ **Example 4.27** Find the value of a solid that lies under the parabloid $z=x^2+y^2$, above the xy-plane and inside the cylinder $x^2+y^2=2x \Leftrightarrow (x-1)^2+y^2=1$.

Solution: In polar coordinates $x^2+y^2-2x \Leftrightarrow r=2\cos(\theta)$ and $z=x^2+y^2=r^2$. Then the volume is

$$V = \iint_D (x^2+y^2)dA = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos(\theta)} r^3 dr d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{4}r^4\Big|_0^{2\cos(\theta)} d\theta - \int_{-\pi/2}^{\pi/2} 4\cos^4(\theta)d\theta$$
$$= 4\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos(2\theta)}{2}\right)^2 d\theta$$
$$= \int_{-\pi/2}^{\pi/2} 1+2\cos(2\theta)+\cos^2(2\theta)d\theta = \int_{-\pi/2}^{\pi/2} 1+2\cos(2\theta)+\frac{1}{2}+\frac{1}{2}\cos(4\theta)d\theta$$
$$= \theta+\sin(2\theta)+\frac{\theta}{2}+\frac{1}{8}\sin(4\theta)\Big|_{-\pi/2}^{\pi/2} = \frac{3\pi}{2}.$$

■

4.5.2 Arc Length in Polar Coordinates

From the Pythagorean Theorem $ds^2 = dr^2 + r^2 d\theta^2 \Rightarrow \frac{ds^2}{d\theta^2} = \frac{dr^2}{d\theta^2} + r^2$. Then $ds = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}d\theta = \sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}dr$. As a check we compute the arclength of a known function such as a circle of radius 1, $x^2+y^2=1$

$$ds = \int_0^{2\pi}\sqrt{r^2}d\theta = \int_0^{2\pi} r d\theta = r\theta\Big|_0^{2\pi} = 2\pi r.$$

4.6 Cylindrical Coordinates

Definition 4.6.1 *(Cylindrical Coordinates)* Recall the relationship between Cartesian (x,y,z) and Cylindrical Coordinates (r,θ,z)

$$r^2 = x^2+y^2, \qquad \theta = \tan^{-1}(y/x), \qquad z = z$$
$$x = r\cos(\theta), \qquad y = r\sin(\theta), \qquad z = z.$$

A typical volume element is $dV = r dr d\theta dz$ and arc length $ds^2 = dr^2 + r^2 d\theta^2 + dz^2$.

■ **Example 4.28** (a) Find the point $(2, \frac{2\pi}{3}, 1)$ in Cartesian coordinates.

$$x = r\cos(\theta) = 2\cos(\frac{2\pi}{3}) = 2(-1/2) = -1$$
$$y = r\sin(\theta) = 2\sin(\frac{2\pi}{3}) = 2(\sqrt{3}/2) = \sqrt{3}$$
$$z = 1.$$

(b) Find the cylindrical coordinates of the point $(3,-3,-7)$.

$$r = \sqrt{x^2+y^2} = \sqrt{9+9} = 3\sqrt{2}$$
$$\tan(\theta) = \frac{y}{x} = \frac{-3}{3} = -1 \quad \Rightarrow \quad \theta = \frac{7\pi}{4} + 2n\pi$$
$$z = -7.$$

■

R Cylindrical coordinates are most useful in problems with symmetry about some axis (e.g., a cylinder). In Cartesian coordinates a cylinder is $x^2 + y^2 = c^2$ and in Cylindrical coordinates $r = c$.

■ **Example 4.29** Describe the surface with cylindrical coordinates $z = r$.

Solution: In Cartesian coordinates this is $z^2 = x^2 + y^2$, which is concentric circles of radius z or a cone. ■

Definition 4.6.2 *(Triple Integrals in Cylindrical Coordinates)*

$$\iiint_E f(x,y,z)dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos(\theta),r\sin(\theta))}^{u_2(r\cos(\theta),r\sin(\theta))} f(r\cos(\theta), r\sin(\theta), z)rdzdrd\theta. \quad (4.17)$$

■ **Example 4.30** A solid E is inside the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. Find the mass of E given the density $\rho(x,y,z) = \sqrt{x^2 + y^2}$.

Solution: Recall the formula for the mass:

$$m = \iiint_E \rho(x,y,z)dV = \int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 rrdzdrd\theta = \int_0^{2\pi} \int_0^1 r^2 z\Big|_{1-r^2}^4 drd\theta = \int_0^{2\pi} \int_0^1 r^2(4-1+r^2)drd\theta$$

$$= \int_0^{2\pi} \int_0^1 r^4 + 3r^2 drd\theta = \int_0^{2\pi} \frac{1}{5}r^5 + r^3\Big|_0^1 d\theta - \int_0^{2\pi} \frac{6}{5}\theta\Big|_0^{2\pi} = \frac{12\pi}{5}.$$

■

■ **Example 4.31** Evaluate $\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 (x^2 + y^2)dzdydx$.

Solution: Solve:

$$\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 (x^2+y^2)dzdydx = \int_0^{2\pi} \int_0^2 \int_r^2 r^2 rdzdrd\theta = \int_0^{2\pi} \int_0^2 r^3 z\Big|_r^2 drd\theta$$

$$= \int_0^{2\pi} \int_0^2 -r^4 + 2r^3 drd\theta = \int_0^{2\pi} -\frac{1}{5}r^5 + \frac{1}{2}r^4\Big|_0^2 d\theta$$

$$= \int_0^{2\pi} -\frac{32}{5} + 8d\theta = \int_0^{2\pi} \frac{8}{5}d\theta = \frac{16\pi}{5}.$$

■

4.7 Cylindrical Coordinates

Definition 4.7.1 *(Spherical Coordinates)* Recall the relationship between Cartesian (x,y,z) and Spherical Coordinates (ρ, θ, ϕ)

$$rho^2 = x^2 + y^2 + z^2, \qquad \theta = \cos^{-1}(z/\rho), \qquad \phi = \cos^{-1}(x/\rho\sin\theta)$$
$$x = \rho\sin(\theta)\cos(\phi), \qquad y = \rho\sin(\theta)\sin(\phi), \qquad z = \rho\cos(\theta).$$

A typical volume element is $dV = \rho^2 \sin^2(\theta)d\rho d\theta d\phi$ and arc length $ds^2 = d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2(\theta)d\phi^2$.

There are many domains which are easier to describe in spherical coordinates such as (i) a sphere $\rho = const$, (ii) the half-plane $\phi = c$, the upper half cone $\theta = c$ for $c < \frac{\pi}{2}$ and lower half cone $\theta = c$ for $c > \frac{\pi}{2}$.

■ **Example 4.32** (a) Find the point $(2, \frac{\pi}{4}, \frac{\pi}{3})$ in Cartesian coordinates.

$$x = \rho\sin(\theta)\cos(\phi) = 2\sin(\frac{\pi}{3})\cos(\frac{\pi}{4}) = 2(\sqrt{3}/2)(1/\sqrt{2}) = \frac{\sqrt{3}}{\sqrt{2}}$$
$$y = \rho\sin(\theta)\sin(\phi) = 2\cos(\frac{\pi}{3})\sin(\frac{\pi}{4}) = 2(\sqrt{3}/2)(1/\sqrt{2}) = \frac{\sqrt{3}}{\sqrt{2}}$$
$$z = \rho\cos(\theta) = 2\cos(\pi/3) = 2(1/2) = 1.$$

(b) Find the spherical coordinates of the point $(0, 2\sqrt{3}, -2)$.

$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{0+12+4} = \sqrt{16} = 4$$
$$\cos(\theta) = \frac{z}{\rho} = \frac{-2}{4} = -\frac{1}{2} \quad \Rightarrow \quad \theta = \frac{2\pi}{3} + 2n\pi$$
$$\cos(\phi) = \frac{x}{\rho\sin(\theta)} = \frac{0}{4\sin(2\pi/3)} = 0.$$

■

Definition 4.7.2 *(Triple Integrals in Spherical Coordinates)*

$$\iiint_E f(x,y,z)dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin(\theta)\cos(\phi), \rho\cos(\theta)\sin(\phi), \rho\cos(\theta))\rho^2\sin(\theta)d\rho d\phi d\theta. \tag{4.18}$$

■ **Example 4.33** Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}dV$ on the unit ball $B = \{x^2+y^2+z^2 \leq 1\}$.

Solution: Setup the appropriate integral

$$\iiint_B e^{(x^2+y^2+z^2)^{3/2}}dV = \int_0^\pi \int_0^{2\pi} \int_0^1 e^{\rho^3}\rho^2\sin(\theta)d\rho d\phi d\theta = \int_0^\pi \sin(\theta)d\theta \int_0^{2\pi} d\phi \int_0^1 \rho^2 e^{\rho^3}d\rho$$
$$= \left[-\cos(\theta)\Big|_0^\pi\right]\left[\phi\Big|_0^{2\pi}\right]\left[\frac{1}{3}e^{\rho^3}\Big|_0^1\right] = 2(2\pi)(\frac{1}{3}e - \frac{1}{3} = \frac{4\pi}{3}(e-1).$$

■

■ **Example 4.34** *"Ice Cream"* Use spherical coordinates to find the volume of he solid that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = z$.

Solution: first we observe that the equation for the sphere can be written as $\rho^2 = \rho\cos(\theta) \Rightarrow \rho = \cos(\theta)$ and the equation of the cone becomes $\rho\cos(\theta) = \rho^2\sin^2(\theta)\cos^2(\phi) + \rho^2\sin^2(\theta)\sin^2(\phi) = \rho\sin(\theta)$. Thus, $\theta = \pi/4$.

$$V(E) = \iiint_E dV = \int_0^{\pi/4} \int_0^{2\pi} \int_0^{\cos(\theta)} \rho^2\sin(\theta)d\rho d\phi d\theta = \int_0^{\pi/4} \int_0^{2\pi} \frac{1}{3}\rho^3\sin(\theta)\Big|_0^{\cos(\theta)} d\phi d\theta$$
$$= \int_0^{\pi/4} \int_0^{2\pi} \frac{1}{3}\cos^3(\theta)\sin(\theta)d\phi d\theta = 2\pi\int_0^{\pi/4} \frac{1}{3}\cos^3(\theta)\sin(\theta)d\theta$$
$$= \frac{2\pi}{3}\left[-\frac{1}{4}\cos^4(\theta)\Big|_0^{\pi/4}\right] = \frac{2\pi}{3}\left[-\frac{1}{16} + \frac{1}{4}\right] = \frac{2\pi}{3}\left[\frac{3}{16}\right] = \frac{\pi}{8}.$$

■

4.7.1 Jacobians

Jacobians describe how a basic area element is scaled when changing coordinates. Consider a transformation T from the xy-plane to the uv-plane, $T(x,y) = (u,v)$. The rectangular area element in (x,y), $dA = dxdy$ will becomes distorted in the uv-plane and have a new area. The scaling of one are to another after a transformation is the Jacobian.

Definition 4.7.3 *(2D Jacobian)* In 2D, when we go from (x,y) to some new coordinates (s,t) we compute the Jacobian using partial derivatives and determinants

$$J = J\left(\frac{x,y}{s,t}\right) = \frac{\partial(x,y)}{\partial(s,t)} := \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix}. \tag{4.19}$$

The area element $dA = dydx$ is replaced buy $|J|dsdt$. *Notice the absolute value.*

■ **Example 4.35** *(Polar Coordinates)*

$$J\left(\frac{x,y}{r,\theta}\right) := \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = r\cos^2(\theta) + r\sin^2(\theta) = r.$$

■

Definition 4.7.4 *(3D Jacobian)* In 3D, when we go from (x,y,z) to some new coordinates (r,s,t) we compute the Jacobian using partial derivatives and determinants

$$J = J\left(\frac{x,y,z}{r,s,t}\right) = \frac{\partial(x,y,z)}{\partial(r,s,t)} := \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{vmatrix}. \tag{4.20}$$

The area element $dA = dzdydx$ is replaced buy $|J|drdsdt$. So $\iiint f(x,y,z)dxdydz = \iiint f(r,s,t)|J|drdsdt$.

■ **Example 4.36** *(Cylindrical Coordinates)*

$$J\left(\frac{x,y,z}{r,\theta,z}\right) := \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2(\theta) + r\sin^2(\theta) = r.$$

■

■ **Example 4.37** *(Spherical Coordinates)*

$$\begin{aligned} J\left(\frac{x,y,z}{\rho,\theta,\phi}\right) &:= \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin(\theta)\cos(\phi) & r\cos(\theta)\cos(\phi) & -r\sin(\theta)\sin(\phi) \\ \sin(\theta)\sin(\phi) & r\cos(\theta)\sin(\phi) & r\sin(\theta)\cos(\phi) \\ \cos(\theta) & -r\sin(\theta) & 0 \end{vmatrix} \\ &= \cos(\theta)\left[r^2\cos^2(\phi)\cos(\theta)\sin(\theta) + r^2\sin^2(\phi)\cos(\theta)\sin(\theta)\right] \\ &\quad + r\sin(\theta)\left[r\sin^2(\theta)\cos^2(\phi) + r\sin^2(\theta)\sin^2(\phi)\right] \\ &= \cos(\theta)\left[r^2\cos(\theta)\sin(\theta)\right] + r\sin(\theta)\left[r\sin^2(\theta)\right] = r^2\sin(\theta). \end{aligned}$$

■

R Express the velocity of a particle in spherical coordinates $v^2 = \left(\frac{ds}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + r^2\sin^2(\theta)\left(\frac{d\phi}{dt}\right)^2$.

4.8 Surface Integrals

Question: What if we want to compute the surface area of an arbitrary object (even if it is not a surface of revolution)?

Let S be a surface defined by $z = f(x,y)$ where f has continuous partial derivatives. We take a small area element defined by the vectors $\mathbf{a}, \mathbf{b}$ each starting a point P_{ij} and lying along the sides of a parallelogram with area ΔA_{ij}. Then

$$\begin{aligned} \text{Area } \Delta A_{ij} &= |\mathbf{a}\times\mathbf{b}| \\ \mathbf{a} &= \Delta x\hat{\mathbf{i}} + f_x(x_i,y_j)\Delta x\hat{\mathbf{k}} \\ \mathbf{b} &= \Delta y\hat{\mathbf{j}} + f_y(x_i,y_j)\Delta y\hat{\mathbf{k}} \end{aligned}$$

where the partial derivatives are the slopes of the tangent lines through the point P_{ij}. Thus,

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \Delta x & 0 f_x\Delta x & \\ 0 & \Delta y & f_y\Delta y \end{vmatrix} = -f_x\Delta x\Delta y\hat{\mathbf{i}} - f_y\Delta x\Delta y\hat{\mathbf{j}} + \Delta x\Delta y\hat{\mathbf{k}}. \tag{4.21}$$

Therefore the area is

$$\Delta A = |\mathbf{a}\times\mathbf{b}| = \sqrt{[f_x]^2+[f_y]^2+1}\Delta x\Delta y \tag{4.22}$$

Definition 4.8.1 *(Area of Surface)* The area of the surface with equation $z = f(x,y)$ for $(x,y)\in D$ where f_x, f_y are continuous is

$$\boxed{A = \iint_D \sqrt{[f_x(x,y)]^2+[f_y(x,y)]^2+1}dA = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}dA}. \tag{4.23}$$

R The formula for the area of a surface is the 3d analogue of the formula for arclength $s = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$.

■ **Example 4.38** Find the surface area of the part of the surface $z = x^2+2y$ that lies above the triangular region in the xy-plane with vertices $(0,0),(1,0),(1,1)$.

Solution: Find the necessary partial derivatives and use the formula for the area:

$$\begin{aligned} A &= \iint_D \sqrt{f_x^2+f_y^2+1}dA = \int_0^1\int_0^x \sqrt{(2x)^2+(2)^2+1}dydx = \int_0^1\int_0^x \sqrt{4x^2+5}dydx \\ &= \int_0^1 \sqrt{4x^2+5}y\Big|_0^x dx = \int_0^1 x\sqrt{4x^2+5}dx = \int_5^9 \frac{1}{8}u^{1/2}du \text{ for } u = 4x^2+5 \text{ and } du = 8xdx \\ &= \frac{1}{12}u^{3/2}\Big|_5^9 = \frac{27}{12} - \frac{5\sqrt{5}}{12} = \frac{1}{12}\left(27-5\sqrt{5}\right). \end{aligned}$$

■

■ **Example 4.39** Find the area of the part of the paraboloid $z = x^2+y^2$ from $z = 0$ to $z = 9$ (this surface is above the disk D with center $(0,0)$ and radius 3.

Solution: Find the necessary partial derivatives and use the formula for the area:

$$A = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\,dA = \iint_D \sqrt{(2x)^2 + (2y)^2 + 1}\,dA = \iint_D \sqrt{4x^2 + 4y^2 + 1}\,dA$$
$$= \int_0^{2\pi}\int_0^3 \sqrt{4r^2+1}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^3 \frac{1}{8}(4r^2+1)^{1/2} 8r\,dr$$
$$= 2\pi\left[\frac{1}{12}(4r^2+1)^{3/2}\Big|_0^3\right] = 2\pi\left[\frac{1}{12}(37)^{3/2} - \frac{1}{12}\right] = \frac{\pi}{6}\left(37\sqrt{37}-1\right).$$

■

■ **Example 4.40** Find the area of the part of the plane $z = 2 + 3x + 4y$ that lies above the rectangle $[0,5] \times [1,4]$.

Solution: Find the necessary partial derivatives and use the formula for the area:

$$A = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\,dA = \iint_D \sqrt{3^2 + 4^2 + 1}\,dA = \int_1^4 \int_0^5 dx\,dy$$
$$= \int_1^4 \sqrt{26}x\Big|_0^5 dy = \int_1^4 5\sqrt{26}\,dy = 20\sqrt{26}.$$

■

■ **Example 4.41** Find the area of the part of the surface $z = \frac{2}{3}\left(x^{3/2} + y^{3/2}\right)$ that lies above the rectangle $[0,1] \times [0,1]$.

Solution: Find the necessary partial derivatives and use the formula for the area:

$$A = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\,dA = \iint_D \sqrt{(x^{1/2})^2 + (y^{1/2})^2 + 1}\,dA = \int_0^1 \int_0^1 \sqrt{x+y+1}\,dy\,dx$$
$$= \int_0^1 \int_0^1 (x+y+1)^{1/2} dy\,dx = \int_0^1 \frac{2}{3}(x+y+1)^{3/2}\Big|_0^1 dx = \int_0^1 \frac{2}{3}(x+2)^{3/2} - \frac{2}{3}(x+1)^{3/2}\,dx$$
$$= \frac{4}{15}(x+2)^{5/2} - \frac{4}{15}(x+1)^{5/2}\Big|_0^1 = \frac{4}{15}3^{5/2} - \frac{4}{15}2^{5/2} - \frac{4}{15}2^{5/2} + \frac{4}{15} = \frac{4}{15}\left(9\sqrt{3} - 8\sqrt{2} + 1\right).$$

■

■ **Example 4.42** Find the area of the part of the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane.

Solution: Find the necessary partial derivatives and use the formula for the area:

$$A = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\,dA = \iint_D \sqrt{(-2x)^2 + (-2y)^2 + 1}\,dA = \iint_D \sqrt{4x^2 + 4y^2 + 1}\,dA$$
$$= \int_0^{2\pi}\int_0^2 \sqrt{4r^2+1}\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \frac{1}{8}(4r^2+1)^{1/2} 8r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{12}(4r^2+1)^{3/2}\Big|_0^2 d\theta$$
$$= \int_0^{2\pi}\left[\frac{1}{12}(17^{3/2}-1)\right] d\theta = \frac{\pi}{6}\left(17\sqrt{17}-1\right).$$

■

5. Vector Analysis

5.1 Applications of Vector Multiplication

Before returning to applications of vectors to physical systems we first briefly summarize everything we know about vectors from Chapter 3.

A vector $\mathbf{x} = (x_1, x_2, ..., x_n)$ is an ordered sequence of real numbers. The number x_n is the nth component of the vector $\mathbf{x}$. The collection of all vectors is called a *vector space* or *linear space*. For concreteness consider two three dimensional vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$. Every vector space has the following properties:

i) **Vector Equality:** If $\mathbf{x} = \mathbf{y}$, then $x_i = y_i$ for all i.

ii) **Vector Addition:** $\mathbf{x} + \mathbf{y} = \mathbf{z} = \mathbf{y} + \mathbf{x}$ where $z_i = x_i + y_i$ (Commutative).

iii) **Scalar Multiplication:** $\alpha \mathbf{x} = (\alpha x_1, \alpha x_2, \alpha x_3)$.

iv) **Zero Vector:** $\mathbf{0} = (0, 0, 0)$.

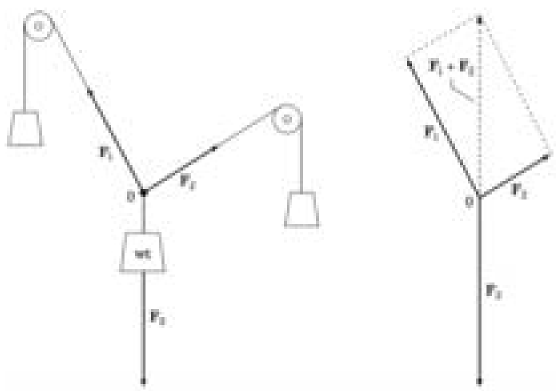


Figure 5.1: Illustration of vector addition with forces.

v) **Associative Addition:** $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$.

vi) **Distributive Law for Scalar Mult.:** $\alpha(\mathbf{x}+\mathbf{y}) = \alpha\mathbf{x}+\alpha\mathbf{y}$.

vii) **Associative Scalar Multiplication:** $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$.

5.1.1 Dot and Cross Products

Also, we recall the definitions of the scalar (dot) product and the vector (cross) product:

- *Scalar Product:* $\mathbf{x}\cdot\mathbf{y} = |\mathbf{x}||\mathbf{y}|\cos(\theta) = x_1y_1+x_2y_2+x_3y_3$
- *Cross Product:* $\mathbf{x}\times\mathbf{y} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \hat{\mathbf{i}}(x_2y_3-x_3y_3)+\hat{\mathbf{j}}(x_3y_1-y_1x_3)+\hat{\mathbf{k}}(x_1y_2-x_2y_1)$,

  and $|\mathbf{x}\times\mathbf{y}| = |\mathbf{x}||\mathbf{y}|\sin(\theta)$.

We now consider various applications of these vector quantities.

Work

A force does *work* if, when acting on a body, there is a displacement of the point of application in the direction of the force. In general force is work times displacement. If the force is applied parallel to the displacement then the work is the magnitude of the force times the distance traveled, but what happens if the force and displacement are not parallel? In physics, we know the component of the force perpendicular to the displacement does no work. So

$$W = |\mathbf{F}||\mathbf{d}|\cos(\theta) = \mathbf{F}\cdot\mathbf{d}.$$

If we want to study a dynamic problem we may need the work done by an applied force over an infinitesimally small distance $dW = \mathbf{F}\cdot d\mathbf{r}$ resulting in a total work $W = \int_a^b \mathbf{F}\cdot d\mathbf{r}$.

Torque

The *torque* or *moment* is the tendency of a force to rotate an object about an axis. Think about a lever balanced on a fulcrum at the origin. Here the torque is the force times the distance. Analogously in vector quantities, the lever arm is the perpendicular distance from the origin to the point the force **F** is applied. Since we need something perpendicular the torque is

$$\tau := \mathbf{r}\times\mathbf{F}, \qquad |\tau| = |\mathbf{F}||\mathbf{r}|\sin(\theta).$$

The torque will act in the direction perpendicular to **r** and **F** as indicated by the *righthand rule*.

Angular Velocity

The angular velocity, ω, is defined as the rate of change of angular displacement. This is a vector quantity acting in the direction along the axis of rotation (by the righthand rule). Consider a point P in a rigid body rotating with angular velocity ω. The linear translational velocity **v** of the point P is

$$\mathbf{v} = \omega\times\mathbf{r}.$$

Shortest Distance

What is the shortest distance of a rocket traveling at a constant velocity $\mathbf{v} = (1,2,3)$ from an observer at $\mathbf{x}_0 = (2,1,3)$? The rocket is launched at time $t = 0$ at the point $\mathbf{x}_1 = (1,1,1)$.

Solution: The rocket (ignoring gravity and air resistance) will follow a straight line

$$\mathbf{x}(t) = \mathbf{x}_0+\mathbf{v}t = \begin{cases} x(t) = 1+t \\ y(t) = 1+2t \\ z(t) = 1+3t \end{cases}. \tag{5.1}$$

We now want to minimize the distance $d = |\mathbf{x} - \mathbf{x}_0|$ from the observer at $\mathbf{x}_0 = (2,1,3)$ from the current position of the rocket at time t, $\mathbf{x}(t)$. Equivalently we can minimize the square of the distance $(\mathbf{x} - \mathbf{x}_0)^2$. To minimize we differentiate the equation of motion with respect to t:

$$\frac{d}{dt}(\mathbf{x} - \mathbf{x}_0)^2 = 2(\mathbf{x} - \mathbf{x}_0) \cdot \dot{\mathbf{x}} = 2[\mathbf{x}_1 - \mathbf{x}_0 + t\mathbf{v}] \cdot \mathbf{v} = 0, \qquad \mathbf{v} = \dot{\mathbf{x}}. \tag{5.2}$$

Since $\dot{\mathbf{x}} = \mathbf{v}$ is the tangent vector of the line, geometrically we say that the shortest distance vector through a point $\mathbf{x}_0$ is perpendicular to the line. Now solving for the time t when the rocket is closest we find

$$t = -\frac{(\mathbf{x}_1 - \mathbf{x}_0) \cdot \mathbf{v}}{\mathbf{v}^2} = \frac{1}{2}.$$

Now substituting t back into (6.202) yields $\mathbf{x}(1/2) = (3/2, 2, 5/2)$ as the point the rocket is closest. So the shortest distance is $d = |\mathbf{x}_0 - (3/2, 2, 5/2)| = |(-1/2, 1, -1/2)| = \sqrt{3/2}$.

Law of Cosines

Let vector $\mathbf{C} = \mathbf{A} + \mathbf{B}$ and take a dot product with itself

$$C^2 = \mathbf{C} \cdot \mathbf{C} = (\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \mathbf{A} + \mathbf{B} \cdot \mathbf{B} + 2\mathbf{A} \cdot \mathbf{B} = A^2 + B^2 + 2|\mathbf{A}||\mathbf{B}|\cos(\theta). \tag{5.3}$$

This is exactly the *Law of Cosines*!

5.2 Triple Products

In the previous sections we reviewed the definitions of the scalar (dot) and vector (cross) products. Now we use these operations in combination to define two useful quantities:

i) Triple Scalar Product, $A \cdot (B \times C)$

ii) Triple Vector Product, $A \times (B \times C)$

The names come from the resulting quantity (scalar, vector respectively).

R Observe that the other possible quantities do not make sense: 1. $(A \cdot B) \times C$ (number $\times$ vector) and 2. $(A \cdot B) \cdot C$ (number $\cdot$ vector).

5.2.1 Triple Scalar Product

First, we give a geometric interpretation of the triple scalar product $A \cdot (B \times C)$ as the volume of a parallelepiped with sides A, B, C. The area of the base is $|B \times C| = |B||C| \sin\theta$ and the height is $|A| \cos\phi$. Thus, the volume

$$V = |A||B||C| \sin\theta \cos\phi = |B \times C||A| \cos\phi = A \cdot (B \times C). \tag{5.4}$$

Observe that if two sides area parallel such that $B \times C = 0$, then both the Triple Scalar Product and the volume of the parallelepiped would be zero.

Observe that the quantities can be rearranged:

$$\begin{aligned} A \cdot (B \times C) &= A_x(B_yC_z - B_zC_y) + A_y(B_zC_x - B_xC_z) + A_z(B_xC_y - B_yC_x) \\ &= B_x(C_yA_z - C_zA_y) + B_y(C_zA_x - C_xA_z) + B_z(C_xA_y - C_yA_x) = B \cdot (C \times A) \\ &= C_x(A_yB_z - A_zB_y) + C_y(A_zB_x - A_xB_z) + C_z(A_xB_y - A_yB_x) = C \cdot (A \times B) \end{aligned}$$

Since the cross product changes sign when the order is reversed, we can only rotate the three quantities A, B, C together clockwise or counter clockwise.

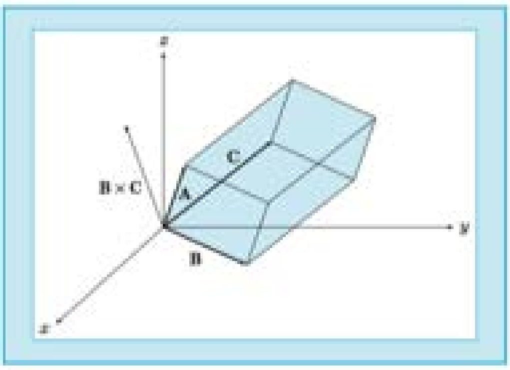


Figure 5.2: Image from Weber et al. Essential Math Methods for Physicists

■ **Example 5.1** A parallelepiped has sides $A = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$, $B = \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $C = \hat{\mathbf{i}} - \hat{\mathbf{j}}$. Find the volume.

Solution: Using the Triple Scalar Product:

$$\begin{aligned} A \cdot (B \times C) &= A \cdot \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} \\ &= A \cdot [\hat{\mathbf{i}}(0-(-1)) + \hat{\mathbf{j}}(1-0) + \hat{\mathbf{k}}(0-1)] \\ &= (1,2,-1) \cdot (1,1,-1) = 1(1) + 2(1) - 1(-1) = 4. \end{aligned}$$

■

There is an alternate definition of the Triple Scalar Product one can use involving determinants (cf. Linear Algebra Ch. 3)

$$A \cdot (B \times C) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = A_x(B_yC_z - B_xC_y) + A_y(B_zC_x - B_xC_z) + A_z(B_xC_y - B_yC_x). \quad (5.5)$$

■ **Example 5.2** Apply the alternate definition of the triple scalar product to the last example.

Solution: Using Cofactor Expansion (*Laplace Development*)

$$A \cdot (B \times C) = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = 1(-1)^{1+1} \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} + 0 + 1(-1)^{3+1} \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 1 + 3 = 4$$

■

■ **Example 5.3** Find the volume of a parallelepiped defined by $A = (0,1,2)$, $B = (1,2,3)$, $C = (-1,-1,-1)$.

Solution: Compute

$$A \cdot (B \times C) = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ -1 & -1 & -1 \end{vmatrix} = 0 + 1(-1)^{1+2} \begin{vmatrix} 1 & 3 \\ -1 & -1 \end{vmatrix} + 2(-1)^{1+3} \begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix} = -2 + 2 = 0.$$

Wait! There is no volume? It turns out that $A - B = C$ forming a linear combination. Thus, we do not have a basis and the three vectors A, B, C lie in the same plane giving a volume of zero. ■

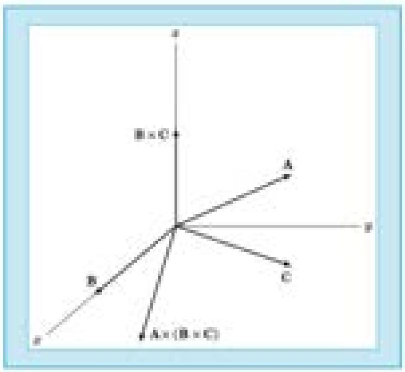


Figure 5.3: Image from Weber et al. Essential Math Methods for Physicists

5.2.2 Triple Vector Product

We now consider the Triple Vector product $A \times (B \times C)$. The first thing to observe is that the location of the parentheses is important! For example,

$$0 = A \times (B \times B) \neq (A \times B) \times B.$$

Begin with the geometric interpretation of $A \times (B \times C)$. What the quantity says is that $B \times C$ is perpendicular to both B and C. Thus the triple product must be perpendicular to A and the vector $B \times C$ resulting in a vector perpendicular to A in the plane spanned by B, C.

So, $A \times (B \times C) = \alpha B + \beta C$ for constants α, β. Take a dot product of both sides with A to find:

$$0 = \alpha(B \cdot A) + \beta(C \cdot A) \qquad \Rightarrow \qquad \alpha = \gamma(C \cdot A), \beta = -\gamma(B \cdot A). \tag{5.6}$$

Choosing $\gamma = 1$ gives $A \times (B \times C) = B(A \cdot C) - C(A \cdot B)$. The so-called *BAC-CAB Rule* for the Triple Vector Product.

■ **Example 5.4** Find $A \times (B \times C)$ for $A = (1, 2, -1)$, $B = (0, 1, 1)$, and $C = (1, -1, 0)$.

Solution: Thus,

$$A \times (B \times C) = A \times \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = A \times (1, 1, -1) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & -1 \\ 1 & 1 & -1 \end{vmatrix} = (-1, 0, -1).$$

Try to change the order of the parentheses

$$(A \times B) \times C = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & -1 \\ 0 & 1 & 1 \end{vmatrix} \times C = (3, -1, 1) \times C = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & -1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = (-1, 1, 2).$$

■

5.2.3 Applications of Triple Scalar Products

Consider the torque of a force about an axis $\tau = \mathbf{r} \times \mathbf{F}$ where $\mathbf{r}, \mathbf{F}$ are in a plane perpendicular to the axis of rotation. The torque at the origin is just $\mathbf{r} \times \mathbf{F}$

We want to find the torque produced by the force $\mathbf{F}$ about any axis L (line). Let $\mathbf{r}$ be the vector from a point on line L to the force $\mathbf{F}$. For simplicity let the line $L = \hat{\mathbf{k}}$ (z-axis). Then the torque about the line L is:

$$\tau_L := \hat{\mathbf{n}} \cdot (\mathbf{r} \times \mathbf{F}) \tag{5.7}$$

where $\hat{\mathbf{n}}$ is the unit normal in the direction of L. We can think of this as the projection of the torque in the direction of L.

Can this be simplified further? Break the vectors $\mathbf{r}$ and $\mathbf{F}$ into components parallel to L and perpendicular to L. Then

$$\begin{aligned}\mathbf{r}\times\mathbf{F} &= (\mathbf{r}_{\|}+\mathbf{r}_{\perp})\times(\mathbf{F}_{\|}+\mathbf{F}_{\perp})\\ &= \mathbf{r}_{\|}\times\mathbf{F}_{\|}+\mathbf{r}_{\perp}\times\mathbf{F}_{\|}+\mathbf{r}_{\|}\times\mathbf{F}_{\perp}+\mathbf{r}_{\perp}\times\mathbf{F}_{\perp}\\ &= 0+\mathbf{r}_{\perp}\times\mathbf{F}_{\|}+\mathbf{r}_{\|}\times\mathbf{F}_{\perp}+\mathbf{r}_{\perp}\times\mathbf{F}_{\perp}\end{aligned}$$

Then we compute

$$\begin{aligned}\hat{\mathbf{n}}\cdot(\mathbf{r}\times\mathbf{F}) &= \hat{\mathbf{n}}\cdot(\mathbf{r}_{\perp}\times\mathbf{F}_{\|})+\hat{\mathbf{n}}\cdot(\mathbf{r}_{\|}\times\mathbf{F}_{\perp})+\hat{\mathbf{n}}\cdot(\mathbf{r}_{\perp}\times\mathbf{F}_{\perp})\\ &= \hat{\mathbf{n}}\cdot(\mathbf{r}_{\perp}\times\mathbf{F}_{\perp}).\end{aligned}$$

The last line follows from the fact that $\mathbf{r}_{\|},\mathbf{F}_{\|}$ are parallel to $\hat{\mathbf{n}}$ so that $\hat{\mathbf{n}}\cdot(\mathbf{r}_{\|}\times\cdot)=0$ and $\hat{\mathbf{n}}\cdot(\mathbf{F}_{\|}\times\cdot)=0$. Thus, the torque about the line L is the torque based on the components perpendicular to L.

■ **Example 5.5** If a force $\mathbf{F}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}}$ acts at the point $P=(1,2,3)$, find the torque of $\mathbf{F}$ about the line $\mathbf{x}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}+(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}})t$.

Solution: First, find the vector torque about a point on the line (observe $\mathbf{x}_0=(2,1,0),\mathbf{v}=(1,2,3)$). This is $\tau=\mathbf{r}\times\mathbf{F}$ where $\mathbf{r}=P-\mathbf{x}_0=(1,2,3)-(2,1,0)=(-1,1,3)$ and thus the torque is

$$\tau=\mathbf{r}\times\mathbf{F}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ -1 & 1 & 3\\ 1 & 2 & -1\end{vmatrix}=(-7,2,-3).$$

Now we can compute the torque about the line L using the unit normal $\hat{\mathbf{n}}=\mathbf{v}/|\mathbf{v}|=(1,2,3)/\sqrt{1^2+2^2+3^2}=\frac{1}{\sqrt{14}}(1,2,3)$,

$$\tau_L=\hat{\mathbf{n}}\cdot(\mathbf{r}\times\mathbf{F})=\frac{1}{\sqrt{14}}(1,2,3)\cdot(-7,2,-3)=\frac{1}{\sqrt{14}}(-7+4-9)=\frac{-12}{\sqrt{14}}.$$

Observe that the negative sign indicated that the torque acts in the direction $-\hat{\mathbf{n}}$. ■

5.2.4 Application of Triple Vector Product

■ **Example 5.6** Suppose a particle of mass m is at rest on a rotating rigid body. Then the angular momentum $\mathbf{L}$ of the particle about the origin, O, is

$$\mathbf{L}:=\mathbf{r}\times(m\mathbf{v})=m\mathbf{r}\times\mathbf{v}, \tag{5.8}$$

where the linear velocity $\mathbf{v}=\omega\times\mathbf{r}$ for angular velocity ω. Thus, the angular momentum becomes

$$\boxed{\mathbf{L}=m\mathbf{r}\times(\omega\times\mathbf{r})}. \tag{5.9}$$

Centripetal acceleration can also be defined likewise, $\mathbf{a}=\omega\times(\omega\times\mathbf{r})$. ■

5.3 Fields

A vector field is a physical quantity, which has a different value at each point in space. Common examples are temperature T or the gravitational force of a satellite on Earth $|\mathbf{F}|=\frac{Gm_1m_2}{r^2}$.

In each example, there is a physical quantity in some region

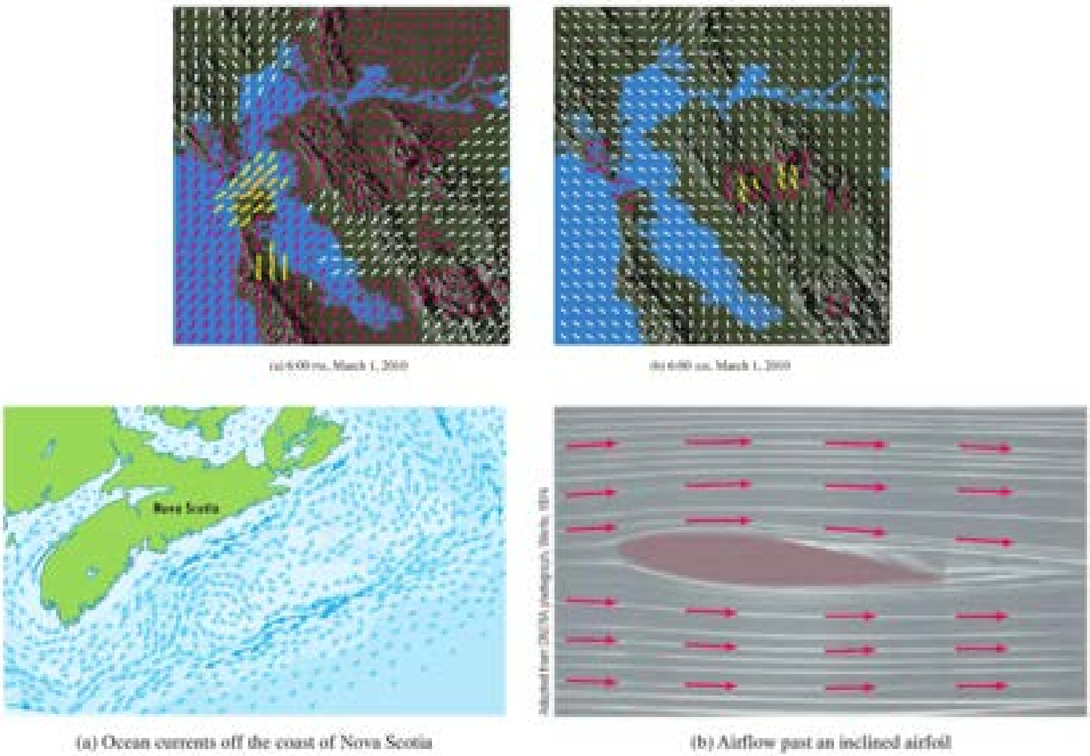


Figure 5.4: Images from Stewart Calculus.

i) If the physical quantity is a scalar, then we have a **scalar field**, e.g., Temperature T.

ii) If the physical quantity is a vector, then we have a **vector field**, e.g., Electric Field $\mathbf{E}$, Magnetic Field $\mathbf{B}$, Force $\mathbf{F}$, velocity $\mathbf{v}$.

Definition 5.3.1 Let D be a region in 2D. A **vector field** is a function $\mathbf{F}$ that assigns to each point (x,y) in D a two-dimensional vector $\mathbf{F}(x,y)$ or in 3D $\mathbf{F}(x,y,z)$. To draw a vector field, place an arrow at equally spaced points (x,y) representing the force $\mathbf{F}(x,y)$.

R One can always write it in terms of component functions

$$\mathbf{F}(x,y) = P(x,y)\hat{\mathbf{i}} + Q(x,y)\hat{\mathbf{j}} = (P(x,y), Q(x,y)) \tag{5.10}$$

where P, Q are scalar functions of two variables, *scalar fields*. In 3D it can be written as

$$\mathbf{F}(x,y,z) = P(x,y,z)\hat{\mathbf{i}} + Q(x,y,z)\hat{\mathbf{j}} + R(x,y,z)\hat{\mathbf{k}} = (P(x,y,z), Q(x,y,z), R(x,y,z)). \tag{5.11}$$

Below are sample vector fields:

■ **Example 5.7** A vector field in 2D is defined by $\mathbf{F} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$. Describe $\mathbf{F}$ by sketching some of the vectors in the vector field (in class). Each arrow is tangent to a circle with its center at the origin (Check: $\mathbf{F}\cdot\mathbf{x} = 0$ implying that the vector field $\mathbf{F}$ is perpendicular to the location $\mathbf{x}$). ■

■ **Example 5.8** Sketch the vector field in 3D given by $\mathbf{F}(x,y,z) = z\hat{\mathbf{k}}$. ■

■ **Example 5.9** Fluid flows along a pipe. Let $\mathbf{V}(x,y,z)$ be the velocity vector at a point. Then the velocity field $\mathbf{V}$ assigns a vector to each point in a certain domain (interior of the pipe). Observe that the velocity spreads out and has a smaller magnitude when the pipe diameter is larger. ■

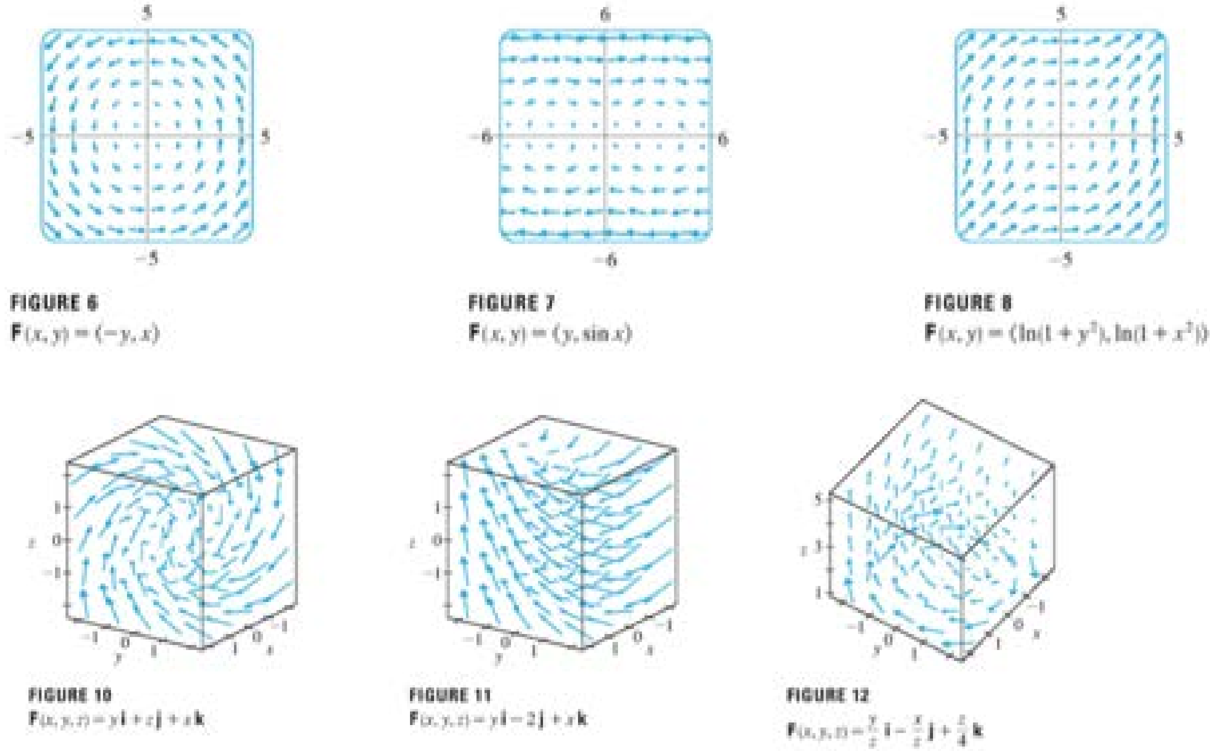


Figure 5.5: Images from Stewart Calculus.

■ **Example 5.10** Recall Newton's Law of Gravitation $|\mathbf{F}| = \frac{GmM}{r^2}$. Assume a mass M is at the origin and little mass m is at location (x,y,z). The gravitational field acts toward the direction of the unit vector $-\frac{\mathbf{x}}{|\mathbf{x}|}$ with a force

$$\mathbf{F} = -\frac{GmM}{|\mathbf{x}|^3}\mathbf{x} = -\frac{GmM}{(x^2+y^2+z^2)^{3/2}}(x,y,z). \tag{5.12}$$

■

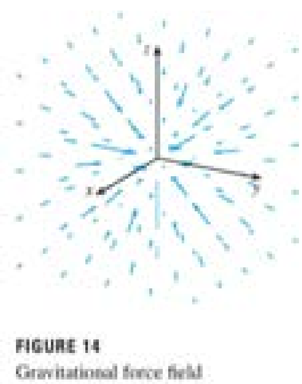


Figure 5.6: Image from Stewart Calculus.

■ **Example 5.11** Assume there is an electric charge, Q, at the origin. By Coulomb's Law, the electric force exerted by this charge on a charge q located at (x,y,z) is

$$\mathbf{F}(x,y,z) = \frac{\varepsilon qQ}{|\mathbf{x}|^3}\mathbf{x} \tag{5.13}$$

where ε is a constant. For like charges $qQ > 0$ (repulsive) and for unlike charges $qQ < 0$ (attractive). ■

5.4 Differentiation of Vectors

Consider the vector

$$\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}} = (v_x, v_y, v_z) \tag{5.14}$$

where each component is a function of time, t. We denote the derivative in time

$$\frac{d\mathbf{v}}{dt} := \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right). \tag{5.15}$$

The derivative of the vector $\mathbf{v}$ is a vector whose components are the derivatives of the components of $\mathbf{v}$.

■ **Example 5.12** Let (x, y, z) be the coordinates of a particle at time t.

$$\begin{aligned} \text{Displacement} \quad & \mathbf{r} = (x, y, z) \\ \text{Velocity} \quad & \mathbf{v} = \frac{d\mathbf{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \\ \text{Acceleration} \quad & \mathbf{z} = \frac{d^2\mathbf{r}}{dt^2} = \frac{d\mathbf{v}}{dt} = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right). \end{aligned}$$

■

One can also show the following relations for a vector $\mathbf{u} = (u_x, u_y, u_z)$ by working with components

i) $\frac{d}{dt}(a\mathbf{u}) = \frac{da}{dt}\mathbf{u} + a\frac{d\mathbf{u}}{dt}$

ii) $\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}) = \frac{d\mathbf{u}}{dt} \cdot \mathbf{v} + \frac{d\mathbf{v}}{dt} \cdot \mathbf{u}$

iii) $\frac{d}{dt}(\mathbf{u} \times \mathbf{v}) = \frac{d\mathbf{u}}{dt} \times \mathbf{v} + \mathbf{u} \times \frac{d\mathbf{v}}{dt}$

■ **Example 5.13** The position vector of a particle is $\mathbf{r} = (4 + 3t)\hat{\mathbf{i}} + t^3\hat{\mathbf{j}} - 5t\hat{\mathbf{k}}$
1. At what time does it pass through the point $(1, -1, 5)$.
2. Find the velocity at this time.
3. Find the equation of the line tangent to its path and plane normal to its path at $(1, -1, 5)$.

Solutions: 1. To find the time the particle passes through the point $(1, -1, 5)$ we set each of these values equal to the corresponding component of $\mathbf{r}(t)$ and solve for t. Thus,

$$\begin{aligned} 4 + 3t &= 1 \\ t^3 &= -1 \\ -5t &= 5. \end{aligned}$$

All the equations are satisfied when $t = -1$.

2. The velocity is $\mathbf{v} = \frac{d\mathbf{r}}{dt} = (3, 3t^2, -5)$. At time $t = -1$, the velocity is $(3, 3, -5)$.

3. The line tangent has equation $\mathbf{x} = \mathbf{x}_0 + \mathbf{v}t$ where $\mathbf{x}_0 = (1, -1, 5)$ and $\mathbf{v} = (3, 3, -5)$. The

plane normal has equation $ax+by+cz+d=0$ where (a,b,c) is the normal to the plane (parallel to the velocity $\mathbf{v}$. Thus,

$$3x+3y-5z+d=0 \qquad \Rightarrow \text{Point on plane is } (x,y,z)=(1,-1,5) \Rightarrow \qquad d=25. \tag{5.16}$$

■

■ **Example 5.14** The position of a particle is $\mathbf{r}(t)=(\cos(t),\sin(t),t)$. Show that the speed $|\mathbf{v}|$ and the acceleration $|\mathbf{a}|$ are constant.

Solution: Compute the velocity and acceleration vectors:

$$\begin{aligned}
\mathbf{v} &= \frac{d\mathbf{r}}{dt} = (-\sin(t),\cos(t),1) \qquad \Rightarrow \qquad |\mathbf{v}| = \sqrt{(-sin(t))^2+(\cos(t))^2+1} = \sqrt{2} \\
\mathbf{a} &= \frac{d\mathbf{r}^2}{dt^2} = (-\cos(t),-\sin(t),0) \qquad \Rightarrow \qquad |\mathbf{a}| = \sqrt{(-\cos(t))^2+(-\sin(t))^2+0} = 1.
\end{aligned}$$

■

5.4.1 Differentiation in Polar Coordinates

Observe that the unit vectors in polar coordinates are:

$$\mathbf{e}_r = (\cos(\theta),\sin(\theta), \qquad \mathbf{e}_\theta = (-\sin(\theta),\cos(\theta)). \tag{5.17}$$

Thus, their corresponding velocities are:

$$\begin{aligned}
\frac{d\mathbf{e}_r}{dt} &= (-\sin\theta\frac{d\theta}{dt},\cos(\theta)\frac{d\theta}{dt}) = \mathbf{e}_\theta\frac{d\theta}{dt} \\
\frac{d\mathbf{e}_\theta}{dt} &= (-\cos(\theta)\frac{d\theta}{dt},-\sin(\theta)\frac{d\theta}{dt}) = -\mathbf{e}_r\frac{d\theta}{dt}
\end{aligned}$$

■ **Example 5.15** Express a vector $\mathbf{u}$ in polar coordinates $\mathbf{u}=u_r\mathbf{e}_r+u_\theta\mathbf{e}_\theta$. Find $\frac{d\mathbf{u}}{dt}$.

Solution: Take the time derivative and use the relationships above

$$\frac{d\mathbf{u}}{dt} = \mathbf{e}_r\frac{d\mathbf{u}_r}{dt} + u_r\frac{d\mathbf{e}_r}{dt} + \mathbf{e}_\theta\frac{du_\theta}{dt} + u_\theta\frac{d\mathbf{e}_\theta}{dt} = \mathbf{e}_r\left[\frac{du_r}{dt} - u_\theta\frac{d\theta}{dt}\right] + \mathbf{e}_\theta\left[u_r\frac{d\theta}{dt} + \frac{du_\theta}{dt}\right]. \tag{5.18}$$

■

■ **Example 5.16** Let $\mathbf{r}=r\mathbf{e}_r$. Find the velocity and acceleration.

Solution: Use the relations above to perform the computations:

$$\begin{aligned}
\mathbf{v} &= \frac{d\mathbf{r}}{dt} = \mathbf{e}_r\frac{dr}{dt} + \mathbf{e}_\theta r\frac{d\theta}{dt} \\
\mathbf{a} &= \frac{d^2\mathbf{r}}{dt^2} = \mathbf{e}_r\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] + \mathbf{e}_\theta\left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right].
\end{aligned}$$

■

5.5 Directional Derivative and Gradient

As discussed previously, temperature $T(x,y,z)$ is a typical example of a scalar field. Suppose one turns on a Jacuzzi with a heater in the center. We want to know how the temperature changes as we move through the water. This depends greatly on the direction one moves. If we move toward the center it gets hotter and the temperature increases; however, if we move away from the center the temperature will decrease.

This leads to two natural questions:

i) What is the temperature change in the specific direction one is heading, $\frac{dT}{ds}$?

ii) Which direction produces the largest/smallest temperature change?

Since heat flows from hot to cold, the heat would follow the direction of the maximal rate of decrease.

Problem: Consider a scalar function $\phi(x,y,z)$ (e.g. Temperature). We want to find the derivative in the direction s, $\frac{d\phi}{dt}$, at a given point (x_0,y_0,z_0) in a given direction.

Suppose $\mathbf{u} = (a,b,c)$ is a unit vector in the s direction. Move a distance s in the direction of $\mathbf{u}$: $(x,y,z) = (x_0,y_0,z_0) + s(a,b,c) = \mathbf{x}_0 + s\mathbf{u}$. Along this line, one can think of x,y,z as functions of only a single variable s. Thus, using the chain rule we find

$$\begin{aligned}\frac{d\phi}{ds} &= \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds}\\ &= \frac{\partial\phi}{\partial x}a + \frac{\partial\phi}{\partial y}b + \frac{\partial\phi}{\partial z}c\\ &= \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)\cdot(a,b,c)\\ &= \nabla\phi\cdot\mathbf{u}\end{aligned}$$

Definition 5.5.1 The vector $\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$ is called the **gradient of** ϕ and may also be denoted $\text{grad}(\phi)$.

$$\nabla\phi := \frac{\partial\phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{j}} + \frac{\partial\phi}{\partial z}\hat{\mathbf{k}} \tag{5.19}$$

Definition 5.5.2 The **directional derivative** in the direction $\mathbf{u}$ is

$$\frac{d\phi}{ds} = \nabla\phi\cdot\mathbf{u} \tag{5.20}$$

R If $\nabla\phi$ is in the direction of $\mathbf{u}$, then it is maximized/minimized: $\nabla\phi = a\mathbf{u}$, then

$$\frac{d\phi}{ds} = \nabla\phi\cdot\mathbf{u} = a\mathbf{u}\cdot\mathbf{u} = a|\mathbf{u}|^2 = a.$$

If the gradient is not in this direction (instead in direction of unit vector $\mathbf{v}$), then

$$\frac{d\phi}{ds} = \nabla\phi\cdot\mathbf{u} = a\mathbf{v}\cdot\mathbf{u} = a|\mathbf{v}||\mathbf{u}|\cos(\theta) = a\cos(\theta) < a.$$

■ **Example 5.17** Find the directional derivative of $\phi = xy^2 + 3yz$ at $(1,0,2)$ in the direction $\mathbf{v} = (1,2,2)$.

Solution:
Step 1: Obtain the unit vector **u** in the direction of **v**

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{1^2+2^2+2^2}}(1,2,2) = \frac{1}{3}(1,2,2)$$

Step 2: Compute the gradient

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = (y^2, 2xy+3z, 3y)$$

and evaluate at the point $(1,0,2)$, $\nabla\phi\Big|_{(1,0,2)} = (0,6,0)$.

Step 3: Compute the directional derivative

$$\frac{d\phi}{ds} = \nabla\phi\cdot\mathbf{u} = (0,6,0)\cdot\frac{1}{3}(1,2,2) = 0+6(2/3)+0 = 4. \tag{5.21}$$

■

The geometric or physical interpretation of the directional derivative requires knowledge of the dot product. The directional derivative $\frac{d\phi}{ds} = \nabla\phi\cdot\mathbf{u} = |\nabla\phi||\mathbf{u}|\cos(\theta)$. Thus, the directional derivative is the projection of the gradient $\nabla\phi$ onto the line in the direction of **u**. Obviously, the projection is largest if $\nabla\phi$ in the direction of **u**, just like the gradient is largest if it is in the direction of **u**.

■ **Example 5.18** Suppose the temperature $T(x,y,z)$ is given by $T = z^3 - x^3 + xyz + 10$. In which direction is the temperature change the greatest at $(-1,1,2)$ and at what rate?

Solution: First, the greatest temperature change is in the direction of the gradient

$$\nabla T = (-3x^2+yz, xz, 3z^2+xy)\Big|_{(-1,1,2)} = (-1,-2,11).$$

The rate of increase/decrease is

$$|\nabla T| = \sqrt{(-1)^2+(-2)^2+11^2} = \sqrt{1+4+121} = \sqrt{126} = 3\sqrt{14}.$$

■

Suppose **u** is tangent to the surface $\phi = const$ at the point $P = (x_0, y_0, z_0)$. Consider $\frac{\Delta\phi}{\Delta s}$ for PA, PB, PC approaching **u**. Since $\phi = const$, P, A, B, C are on the surface, $\Delta\phi = 0$. Thus,

$$\frac{\Delta\phi}{\Delta s} = 0 \rightarrow_{\Delta s=0} \frac{d\phi}{ds} = 0 \quad \Rightarrow \quad \nabla\phi\cdot\mathbf{u} = 0 \quad \Rightarrow \quad \nabla\phi \perp \mathbf{u}. \tag{5.22}$$

Definition 5.5.3 The vector $\nabla\phi$ is normal to the surface $\phi = const$.

Since $|\nabla\phi|$ is the value of the directional derivative normal to the surface, then the **normal derivative** $\frac{d\phi}{dn} = |\nabla\phi|$. In temperature problems, the direction of largest change in temperature is normal to the isothermal lines (constant temperature).

■ **Example 5.19** Given the surface $xyz^2 = 4$, find the equation of the tangent plane and normal line at the point $(2,2,-1)$.

Solution: The level surface is $w = xyz^2$, so the normal direction is in the direction of the gradient

$$\nabla w = (yz^2, xz^2, 2xyz)\Big|_{(2,2,-1)} = (2,2,-8).$$

The tangent plane has the following equation (since the gradient is normal to the surface)

$$\frac{\partial w}{\partial x}x + \frac{\partial w}{\partial y}y + \frac{\partial w}{\partial z}z + d = 0 \quad \Rightarrow \quad 2x + 2y - 8z + d = 0 \quad \Rightarrow \quad x + y - 4z = 8.$$

Found d by plugging in the point $(x,y,z) = (2,2,-1)$. The normal line has equation

$$\frac{x-2}{2} = \frac{y-2}{2} = \frac{z+1}{-8}, \qquad \begin{cases} x = 2 + 2t \\ y = 2 + 2t \\ z = -1 - 8t \end{cases}$$

■

5.5.1 Gradients in Other Coordinate Systems

$$\text{Cylindrical (Polar if } z = 0) \qquad \nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z$$

$$\text{Spherical} \qquad \nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\phi}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi.$$

5.5.2 Physical Significance

The gradient of a scalar, $\nabla\phi$, is extremely important in physics and engineering. Given a potential energy U, the associated force is $\mathbf{F} = -\nabla U$ (e.g., gravity, electrostatics, etc.). If a force can be described by a single scalar function U, we call U the **potential**.

R Since the force is the directional derivative of U, we can find U by integrating the force along a path. The work of the force $\mathbf{F}$ along the $d\mathbf{r}$ is

$$W = \int dU = \int \nabla U \cdot d\mathbf{r} = \int -\mathbf{F}\cdot d\mathbf{r} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} dU = U(\mathbf{r}_2) - U(\mathbf{r}_1).$$

Definition 5.5.4 Such forces that behave in this way are call **conservative**. (More on this in the coming sections).

■ **Example 5.20** Find the gradient for a function of position $\mathbf{r}$ (Central Forces) $r = \sqrt{x^2+y^2+z^2}$.

Solution: Then $\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$. Then

$$\begin{aligned}\nabla f(\mathbf{r}) &= \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}} \\ &= \frac{\partial f}{\partial r}\frac{\partial r}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial r}\frac{\partial r}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial r}\frac{\partial r}{\partial z}\hat{\mathbf{k}} \\ &= \frac{\partial f}{\partial r}\left[\frac{x}{r}\hat{\mathbf{i}} + \frac{y}{r}\hat{\mathbf{j}} + \frac{z}{r}\hat{\mathbf{k}}\right] = \frac{\mathbf{r}}{|\mathbf{r}|}\frac{\partial f}{\partial r},\end{aligned}$$

which is made up of the unit vector in the radial direction multiplied by the directional derivative in the radial direction. ■

If a vector function depends on space (x,y,z) and time t, the from the total differential we see

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz + \frac{\partial F}{\partial t}dt = (d\mathbf{r}\cdot\nabla)F + \frac{\partial F}{\partial t}dt. \tag{5.23}$$

Divide the result by dt to get the so-called *material derivative*

$$\frac{dF}{dt} = \left(\frac{d\mathbf{r}}{dt}\cdot\nabla\right)F + \frac{\partial F}{\partial t}. \tag{5.24}$$

This expression comes up a lot in physics, but most famously in the Navier-Stokes equations for fluid flow

$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{u} \tag{5.25}$$

where $\mathbf{u}$ is the fluid velocity and $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ is the velocity of an individual fluid particle.

5.6 Some Other Expressions Involving ∇

In this section we take a deeper look at the gradient operator and where else it can appear.

Definition 5.6.1 The symbol ∇ is used to represent a vector operator and is defined as

$$\nabla = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}. \tag{5.26}$$

This can be compared and contrasted with the scalar differential operator $\frac{d}{dx}$.

So far we have considered the gradient of a scalar function, $\nabla\phi$. Here an operation is performed on the scalar ϕ resulting in a vector. Can the gradient operator, ∇, be applied to a vector?

Given a vector function $\mathbf{V}(x,y,z) = (V_x,V_y,V_z)$. **Question:** How can we now compute the vector operator ∇ applied to the vector $\mathbf{V}$ in different ways?

5.6.1 Divergence, $\nabla\cdot\mathbf{V}$

Definition 5.6.2 The divergence of a vector is defined as the scalar quantity

$$\nabla\cdot\mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}. \tag{5.27}$$

This is represents the outward flow. If the divergence is positive there is a net outward flow, if negative there is a net inward flow. The special case where the divergence is zero the media is referred to as *incompressible* or **solenoidal**.

Special Case: The divergence of a central force field. Let $\mathbf{r}$ be the position vector and ϕ a scalar function. Then

$$\begin{aligned} \nabla\cdot(\mathbf{r}\phi) &= \frac{\partial}{\partial x}[x\phi] + \frac{\partial}{\partial y}[y\phi] + \frac{\partial}{\partial z}[z\phi] \\ \text{Product Rule} \quad &= \phi + x\frac{\partial \phi}{\partial x} + \phi + \frac{\partial \phi}{\partial y} + \phi + \frac{\partial \phi}{\partial z} \\ &= 3\phi + \mathbf{r}\cdot(\nabla\phi) \end{aligned}$$

Definition 5.6.3 In general, we have the following "Product Rule for Divergence"

$$\nabla \cdot (\phi \mathbf{v}) = \nabla \phi \cdot \mathbf{v} + \phi (\nabla \cdot \mathbf{v}). \tag{5.28}$$

5.6.2 Physical Interpretation

Consider the quantity $\nabla \cdot (\phi \mathbf{v})$ where $\mathbf{v}$ is the fluid velocity and ρ is the density of the fluid. Consider a small volume $dxdydz$ as a rectangular prism $ABCDEFGH$. Then the total flow through the face $EFGH$ is

$$\rho v_x \Big|_{x=0} dydz,$$

which is the tangential component of the force while $\rho v_y, \rho v_z$ contribute nothing. The flow out the opposite face $ABCD$ is

$$\rho v_x \Big|_{x=dx} dydz = \left[\rho v_x + \frac{\partial}{\partial x}(\rho v_x)dx\right]\Big|_{x=0} dydz.$$

Thus, the net rate of flow in the x-direction is the flow in minus the flow out

$$\rho v_x \Big|_{x=0} dydz - \left[\rho v_x + \frac{\partial}{\partial x}(\rho v_x)dx\right]\Big|_{x=0} dydz = -\frac{\partial}{\partial x}(\rho v_x)dxdydz$$

Similarly we can find the net rate of flow in the y and z directions:

$$y\text{-direction} \qquad -\frac{\partial}{\partial y}(\rho v_y)dxdydz$$

$$z\text{-direction} \qquad -\frac{\partial}{\partial z}(\rho v_z)dxdydz.$$

Therefore, the net flow per unit time is:

$$-\left[\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z)\right] dxdydz = -\nabla \cdot (\rho \mathbf{v})dxdydz.$$

A direct application of this idea results in the **continuity equation**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{5.29}$$

This equation says that the net flow out of a volume results in a decreased density inside that volume.

■ **Example 5.21** If $\mathbf{F} = (xz, xyz, -y^2)$. Find $\nabla \cdot \mathbf{F} = \text{div}(\mathbf{F})$.

Solution: Compute

$$\begin{aligned}\text{div}(\mathbf{F}) &= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2) \\ &= z + xz + 0 = z + xz.\end{aligned}$$

■

5.6.3 Curl $\nabla \times \mathbf{V}$

Definition 5.6.4 The **curl** of a vector field is defined as

$$\nabla\times\mathbf{V}=\text{curl}(\mathbf{V})=\hat{\mathbf{i}}\left(\frac{\partial V_z}{\partial y}-\frac{\partial V_y}{\partial z}\right)+\hat{\mathbf{j}}\left(\frac{\partial V_x}{\partial z}-\frac{\partial V_z}{\partial x}\right)+\hat{\mathbf{k}}\left(\frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial y}\right)=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ V_x & V_y & V_z\end{vmatrix},\tag{5.30}$$

resulting in a vector.

■ **Example 5.22** Find $\nabla\times\mathbf{V}$ where $\mathbf{V}=(xz,xyz,-y^2)$.

$$\nabla\times\mathbf{V}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ V_x & V_y & V_z\end{vmatrix}=(-2y-xy,x,yz).$$

■

■ **Example 5.23** Compute the curl of a central force

$$\nabla\times\mathbf{r}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x & y & z\end{vmatrix}=0.$$

The physical significance can be seen by considering the circulation of a fluid around a rectangle in the xy-plane with corners $(x_0,y_0),(x_0+dx,y_0),(x_0+dx,y_0+dy),(x_0,y_0+dy)$.

$$\begin{aligned}Circ_{1234}&=v_x(x_0,y_0)dx+\left[v_y(x_0,y_0)+\frac{\partial v_y}{\partial x}dx\right]dy+\left[v_x(x_0,y_0)+\frac{\partial v_x}{\partial y}dy\right](-dx)+v_y(x_0,y_0)(-dy)\\ &=\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)dxdy.\end{aligned}$$

Divide by the area $dxdy$ to find $\frac{Circ}{area}=\nabla\times\mathbf{v}\Big|_{z-component}$. ■

Special Case: When the curl is zero $\nabla\times\mathbf{v}=0$ the flow is called **irrotational** or **conservative**. Physically this means that the fluid velocity only moves in the radial direction with no rotation. Examples of this include gravitational and electrostatic forces: $\mathbf{v}=C\frac{\mathbf{r}}{|\mathbf{r}|}$ where $C=-Gm_1m_2$ for gravitational forces and $C=\frac{q_1q_2}{4\pi\varepsilon_0}$ for electrostatic Coulomb forces.

Consider the quantity $\nabla\times(\phi\mathbf{v})$, then

$$\begin{aligned}\nabla\times(\phi\mathbf{v})&=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \phi v_x & \phi v_y & \phi v_z\end{vmatrix}\\ &=\hat{\mathbf{i}}\left(\frac{\partial\phi}{\partial y}v_z+f\frac{\partial v_z}{\partial y}-\frac{\partial\phi}{\partial z}v_y-\phi\frac{\partial v_y}{\partial z}\right)+\hat{\mathbf{j}}\left(\frac{\partial\phi}{\partial z}v_x+f\frac{\partial v_x}{\partial z}-\frac{\partial\phi}{\partial x}v_z-\phi\frac{\partial v_z}{\partial x}\right)\\ &\qquad+\hat{\mathbf{k}}\left(\frac{\partial\phi}{\partial x}v_y+f\frac{\partial v_y}{\partial x}-\frac{\partial\phi}{\partial y}v_x-\phi\frac{\partial v_x}{\partial y}\right)\\ &=\phi(\nabla\times\mathbf{v})+(\nabla\phi)\times\mathbf{v}.\end{aligned}$$

The last line represents to so-called “Product Rule for Curl".

■ **Example 5.24** Consider the vector potential of a constant **B**-field In electrodynamics, if $\nabla\cdot\mathbf{B}=0$, then the **B**-field can be represented as $\mathbf{B}=\nabla\times\mathbf{A}$ where **A** is a **vector potential**. This is seen to be true by doing the following computation:

$$\nabla\cdot\mathbf{B}=\nabla\cdot(\nabla\times\mathbf{A})=\mathbf{A}\cdot(\nabla\times\nabla)=0.$$

One possible form of $\mathbf{A}=\frac{1}{2}(\mathbf{B}\times\mathbf{r})$. Thus,

$$2(\nabla\times\mathbf{A})=\nabla\times(\mathbf{B}\times\mathbf{r})=\mathbf{B}(\nabla\cdot\mathbf{r})-(\mathbf{B}\cdot\nabla)\mathbf{r}=3\mathbf{B}-\mathbf{B}=2\mathbf{B}\qquad\Rightarrow\qquad\mathbf{B}=(\nabla\times\mathbf{A}).\quad(5.31)$$

■

The other interesting observation is that the $BAC-CAB$ Rule still holds when using the gradient operator, ∇.

■ **Example 5.25** Evaluate

$$\nabla\times(\nabla\times\mathbf{V})=\nabla(\nabla\cdot\mathbf{V})-(\nabla\cdot\nabla)\mathbf{V}=\nabla(\nabla\cdot\mathbf{V})-\nabla^2\mathbf{V}$$

■

Definition 5.6.5 *(Laplacian)* The Laplacian operator is defined as

$$\Delta=\nabla^2=(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}).\qquad(5.32)$$

The Laplacian of a scalar is $\Delta\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$ resulting in a scalar. If we apply the Laplacian to a vector

$$\Delta\mathbf{V}=\nabla^2\mathbf{V}=\left(\frac{\partial^2V_x}{\partial x^2}+\frac{\partial^2V_x}{\partial y^2}+\frac{\partial^2V_x}{\partial z^2},\frac{\partial^2V_y}{\partial x^2}+\frac{\partial^2V_y}{\partial y^2}+\frac{\partial^2V_y}{\partial z^2},\frac{\partial^2V_z}{\partial x^2}+\frac{\partial^2V_z}{\partial y^2}+\frac{\partial^2V_z}{\partial z^2}\right),$$

the result is a vector with each component the Laplacian applied to each scalar component.

There are famous equations involving the Laplacian we will study in detail in Chapter 13:
1. Laplace's Equation $\Delta\phi=0$ (Elasticity)
2. Heat Equation $\Delta\phi=\frac{1}{a^2}\frac{\partial\phi}{\partial t}$ (Temperature Distribution, Diffusion, Schrödinger)
3. Wave Equation $\Delta\phi=\frac{1}{a^2}\frac{\partial^2\phi}{\partial t^2}$ (Vibration, Waves).

5.6.4 Solenoidal and Irrotational

If a vector **v** is solenoidal, then $\nabla\cdot\mathbf{v}=0$ and there exists a vector **A** such that $\mathbf{v}=\nabla\times\mathbf{A}$.

If a vector **v** is irrotational or conservative, then $\nabla\times\mathbf{v}=0$ and there exists a scalar function ϕ such that $\mathbf{v}=\nabla\phi$.

Some vectors always fit into one of these categories. For example, the curl of a vector curl(**v**) is always solenoidal since $\nabla\cdot\text{curl}(\mathbf{v})=0$. Also, every vector can be written as a conservative vector field added to a solenoidal vector field. Given a vector **F**, then

$$\mathbf{F}=-\nabla\phi+\nabla\times\mathbf{A}$$

for some scalar ϕ and vector **A**.

We can also combine these operations to get other useful relationship (no need to memorize):

$$\nabla(\mathbf{A}\cdot\mathbf{B})=(\mathbf{B}\cdot\nabla)\mathbf{A}+(\mathbf{A}\cdot\nabla)\mathbf{B}+\mathbf{B}\times(\nabla\times\mathbf{A})+\mathbf{A}\times(\nabla\times\mathbf{B})$$
$$\mathbf{A}\times(\nabla\times\mathbf{B})=\nabla(\mathbf{A}\cdot\mathbf{B})-(\mathbf{A}\cdot\nabla)\mathbf{B}$$
$$\mathbf{B}\times(\nabla\times\mathbf{A})=\nabla(\mathbf{A}\cdot\mathbf{B})-(\mathbf{B}\cdot\nabla)\mathbf{A}.$$

5.6.5 Divergence and Laplacian in Other Coordinate Systems

In Cylindrical (r,θ,z) coordinates we have

$$\nabla\cdot\mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(v_\theta) + \frac{\partial}{\partial z}v_z$$
$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial f}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2}.$$

In Spherical (rho,θ,ϕ) coordinates we have

$$\nabla\cdot\mathbf{v} = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2 v_\rho) + \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta) + \frac{1}{\rho\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$
$$\Delta f = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2\frac{\partial f}{\partial\rho}) + \frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial f}{\partial\theta}) + \frac{1}{\rho^2}\sin^2\theta\frac{\partial^2 f}{\partial\phi^2}.$$

5.7 Line Integrals

Recall that infinitesimal work can be written $dW = \mathbf{F}\cdot d\mathbf{r}$. Suppose that the object is moving along some path (from A to B). Along this curve there is only one independent variable (the parameterization of the curve). Therefore, the force field $\mathbf{F}$ and $d\mathbf{r} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}$ are functions of a single variable. One can think of breaking the curve into equal size arc of length Δs such that the work done is summed over each segment

$$W = \sum_i^N f(x_i,y_i)\Delta s_i \to_{\Delta s\to 0} \int_C f\cdot ds.$$

Definition 5.7.1 *(Line Integral)* The line integral (in this case for work) can be expressed as

$$W = \int_C \mathbf{F}\cdot d\mathbf{r}, \tag{5.33}$$

for any curve C moving counterclockwise. If moving clockwise, then a $-$ appears in front of the integral.

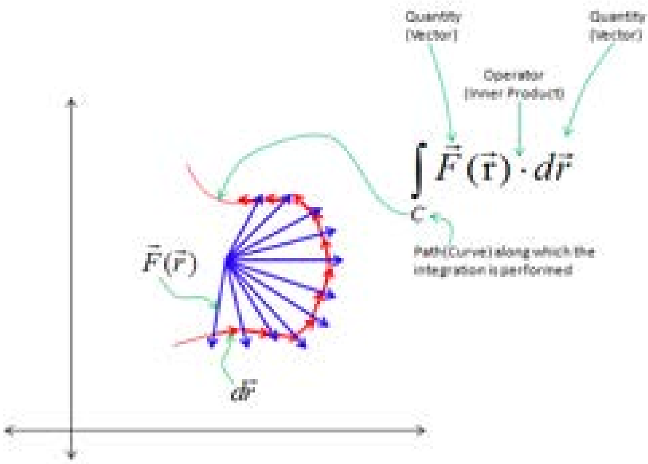


Figure 5.7: Depiction of Line Integral for a force field $\mathbf{F}$.

■ **Example 5.26** Given the force $\mathbf{F} = (x^2, -xy)$ find the work done by $\mathbf{F}$ along the paths between $(0,0)$ and $(2,1)$: 1. Line, 2. parabola, 3. broken line (vertical up then right), and 4. $x = 2t^3$ and $y = t^2$.

Solution: Since $\mathbf{r} = (x,y)$ and $dr = (dx,dy)$, then $\mathbf{F} \cdot d\mathbf{r} = x^2dx - xydy$ and the total work is

$$W = \int x^2 dx - xydy.$$

Thus, the work along path 1 (the line $y = \frac{1}{2}x \Rightarrow dy = \frac{1}{2}dx$) gives

$$W_1 = \int_0^2 \left[x^2 dx - \frac{1}{4}x^2 dx \right] = \int_0^2 \frac{3}{4}x^2 dx = \frac{1}{4}x^3 \Big|_0^2 = 2.$$

Thus, the work along path 2 (the parabola $y = \frac{1}{4}x^2 \Rightarrow dy = \frac{1}{2}xdx$) gives

$$W_2 = \int_0^2 \left[x^2 dx - \frac{1}{8}x^4 dx \right] = \frac{1}{3}x^3 - \frac{1}{40}x^5 \Big|_0^2 = \frac{8}{3} - \frac{32}{40} = \frac{28}{15}.$$

Thus, the work along path 3 (the broken line, up: $dx = 0, x = 0$, right: $dy = 0, y = 1$) gives

$$W_3 = \int_0^1 0 - 0dy + \int_0^2 x^2 dx + 0 = \frac{1}{3}x^3 \Big|_0^2 = \frac{8}{3}.$$

Thus, the work along path 4 ($x = 2t^3 \Rightarrow dx = 6t^2dt$ and $y = t^2 \Rightarrow dy = 2tdt$ for $0 \le t \le 1$) gives

$$W_4 = \int_0^1 (2t^3)^2(6t^2dt) - 2t^5(2tdt) = \int_0^1 \left[24t^8 - 4t^6\right] dt = \frac{24}{9}t^9 - \frac{4}{7}t^6 \Big|_0^1 = \frac{43}{21}.$$

Observe that the most work is required to move the object is along the broken path and the least work is along the parabola path 2 favoring x. ■

R An analogous procedure can be carried out in $3D$ to compute the work

$$W = \int \mathbf{F} \cdot d\mathbf{r} = \int F_x dx + \int F_y dy + \int F_z dz. \tag{5.34}$$

■ **Example 5.27** *(Path Dependent Work)* The force exerted on a body $\mathbf{F} = (-y, x)$. Find the work required to move an object from $(0,0)$ to $(1,1)$ moving along the broken line right then up.

$$W = \int_{(0,0)}^{(1,1)} \mathbf{F} \cdot d\mathbf{r} = \int_{(0,0)}^{(1,1)} (-ydx + xdy) = -\int_0^1 ydx + \int_0^1 xdy = 0 + \int_0^1 dy = y\Big|_0^1 = 1,$$

or along the broken line up then right

$$W = \int_{(0,0)}^{(1,1)} \mathbf{F} \cdot d\mathbf{r} = \int_{(0,0)}^{(1,1)} (-ydx + xdy) = -\int_0^1 ydx + \int_0^1 xdy = -\int_0^1 dx + 0 = -x\Big|_0^1 = -1.$$

The amount of work required depends on the choice of path! ■

■ **Example 5.28** *(Line Integral for Work)* Find the work done on the unit circle clockwise from 0 to $-\pi$ for the force $\mathbf{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$.

Solution: Parameterize the circle: $x = \cos(\varphi), y = \sin(\varphi)$, then $dx = -\sin(\varphi)d\varphi, dy = \cos(\varphi)d\varphi$ and $\mathbf{F} = (-\sin(\varphi), \cos(\varphi))$

$$W = -\int_C \frac{xdy - ydx}{x^2+y^2} = \int_0^{-\pi} (-\sin^2(\varphi) - \cos^2(\varphi))d\varphi = -\int_0^{-\pi} d\varphi = -\varphi\Big|_0^{-\pi} = \pi.$$

Now using the same force compute the integral around the square from $(1,0) \to (1,-1) \to (-1,-1) \to (-1,0)$.

$$\begin{aligned}
W &= -\int \mathbf{F}\cdot d\mathbf{r} = -\int_0^{-1} F_y dy\Big|_{x=1} - \int_1^{-1} F_x dx\Big|_{y=-1} - \int_{-1}^{0} F_y dy\Big|_{x=-1} \\
&= -\int_0^{-1} \frac{1}{1+y^2}dy - \int_1^{-1} \frac{1}{x^2+(-1)^2}dx - \int_{-1}^{0} \frac{-1}{(-1)^2+y^2}dy \\
&= -int_{-1}^{0} \frac{1}{1+y^2}dy + \int_{-1}^{1} \frac{1}{x^2+1}dx + \int_{-1}^{0} \frac{1}{1+y^2}dy \\
&= \tan^{-1}(y)\Big|_{-1}^{0} + \tan^{-1}(x)\Big|_{-1}^{1} + \tan^{-1}(y)\Big|_{-1}^{0} = \tan^{-1}(1) + \tan^{-1}(1) - \tan^{-1}(-1) - \tan^{-1}(-1) = 4\frac{\pi}{4} = \pi.
\end{aligned}$$

This is the same result as the circular path!! ■

Question: What is special about this force field $\mathbf{F}$ that allows the work to be the same regardless of path?

Solution: Conservative vector fields!! In example 1 the answer depended on the path and in the second example it was independent of the path. For example 1 consider a lady with a box on a truck. She performed two paths: 1. Drag the box right then lift to desired point. 2. Lift immediately then carry to desired point. The only work done on the second path is lifting.

In physics, if there is friction, then the work depends on the path. This is "non-conservative" where energy is dissipated by friction. In the second example the only work was lifting, this is "conservative" where no energy is dissipated due to friction.

R Thus, to be path-independent the force field must be curl free, $\nabla \times \mathbf{F} = \text{curl}(\mathbf{F}) = \mathbf{0}$.

■ **Example 5.29** Let the electric field $\mathbf{E} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$. Is this field conservative and therefore any line integral would be path independent?

Solution: Compute

$$\text{curl}(\mathbf{E}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix} = \left(0, 0, \frac{x^2+y^2-x(2x)}{(x^2+y^2)^2} - \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2}\right) = (0,0,0).$$

Yes, the electric field is conservative! ■

Consider another case. Suppose that there exists a function $W(x,y,z)$ such that $\mathbf{F} = \nabla W$. This implies that

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}, \frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y}$$

and

$$\text{curl}(\mathbf{F}) = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) = \mathbf{0}.$$

Then the quantity

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B \nabla W \cdot d\mathbf{r} = \int_A^B \frac{\partial W}{\partial x} dx + \frac{\partial W}{\partial y} dy + \frac{\partial W}{\partial z} dz = \int_A^B dW = W(B) - W(A).$$

In words this says that when a vector field is conservative, then the total work done is the work at the end point minus the work at the starting point *independent of the path*. This is referred to as **The Fundamental Theorem of Line Integrals**.

R In the previous explanation we used the exact differential of W, $dW = \frac{\partial W}{\partial x} dx + \frac{\partial W}{\partial y} dy + \frac{\partial W}{\partial z} dz$. In other words, the exact differential exists if and only if the curl of the force field $\mathbf{F}$ is zero, $\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \mathbf{0}$.

■ **Example 5.30** Is $\mathbf{F} = (yze^{xz}, e^{xz}, xye^{xz})$ an exact differential?

Solution: First computing the curl

$$0 = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yze^{xz} & e^{xz} & xye^{xz} \end{vmatrix} = \hat{\mathbf{i}}(xe^{xz} - xe^{xz}) + \hat{\mathbf{j}}(ye^{xz} + xyze^{xz} - ye^{xz} - xyze^{xz}) + \hat{\mathbf{k}}(ze^{xz} - ze^{xz}) = \mathbf{0}.$$

Yes! This force field is an exact differential. ■

R In two-dimensions we only need that $\frac{\partial W}{\partial x \partial y} = \frac{\partial W}{\partial y \partial x}$. So if $\mathbf{F} = \left(\frac{\partial W}{\partial x}, \frac{\partial W}{\partial y}, 0 \right)$, then $\mathbf{F}$ is an exact differential. For example $\mathbf{F} = (e^x \sin(y), e^x \cos(y))$ is conservative.

5.7.1 Potentials

In mechanics, if $\mathbf{F} = \nabla W$ (conservative), then W is the work done by the force $\mathbf{F}$. If a mass falls a distance z, then the work done is $W = mgz$. If a mass is lifted a distance z, then the work done is $W = -mgz$ (direction opposite the force of gravity). The total increase in potential energy when lifting the object is $\phi = mgz$ implying that $\phi = -W$. Thus, the force $\mathbf{F} = -\nabla \phi$ where ϕ is the potential energy or scalar potential function.

R In general, if $\text{curl}(\mathbf{v}) = 0$, then there exists a scalar function ϕ such that $\mathbf{v} = -\nabla \phi$. One special case where the sign is opposite is hydrodynamics where $\mathbf{v} = \nabla \phi$, but we ignore this case for now.

Now, suppose that the $\text{curl}(\mathbf{F}) = \mathbf{0} \Rightarrow \mathbf{F} = \nabla W$. **Questions:** How can we find the function W?

Solution: We calculate the line integral from A to B along a convenient path (since the integral is path independent).

■ **Example 5.31** Show that the force $\mathbf{F} = (3 + 2xy, x^2 - 3y^2, 0)$ is conservative, then find the scalar potential ϕ such that $\mathbf{F} = -\nabla \phi$.

Solution: First, show the force field is conservative by computing the curl

$$0 = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3+2xy & x^2-3y^2 & 0 \end{vmatrix} = \hat{\mathbf{i}}(0-0) + \hat{\mathbf{j}}(0-0) + \hat{\mathbf{k}}(2x-2x) = \mathbf{0}.$$

So the force field is conservative!

Now consider the path in 3D from $(0,0,0) \to (x,0,0) \to (x,y,0) \to (x,y,z)$ and compute the work

$$\begin{aligned} W &= \int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B (3+2xy)dx + (x^2-3y^2)dy + 0dz \\ &= \text{Integral of Path 1 + Path 2 + Path 3} \\ &= \int_0^x 3dx + \int_0^y (x^2-3y^2)dy + \int_0^z dz \\ &= 3x + x^2y - y^3 + 0. \end{aligned}$$

Thus, $W = 3x + x^2y - y^3$ and $\phi = -W = -3x - x^2y + y^3$. ■

5.7.2 Alternate Approach to Finding Scalar Potential ϕ

■ **Example 5.32** Consider the force field $\mathbf{F} = (y^2, 2xy + e^{3z}, 3ye^{3z})$. Show that $\mathbf{F}$ is conservative and find the scalar potential ϕ such that $\mathbf{F} = -\nabla\phi$.

Solution: First, show the force field is conservative by computing the curl

$$0 = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy+e^{3z} & 3ye^{3z} \end{vmatrix} = \hat{\mathbf{i}}(3e^{3z} - 3e^{3z}) + \hat{\mathbf{j}}(0-0) + \hat{\mathbf{k}}(2y-2y) = \mathbf{0}.$$

So the force field is conservative!

Now we need to find W. We know that $\frac{\partial W}{\partial x} = F_x = y^2$. If we integrate in x we will get an expression for W, $W = xy^2 + g(y,z)$. Now take the y derivative to find $\frac{\partial W}{\partial y} = 2xy + g_y(y,z) = 2xy + e^{3z} = F_y$. This implies that $g_y(y,z) = e^{3z}$. Integrating this in y gives $g(y,z) = ye^{3z} + h(z)$.

Now we have that $W = xy^2 + ye^{3z} + h(z)$. Take a derivative in z to find $\frac{\partial W}{\partial z} = 3ye^{3z} + h'(z) = 3ye^{3z} = F_z$. Thus, $h'(z) = 0 \Rightarrow h(x) = C_1$ (a constant). Therefore, $W = xy^2 + ye^{3z} + C_1$ for any constant C_1 and $\phi = -W$. ■

5.8 Green's Theorem in the Plane

Recall the *Fundamental Theorem of Calculus*:

$$\int_a^b \frac{d}{dt} f(t)dt = f(b) - f(a). \tag{5.35}$$

We want to now generalize this idea to multiple dimensions in the form of the celebrated Divergence and Stokes Theorems in 3D. We start, however, with the 2D versions of these theorems known as Green's Theorem. The idea is to relate an area integral to the line integral around its boundary.

Let $P(x,y)$ and $Q(x,y)$ be continuous functions with continuous first derivatives. Let $x = a$ and $x = b$ be the left and right most x-coordinates of area A. Let y_u describe the upper curve and

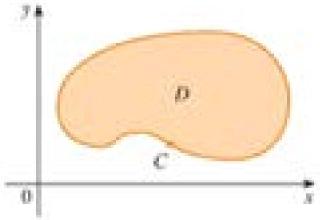


FIGURE 1

Figure 5.8: Positively oriented (counter-clockwise) curve around boundary. Figure from Stewart Calculus.

y_l describe the lower curve between a and b. We want to show that the integral over the area is equivalent to the line integral around the boundary

$$\iint_A \frac{\partial P}{\partial y} dA = \oint_C P ds. \tag{5.36}$$

Starting from the left-hand side:

$$\begin{aligned}\iint_A \frac{\partial P}{\partial y} dydx &= \int_a^b \left[\int_{y_l}^{y^u} \frac{\partial P(x,y)}{\partial y} dy\right] dx \\ &= \int_a^b \left[P(x,y_u) - P(x,y_l)\right] dx \\ &= -\int_b^a P(x,y_u)dx - \int_a^b P(x,y_l)dx = -\oint_C P ds.\end{aligned}$$

Repeating the calculation, but using functions of y instead of x gives: $\iint_A \frac{\partial q}{\partial x} dxdy = \oint_C Q dy$.

Theorem 5.8.1 *(Green's Theorem in the Plane (2D))* Let $P(x,y)$ and $Q(x,y)$ be continuous functions with continuous first derivatives defined on the area A, then

$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint_{\partial A} (Pdx + Qdy), \tag{5.37}$$

with the line integral oriented counter-clockwise around the boundary ∂A.

■ **Example 5.33** Evaluate $\oint x^4 dx + xydy$ where C is a triangle from with vertices $(0,0),(1,0),(0,1)$.

Solution: First note the region under consideration can be written as a Type I or Type II region (see Chapter on Multiple Integration). Also, given the vertices the triangle is bounded from above by the line $y = 1-x$ and below by $y = 0$ between $x = 0$ and $x = 1$. From the initial function we see that $P(x,y) = x^4$ and $Q(x,y) = xy$. Thus, using Green's Theorem

$$\begin{aligned}\oint x^4 dx + xydy &= \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_0^1 \int_0^{1-x} (y-0) dydx \\ &= \int_0^1 \frac{1}{2} y^2 \Big|_0^{1-x} dx = \int_0^1 \frac{1}{2}(1-x)^2 dx = -\frac{1}{6}(1-x)^3 \Big|_0^1 = \frac{1}{6}.\end{aligned}$$

■

■ **Example 5.34** Evaluate $\oint (3y - e^{\sin(x)})dx + (7x + \sqrt{y^4+1})dy$ where C is the ellipse defined by $x^2 + y^2 = 9$.

Solution: From the initial function we see that $P(x,y) = 3y - e^{\sin(x)}$ and $Q(x,y) = 7x + \sqrt{y^4+1}$. Thus, using Green's Theorem

$$\begin{aligned}\oint (3y - e^{\sin(x)})dx + (7x + \sqrt{y^4+1})dy &= \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \iint_A (7-3)dxdy \\ &= \iint_D 4dA =_{Polar} \int_0^{2\pi}\int_0^3 4rdrd\theta = \int_0^{2\pi} 2r^2\Big|_0^3 d\theta \\ &= \int_0^{2\pi} 18d\theta = 36\pi.\end{aligned}$$

■

(R) In the prior two examples it is easier to do the double integral than the line integral. Sometimes the line integral is easier, so remember to use the theorem both ways!

This theorem can be used to find the area A of some objects. Consider $A = \iint_A 1dA$. Then one can choose P, Q such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$. For example, a) $P = 0, Q = x$, b) $P = -y, Q = 0$, c) $P = -\frac{1}{2}y, Q = \frac{1}{2}x$. Then using Green's Theorem we derive *Green's Theorem for Areas*:

$$A = \oint_C xdy = -\oint_C ydx = \frac{1}{2}\oint_C xdy - ydx. \tag{5.38}$$

■ **Example 5.35** Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution: Using Green's Theorem for Areas: (Let $x = a\cos(t), y = b\sin(t), dx = -a\sin(t)dt, dy = b\cos(t)dt$)

$$\begin{aligned}A &= \frac{1}{2}\oint_C xdy - ydx = \frac{1}{2}\int_0^{2\pi} ab\cos^2(t)dt + ab\sin^2(t)dt \\ &= \frac{ab}{2}\int_0^{2\pi} dt = \pi ab.\end{aligned}$$

■

■ **Example 5.36** Let $\mathbf{F} = (x^2, -xy)$ and consider the area bounded from above by $y = 1$ and below by $y = \frac{1}{4}x^2$. Find the work done in moving around this curve. (From last section it is $W_2 - W_3 = \frac{-4}{5}$).

Solution: We can use Green's Theorem to compute the work!

$$\begin{aligned}W &= \oint_{\partial A} x^2dx - xydy = \iint_A \left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^2)\right] dxdy \\ &= \iint_A -ydxdy = \int_0^1\int_0^{2\sqrt{y}} -ydxdy = \int_0^1 -xy\Big|_0^{2\sqrt{y}} dy \\ &= \int_0^1 -2y^{3/2}dy = -\frac{4}{5}y^{5/2}\Big|_0^1 = -\frac{4}{5}.\end{aligned}$$

■

■ **Example 5.37** Let's examine conservative forces, $\nabla \times \mathbf{F} = \mathbf{0}$, or in 2D $\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0$. From Green's Theorem we compute the work

$$W = \oint_{\partial A} F_x dx + F_y dy = \iint_A \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dxdy = 0.$$

The work required to move an object around any closed path for a conservative force is zero! This results from the fact that the work in moving an object from one point to another is independent of the path. ■

■ **Example 5.38** Let $Q = V_x, P = -V_y$ where $\mathbf{V} = (V_x, V_y)$ Then $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = \text{div}(\mathbf{V})$ with $V_z = 0$.

The infinitesimal tangent at any point is $d\mathbf{r} = (dx, dy)$ and the normal is $\mathbf{n}ds = (dy, -dx)$ with $ds = \sqrt{dx^2 + dy^2}$. Thus, $Pdx + Qdy = -V_y dx + V_x dy = (V_x, V_y) \cdot (dy, -dx) = \mathbf{V} \cdot \mathbf{n}ds$. Therefore,

$$\text{2D Divergence Theorem} \qquad \iint_A \text{div}(\mathbf{V}) dA = \oint_{\partial A} \mathbf{V} \cdot \mathbf{n} ds \tag{5.39}$$

■

■ **Example 5.39** Let $Q = V_y, P = V_x$ where $\mathbf{V} = (V_x, V_y)$ Then $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} = \text{curl}(\mathbf{V}) \cdot \hat{\mathbf{k}}$ with $V_z = 0$.

Thus, $Pdx + Qdy = V_x dx + V_y dy = (V_x, V_y) \cdot (dx, dy) = \mathbf{V} \cdot d\mathbf{r}$. Therefore,

$$\text{2D Stokes Theorem} \qquad \iint_A \text{corl}(\mathbf{V}) \cdot \hat{\mathbf{k}} dA = \oint_{\partial A} \mathbf{V} \cdot d\mathbf{r} \tag{5.40}$$

■

Question: What about non-simple regions made up of many areas?

■ **Example 5.40** Evaluate $\oint_C y^2 dx + 3xy dy$ where C is the curve around the half annulus with outer radius 2 and inner radius 1.

Solution: Compute

$$\begin{aligned} \oint_C y^2 dx + 3xydy &= \iint_A \left[\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(y^2) \right] = \iint_A y dA = \int_0^\pi \int_1^2 r \sin(\theta) r dr d\theta \\ &= \int_0^\pi \frac{1}{3} r^3 \sin(\theta) \Big|_1^2 d\theta = \int_0^\pi \frac{7}{3} \sin(\theta) d\theta = -\frac{7}{3} \cos(\theta) \Big|_0^\pi = \frac{14}{3}. \end{aligned}$$

■

R What if the region under consideration has a hole? Consider the outer boundary curve C_1 and the inner hole has boundary C_2. Then the total line integral around the boundary

$$\oint_{C_1} Pdx + Qdy + \oint_{-C_2} Pdx + Qdy = \oint_{C_1} Pdx + Qdy - \oint_{C_2} Pdx + Qdy. \tag{5.41}$$

Here we just subtract the area inside the hole from the total!

■ **Example 5.41** Let $\mathbf{F} = (y^2, 3xy)$ be the force and compute the work done around two circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution: Compute

$$\oint_{C_1} y^2dx+3xydy-\oint_{C_2} y^2dx+3xydy = \int_0^{2\pi}\int_0^2 r^2\sin(\theta)drd\theta - \int_0^{2\pi}\int_0^1 r^2\sin(\theta)drd\theta$$
$$= \int_0^{2\pi}\int_1^2 r^2\sin(\theta)drd\theta = \int_0^{2\pi}\sin(\theta)d\theta\int_1^2 r^2dr$$
$$= -\cos(\theta)\Big|_0^{2\pi}\frac{1}{3}r^3\Big|_1^2 = 0.$$

■

5.9 The Divergence (Gauss) Theorem

Recall the divergence of a vector field $\mathbf{v}(x,y,z)$

$$\text{div}(\mathbf{v}) = \nabla\cdot\mathbf{v} = \frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}.$$

The idea behind the divergence theorem relies on fluid in a region R. Let $\mathbf{v}$ be the fluid velocity at a point inside R. The divergence is the amount of substance flowing out of a given volume. Consider the cross-sectional area A. The amount of water flowing through this region at time t over the area A' perpendicular to the flow is the water in a cylindrical cross-section of area A' and length vt for a total amount of $vtA'\rho$ where ρ is the density. For cross-section A (inclined a angle θ to $\mathbf{v}$) we have $vtA'\rho = vt\rho A\cos(\theta)$. This gives a unit area in unit time of $v\rho\cos(\theta) = v\rho\mathbf{n}$ where $\mathbf{n}$ is the unit normal.

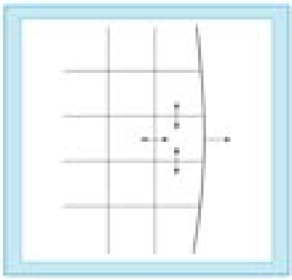


Figure 5.9: Depiction of fluid flow in and out of a region. Image from Weber Essential Math Methods for Physicists.

Imagine a volume V is subdivided into parallelepipeds. For each parallelepipeds

$$\sum_{six faces}\mathbf{v}\cdot dA = \nabla\cdot\mathbf{v}dV.$$

Adding up this quantity for all parallelepipeds results in just the divergence at the boundary (the amount flowing in/out of interior boundaries cancels due to same magnitude in opposite directions)

$$\sum_{all parallelepipeds}\mathbf{v}\cdot dA = \sum_{exterior surface}\mathbf{V}\cdot dA = \sum_{Volumes}\nabla\cdot\mathbf{v}dV.$$

Theorem 5.9.1 *(Gauss (Divergence) Theorem)* Given a vector field $\mathbf{v}$ we have the following relation between the volume and surface integrals

$$\iint_A \mathbf{v}\cdot dA = \iiint_V \nabla\cdot\mathbf{v}dV, \qquad \iint_A \mathbf{v}\cdot\mathbf{n}dA + \iiint_V \nabla\cdot\mathbf{v}dV. \tag{5.42}$$

Here V is the volume, A is the associated surface area, and $\mathbf{n}$ is the unit normal to the surface. This holds for simple solid regions with no holes.

■ **Example 5.42** Let $\mathbf{B} = \nabla \times \mathbf{A}$. Show that $\iint_S \mathbf{B} \cdot dA = 0$.

Solution: Compute

$$\iint_S \nabla \times \mathbf{A} =_{Div.Thm} \int_V \nabla \cdot (\nabla \times \mathbf{A}) dV = \int_V 0 dV = 0.$$

■

■ **Example 5.43** Over a volume V, let ψ solve Laplace's equation ($\Delta\psi = 0$). Show that the integral over the closed surface in V of the normal derivative of ψ ($\frac{\partial \psi}{\partial n} = \nabla\psi \cdot \mathbf{n}$) is zero

$$\int_S \frac{\partial \psi}{\partial n} dA = \int_S \nabla\psi \cdot \mathbf{n} dA =_{Div.Thm}= \int_V \nabla \cdot (\nabla\psi) dV = \int_V \Delta\psi dV = 0.$$

■

Definition 5.9.1 The flux of a vector field through the surface an object is described by the divergence

$$\iint_S \mathbf{F} \cdot dA = \iiint_V \nabla \cdot \mathbf{F} dV. \tag{5.43}$$

■ **Example 5.44** Find the flux of the vector field $\mathbf{F}(x,y,z) = (z,y,x)$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution: First compute the divergence $\nabla \cdot \mathbf{F} = 1$. Then the flux

$$\text{Flux} = \iint_S \mathbf{F} \cdot dA = \iiint_V \nabla \cdot \mathbf{F} dV = \iiint_V 1 dV = \text{Vol}(V) = \frac{4\pi}{3}.$$

■

■ **Example 5.45** Find the flux when $\mathbf{F} = (xy, y^2 + e^{xz^2}, \sin(xy))$, through the surface bounded by the parabolic cylinder $z = 1 - x^2$ and planes $z = 0, y = 0, y + z = 2$.

Solution: First compute the divergence $\nabla \cdot \mathbf{F} = y + 2y + 0 = 3y$. Then the flux

$$\begin{aligned}
\text{Flux} &= \iint_S \mathbf{F} \cdot dA =_{DivThm} \iiint_V \nabla \cdot \mathbf{F} dV = \int_{-1}^{1} \int_0^{1-x^2} \int_0^{2-z} 3y dy dz dx \\
&= 3 \int_{-1}^{1} \int_0^{1-x^2} \frac{1}{2} y^2 \Big|_0^{2-z} dz dx = \frac{3}{2} \int_{-1}^{1} \int_0^{1-x^2} (2-z)^2 dz dx \\
&= \frac{3}{2} \int_{-1}^{1} -\frac{1}{3}(2-z)^3 \Big|_0^{1-x^2} dx = -\frac{1}{2} \int_{-1}^{1} (1+x^2)^3 - 8 dx = -\frac{1}{2} \int_{-1}^{1} (x^6 + 3x^4 + 3x^2 + 1) - 8 dx \\
&= -\frac{1}{2} \int_{-1}^{1} x^6 + 3x^4 + 3x^2 - 7 dx = -\frac{1}{2} \left[\frac{1}{7}x^7 + \frac{3}{5}x^5 + x^3 - 7x \Big|_{-1}^{1} \right] = \frac{184}{35}.
\end{aligned}$$

■

(R) The divergence theorem is also true for unions of simple regions!

5.9.1 Gauss Law for Electricity

Recall Coulomb's Law $\mathbf{E} = \frac{q}{4\pi\varepsilon_0 r^2}\mathbf{e}_r$ where ε_0 is the permitivity of free space derived from the Electric displacement $\mathbf{D} = \varepsilon_0\mathbf{E} = \frac{q}{4\pi r^2}\mathbf{e}_r$. Let S be a closed surface surrounding a point close to a charge q at the origin. Then

$$\oint_S \mathbf{D}\cdot\mathbf{n}dA = \frac{q}{4\pi}\int_{AreaofCircle} 1d\Omega.$$

Definition 5.9.2 Gauss Law: The total charge inside a closed surface

$$\oint \mathbf{D}\cdot\mathbf{n}dA = \iiint_V \rho dV$$

where ρ is the charge density / charge distribution

$$\oint_S \mathbf{D}\cdot\mathbf{n}dA = \sum_i \oint_{S_i} \mathbf{D}\cdot\mathbf{n}dA = \sum_i q_i, \tag{5.44}$$

where the total charge over the region S is the sum of the isolated charges. Using the divergence theorem we find

$$\iint_S \nabla\cdot\mathbf{D}dV = \int \rho dV, \tag{5.45}$$

which is Maxwell's equation, $\nabla\cdot\mathbf{D} = \rho$, in non-local form.

■ **Example 5.46** Let $\mathbf{D} = \frac{q}{4\pi r^2}\mathbf{e}_r$ where $\mathbf{e}_r = \frac{\mathbf{r}}{|\mathbf{r}|}$. Show that the electric flux through any closed surface surrounding the origin is

$$\iint_S \mathbf{D}\cdot dA = q.$$

Solution: Let $\tilde{S}$ be a small sphere of radius a around the origin. Then

$$\begin{aligned}\iint_S \mathbf{D}\cdot dA &= \iint_{\tilde{S}} \mathbf{D}\cdot dA = \iint_{\tilde{S}} \mathbf{D}\cdot\mathbf{n}dA \\ &= \iint_{\tilde{S}} \frac{q}{4\pi|\mathbf{r}|^3}\mathbf{r}\cdot\frac{\mathbf{r}}{|\mathbf{r}|}dA = \iint_{\tilde{S}} \frac{q}{4\pi|\mathbf{r}|^2}dA \\ &= \iint_{\tilde{S}} \frac{q}{4\pi a^2}dA = \frac{q}{4\pi a^2}\text{Area}(\tilde{S}) = \frac{q}{4\pi a^2}4\pi a^2 = q.\end{aligned}$$

■

5.10 The Stokes (Curl) Theorem

Recall the definition of the curl of a vector field $\mathbf{v}$:

$$\text{curl}(\mathbf{v}) = \nabla\times\mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}.$$

For a rigid body we have seen that the translational velocity $\mathbf{v} = \omega\times\mathbf{r}$ where ω is the angular velocity. Then

$$\begin{aligned}\text{curl}(\mathbf{v}) &= \nabla\times(\omega\times\mathbf{r}) =_{BAC-CAB} (\nabla\cdot\mathbf{r})\omega - (\omega\cdot\nabla)\mathbf{r} \\ &= 3\omega - \omega = 2\omega.\end{aligned}$$

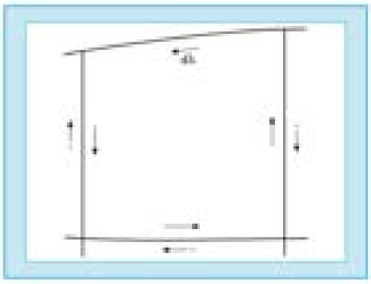


Figure 5.10: Depiction of local cells from Weber Essential Math Methods.

Thus, the angular velocity $\omega = \frac{1}{2}(\nabla \times \mathbf{v})$ (sometimes $\text{curl}(\mathbf{v})$ is denoted $\text{rot}(\mathbf{v})$. Also, recall that if a vector field is conservative, then it is curl free and called *irrotational*.

Consider four basic flow patterns: a) Vortex, b) Parallel, c) Parallel with variable velocity (shear), d) flow around a corner. Then we can determine the degree to which the fluid is rotating by computing the *circulation*.

Definition 5.10.1 *(Circulation)* The circulation of a fluid with fluid velocity V through the surface S is

$$\oint_S \mathbf{v} \cdot d\mathbf{r} \tag{5.46}$$

In flow a) $\nabla \times \mathbf{v} \neq 0$ at the center, b) $\nabla \times \mathbf{v} = 0$, c) $\nabla \times \mathbf{v} \neq 0$, d) $\nabla \times \mathbf{v} = 0$.

Consider the circulation around the sub-cells of a surface

$$\sum_{4 sides of cell} \mathbf{v} \cdot d\mathbf{r} = \sum_{exterior} \mathbf{v} \cdot d\mathbf{r} = \sum_{rectangles} \nabla \times \mathbf{v} \cdot dA.$$

Taking the limit of both sides gives Stokes Theorem

Theorem 5.10.1 *(Stokes (Curl) Theorem)* For a vector field $\mathbf{v}$ the following relation holds for a surface S with boundary C

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{v}) \cdot \mathbf{n} dA. \tag{5.47}$$

■ **Example 5.47** Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = (-y^2, x, z^2)$ and C is the intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$.

Solution: Compute

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &=_{Stokes} \iint_S \text{curl}(\mathbf{F}) \cdot dS = \iint_S (1+2y) dA \\
&= \int_0^{2\pi} \int_0^1 (1_2 r \sin(\theta)) r dr d\theta = \int_0^{2\pi} \frac{1}{2} r^2 + \frac{2}{3} r^3 \sin(\theta) \Big|_0^1 d\theta \\
&= \int_0^{2\pi} \frac{1}{2} + \frac{2}{3} \sin(\theta) d\theta = \frac{\theta}{2} - \frac{2}{3} \cos(\theta) \Big|_0^{2\pi} = \pi.
\end{aligned}$$

■

■ **Example 5.48** Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = (x + y^2, y + z^2, z + x^2)$ and C is the triangle with vertices $(1,0,0), (0,1,0), (0,0,1)$.

Solution: Compute

$$\oint_C \mathbf{F}\cdot d\mathbf{r} =_{Stokes} \iint_S \text{curl}(\mathbf{F})\cdot\mathbf{n} = \int_0^1\int_0^{1-x} -2y\,dy\,dx$$
$$= \int_0^1 -y^2\Big|_0^{1-x} dx = \int_0^1 -(1-x)^2 dx = \frac{1}{3}(1-x)^3\Big|_0^1 = -\frac{1}{3}.$$

■

5.10.1 Ampere's Law

Ampere's Law relates the current to the magnetic field

$$\oint_C \mathbf{H}\cdot d\mathbf{r} = I, \tag{5.48}$$

where I is the current, $\mathbf{H} = \frac{\mathbf{B}}{\mu_0}$, B is the magnetic field, and μ_0 is the constant permeability. Using Ampere's Law we find

$$I = \oint_C \mathbf{H}\cdot d\mathbf{r} = \int_0^{2\pi} |\mathbf{H}| r\,d\theta = |\mathbf{H}| r 2\pi \quad\Rightarrow\quad |\mathbf{H}| = \frac{I}{2\pi r}.$$

Using the current density $\mathbf{J}$ we can find the total current: $I = \iint_S \mathbf{J}\cdot\mathbf{n}\,dA$. Combining this with Ampere's Law gives

$$\iint_S \mathbf{J}\cdot\mathbf{n}\,dA = \oint_C \mathbf{H}\cdot d\mathbf{r} =_{Stokes} \iint_S (\nabla\times\mathbf{H})\cdot\mathbf{n}\,dA.$$

This implies that $\nabla\times\mathbf{H} = \mathbf{J}$, another one of Maxwell's equations.

5.10.2 Conservative Fields

Consider simply connected regions with no holes (a region is simply connected if any closed curve can be shrunk to a point without including points outside the region).

Theorem 5.10.2 Let the vector field $\mathbf{F}$ be continuous with continuous first partial derivatives in a simply connected region S. Then all the following statements are true or false:

a) $\text{curl}(\mathbf{F}) = \mathbf{0}$ at every point

b) $\oint \mathbf{F}\cdot d\mathbf{r} = 0$ around every simple closed curve in the region

c) $\mathbf{F}$ is conservative, and any work done is path independent

d) $\mathbf{F}\cdot d\mathbf{r}$ is an exact differential of a single valued function

e) $\mathbf{F} = -\nabla\phi$ where ϕ is a single-valued, scalar potential field.

To briefly review:
1. Irrotational ⇒ $\nabla\times\mathbf{v} = 0$ ⇒ there exists a scalar field ϕ such that $\mathbf{v} = -\nabla\phi$.
2. Solenoidal ⇒ $\nabla\cdot\mathbf{v} = 0$ ⇒ there exists a vector field $\mathbf{A}$ such that $\mathbf{v} = \nabla\times\mathbf{A}$.

■ **Example 5.49** Given a vector field $\mathbf{v} = (x^2 - yz, -2yz, z^2 - 2xz)$. Find $\mathbf{A}$ such that $\mathbf{v} = \nabla\times\mathbf{A}$.

Solution: First check that $\mathbf{v}$ is solenoidal: $\nabla\cdot\mathbf{v} = 2x - 2z + 2z - 2x = 0$. Yes!

Knowing the curl of **A** must be **v** we have infinitely many choices of **A**, we need to find one. Start by assuming $A_x = 0$. Then $-2yz == \frac{\partial A_z}{\partial x}$ and $z^2 - 2xz = \frac{\partial A_y}{\partial x}$. Integrating each with respect to x gives

$$\begin{aligned} A_y &= z^2x - x^2z + f_1(y,z) \\ A_z &= 2xyz + f_2(y,z). \end{aligned}$$

Now using the first component of **v**:

$$x^2 - yz = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = 2xz + \frac{\partial f_2}{\partial y} - 2xz + x^2 - \frac{\partial f_1}{\partial z}.$$

In particular, pick f_1, f_2 to satisfy this. Some examples are: 1. $f_1 = \frac{1}{2}yz^2, f_2 = 0$, 2. $f_1 = 0, f_2 = -\frac{1}{2}y^2z$. Using the latter we find $\mathbf{A} = (0, xz^2 - x^2z, 2xyz - \frac{1}{2}y^2z)$. ■

R If we have one **A**, then all others have the form $\mathbf{A} + \nabla u$ for any scalar function u. This is true since $\nabla \times (\mathbf{A} + \nabla u) = \nabla \times \mathbf{A} + \nabla \times (\nabla u) = \nabla \times \mathbf{A}$.

IV

Part Four: Ordinary Differential Equations

6. Ordinary Differential Equations

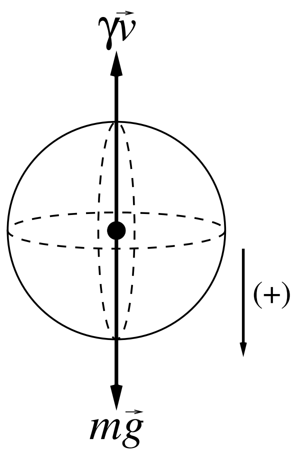
6.1 Introduction to ODEs

6.1.1 Some Basic Mathematical Models; Direction Fields

Definition 6.1.1 A **differential equation** is an equation containing derivatives.

Definition 6.1.2 A differential equation that describes some physical process is often called a **mathematical model**

■ **Example 6.1** *(Falling Object)*



Consider an object falling from the sky. From Newton's Second Law we have

$$F = ma = m\frac{dv}{dt} \tag{6.1}$$

When we consider the forces from the free body diagram we also have

$$F = mg - \gamma v \tag{6.2}$$

where γ is the **drag coefficient**. Combining the two

$$m\frac{dv}{dt} = mg - \gamma v \tag{6.3}$$

Suppose $m = 10kg$ and $\gamma = 2kg/s$. Then we have

$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \tag{6.4}$$

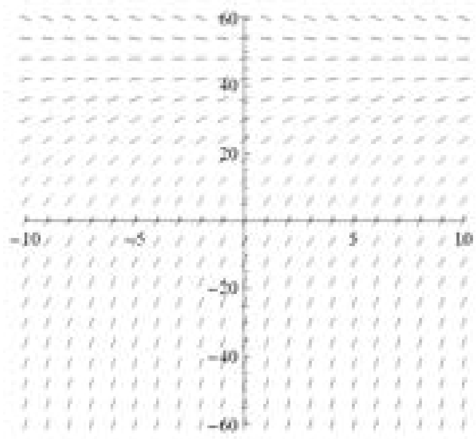


Figure 6.1: Direction field for above example

It looks like the direction field tends towards $v = 49m/s$. We plot the direction field by plugging in the values for v and t and letting dv/dt be the slope of a line at that point. ■

Direction Fields are valuable tools in studying the solutions of differential equations of the form

$$\frac{dy}{dt} = f(t, y) \tag{6.5}$$

where f is a given function of the two variables t and y, sometimes referred to as a **rate function**. At each point on the grid, a short line is drawn whose slope is the value of f at the point. This technique provides a good picture of the overall behavior of a solution.

Two Things to keep in mind:
1. In constructing a direction field we never have to solve the differential equation only evaluate it at points.

2. This method is useful if one has access to a computer because a computer can generate the plots well.

■ **Example 6.2** *(Population Growth)* Consider a population of field mice, assuming there is nothing to eat the field mice, the population will grow at a constant rate. Denote time by t (in months) and the mouse population by $p(t)$, then we can express the model as

$$\frac{dp}{dt} = rp \tag{6.6}$$

where the proportionality factor r is called the **rate constant** or **growth constant**. Now suppose owls are killing mice (15 per day), the model becomes

$$\frac{dp}{dt} = 0.5p - 450 \tag{6.7}$$

note that we subtract 450 rather than 15 because time was measured in months. In general

$$\frac{dp}{dt} = rp - k \tag{6.8}$$

where the growth rate is r and the predation rate k is unspecified. Note the equilibrium solution would be k/r.

Definition 6.1.3 The **equilibrium solution** is the value of $p(t)$ where the system no longer changes, $\frac{dp}{dt} = 0$.

In this example solutions above equilibrium will increase, while solutions below will decrease.

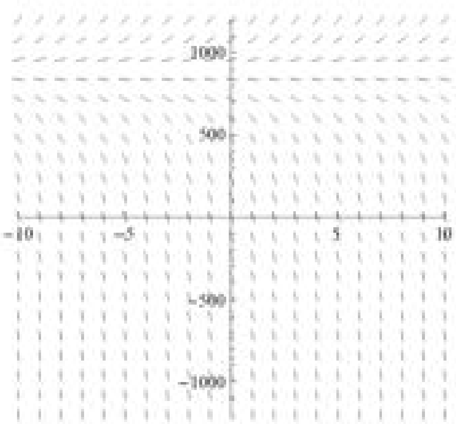


Figure 6.2: Direction field for above example

■

Steps to Constructing Mathematical Models:
1. Identify the independent and dependent variables and assign letters to represent them. Often the independent variable is time.
2. Choose the units of measurement for each variable.
3. Articulate the basic principle involved in the problem.
4. Express the principle in the variables chosen above.
5. Make sure each term has the same physical units.
6. We will be dealing with models in this chapter which are single differential equations.

■ **Example 6.3** Draw the direction field for the following, describe the behavior of y as $t \to \infty$. Describe the dependence on the initial value:

$$y' = 2y + 3 \tag{6.9}$$

Ans: For $y > -1.5$ the slopes are positive, and hence the solutions increase. For $y < -1.5$ the slopes are negative, and hence the solutions decrease. All solutions appear to diverge away from the equilibrium solution $y(t) = -1.5$. ■

■ **Example 6.4** Write down a DE of the form $dy/dt = ay + b$ whose solutions have the required behavior as $t \to \infty$. It must approach $\frac{2}{3}$.
Answer: For solutions to approach the equilibrium solution $y(t) = 2/3$, we must have $y' < 0$ for $y > 2/3$, and $y' > 0$ for $y < 2/3$. The required rates are satisfied by the DE $y' = 2 - 3y$. ■

■ **Example 6.5** Find the direction field for $y' = y(y-3)$

■

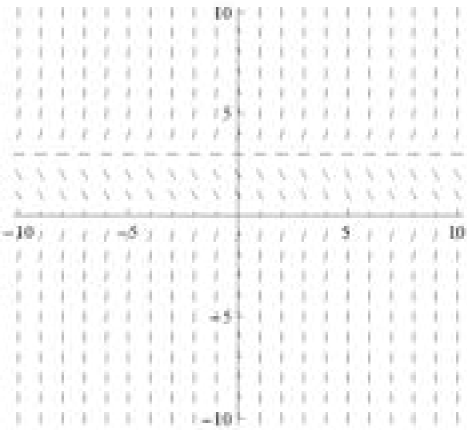


Figure 6.3: Direction field for above example

6.1.2 Solutions of Some Differential Equations

Last Time: We derived two formulas:

$$m\frac{dv}{dt} = mg - \gamma v \quad \text{(Falling Bodies)} \tag{6.10}$$

$$\frac{dp}{dt} = rp - k \quad \text{(Population Growth)} \tag{6.11}$$

Both equations have the form:

$$\frac{dy}{dt} = ay - b \tag{6.12}$$

■ **Example 6.6** *(Field Mice / Predator-Prey Model)*
Consider

$$\frac{dp}{dt} = 0.5p - 450 \tag{6.13}$$

we want to now solve this equation. Rewrite equation (6.13) as

$$\frac{dp}{dt} = \frac{p-900}{2}. \tag{6.14}$$

Note $p = 900$ is an equilbrium solution and the system does not change. If $p \neq 900$

$$\frac{dp/dt}{p-900} = \frac{1}{2} \tag{6.15}$$

By Chain Rule we can rewrite as

$$\frac{d}{dt}\left[\ln|p-900|\right] = \frac{1}{2} \tag{6.16}$$

So by integrating both sides we find

$$\ln|p-900| = \frac{t}{2} + C \tag{6.17}$$

Therefore,

$$p = 900 + Ce^{t/2} \tag{6.18}$$

Thus we have infinitely many solutions where a different arbitrary constant C produces a different solution. What if the initial population of mice was 850. How do we account for this? ■

Definition 6.1.4 The additional condition, $p(0) = 850$, that is used to determine C is an example of an **initial condition**.

Definition 6.1.5 The differential equation together with the initial condition form the **initial value problem**

Consider the general problem

$$\frac{dy}{dt} = ay - b \tag{6.19}$$

$$y(0) = y_0 \tag{6.20}$$

The solution has the form

$$y = (b/a) + [y_0 - (b/a)]e^{at} \tag{6.21}$$

when $a \neq 0$ this contains all possible solutions to the general equation and is thus called **the general solution** The geometric representation of the general solution is an infinite family of curves called **integral curves**.

■ **Example 6.7** *(Dropping a ball)* System under consideration:

$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \tag{6.22}$$

$$v(0) = 0 \tag{6.23}$$

From the formula above we have

$$v = \left(\frac{-9.8}{-1/5}\right) + \left[0 - \frac{-9.8}{-1/5}\right]e^{-\frac{t}{5}} \tag{6.24}$$

and the general solution is

$$v = 49 + Ce^{-t/5} \tag{6.25}$$

with the I.C. $C = -49$. ■

6.1.3 Classifications of Differential Equations

Last Time: We solved some basic differential equations, discussed IVPs, and defined the general solution.

Now we want to classify two main types of differential equations.

Definition 6.1.6 If the unknown function depends on a single independent variable where only ordinary derivatives appear, it is said to be an **ordinary differential equation**. Example

$$y'(x) = xy \tag{6.26}$$

Definition 6.1.7 If the unknown function depends on several variables, and the derivatives are partial derivatives it is said to be a **partial differential equation**.

One can also have a **system of differential equations**

$$dx/dt = ax - \alpha xy \quad (6.27)$$
$$dy/dt = -cy + \gamma xy \quad (6.28)$$

Note: Questions from this section are common on exams.

Definition 6.1.8 The **order** of a differential equation is the order of the highest derivative that appears in the equation.

Ex 1: $y''' + 2e^t y'' + yy' = 0$ has order 3.
Ex 2: $y^{(4)} + (y')^2 + 4y''' = 0$ has order 4. Look at derivatives not powers.

Another way to classify equations is whether they are linear or nonlinear:

Definition 6.1.9 A differential equation is said to be **linear** if $F(t, y, y', y'', ..., y^{(n)}) = 0$ is a linear function in the variables $y, y', y'', ..., y^{(n)}$. i.e. none of the terms are raised to a power or inside a sin or cos.

■ **Example 6.8** a) $y' + y = 2$
b) $y'' = 4y - 6$
c) $y^{(4)} + 3y' + \sin(t)y$ ■

Definition 6.1.10 An equation which is not linear is **nonlinear**.

■ **Example 6.9** a) $y' + t^4 y^2 = 0$
b) $y'' + \sin(y) = 0$
c) $y^{(4)} - \tan(y) + (y''')^3 = 0$ ■

■ **Example 6.10** $\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin(\theta) = 0.$

The above equation can be approximated by a linear equation if we let $\sin(\theta) = \theta$. This process is called **linearization**. ■

Definition 6.1.11 A **solution** of the ODE on the interval $\alpha < t < \beta$ is a function ϕ that satisfies

$$\phi^{(n)}(t) = f[t, \phi(t), ..., \phi^{(n-1)}(t)] \quad (6.29)$$

Common Questions:
1. (Existence) Does a solution exist? Not all Initial Value Problems (IVP) have solutions.

2. (Uniqueness) If a solution exists how many are there? There can be none, one or infinitely many solutions to an IVP.

3. How can we find the solution(s) if they exist? This is the key question in this course. We will develop many methods for solving differential equations the key will be to identify which method to use in which situation.

6.2 Separable Equations

In general, we want to solve first order differential equations of the form

$$\frac{dy}{dt} = f(t,y). \tag{6.30}$$

We begin with equations that have a special form referred to as **separable** equations of the form

$$\frac{dy}{dx} = f(y)g(x) \tag{6.31}$$

The General Solution Method:

$$\text{Step 1: (Separate)} \quad \frac{1}{f(y)}dy = g(x)dx \tag{6.32}$$

$$\text{Step 2: (Integrate)} \quad \int \frac{1}{f(y)}dy = \int g(x)dx \tag{6.33}$$

$$\text{Step 3: (Solve for } y\text{)} \quad F(y) = G(x) + c \tag{6.34}$$

Note only need a constant of integration of one side, could just combine the constants we get on each side. Also, we only solve for y if it is possible, if not leave in **implicit form**.

Definition 6.2.1 An **equilibrium solution** is the value of y which makes $dy/dx = 0$, y remains this constant forever.

■ **Example 6.11** *(Newton's Law of Cooling)* Consider the ODE, where E is a constant:

$$\frac{dB}{dt} = \kappa(E - B) \tag{6.35}$$

with initial condition (IC) $B(0) = B_0$. This is separable

$$\int \frac{dB}{E-B} = \int \kappa dt \tag{6.36}$$

$$-\ln|E-B| = \kappa t + c \tag{6.37}$$

$$E - B = e^{-\kappa t + c} = Ae^{-\kappa t} \tag{6.38}$$

$$B(t) = E - Ae^{-\kappa t} \tag{6.39}$$

$$B(0) = E - A \tag{6.40}$$

$$A = E - B_0 \tag{6.41}$$

$$B(t) = E - \frac{E - B_0}{e^{\kappa t}} \tag{6.42}$$

■

■ **Example 6.12**

$$\frac{dy}{dt} = 6y^2 x, \quad y(1) = \frac{1}{3}. \tag{6.43}$$

Separate and Solve:

$$\int \frac{dy}{y^2} = \int 6x dx \tag{6.44}$$
$$-\frac{1}{y} = 3x^2 + c \tag{6.45}$$
$$y(1) = 1/3 \tag{6.46}$$
$$-3 = 3(1) + c \Rightarrow c = -6 \tag{6.47}$$
$$-\frac{1}{y} = 3x^2 - 6 \tag{6.48}$$
$$y(x) = \frac{1}{6 - 3x^2} \tag{6.49}$$

What is the **interval of validity** for this solution? Problem when $6 - 3x^2 = 0$ or when $x = \pm\sqrt{2}$. So possible intervals of validity: $(-\infty, -\sqrt{2}), (-\sqrt{2}, \sqrt{2}), (\sqrt{2}, \infty)$. We want to choose the one containing the initial value for x, which is x = 1, so the interval of validity is $(-\sqrt{2}, \sqrt{2})$. ■

■ **Example 6.13**

$$y' = \frac{3x^2 + 2x - 4}{2y - 2}, \quad y(1) = 3 \tag{6.50}$$

There are no equilibrium solutions.

$$\int 2y - 2dy = \int 3x^2 + 2x - 4dx \tag{6.51}$$
$$y^2 - 2y = x^3 + x^2 - 4x + c \tag{6.52}$$
$$y(1) = 3 \Rightarrow c = 5 \tag{6.53}$$
$$y^2 - 2y + 1 = x^3 + x^2 - 4x + 6 \quad \text{(Complete the Square)} \tag{6.54}$$
$$(y-1)^2 = x^3 + x^2 - 4x + 6 \tag{6.55}$$
$$y(x) = 1 \pm \sqrt{x^3 + x^2 - 4x + 6} \tag{6.56}$$

There are two solutions we must choose the appropriate one. Use the IC to determine only the positive solution is correct.

$$y(x) = 1 + \sqrt{x^3 + x^2 - 4x + 6} \tag{6.57}$$

We need the terms under the square root to be positive, so the interval of validity is values of x where $x^3 + x^2 - 4x + 6 \geq 0$. Note $x = 1$ is in here so IC is in interval of validity. ■

■ **Example 6.14**

$$\frac{dy}{dx} = \frac{xy^3}{1 + x^2}, \quad y(0) = 1 \tag{6.58}$$

One equilibrium solution, $y(x) = 0$, which is not our case (since it does not meet the IC). So separate:

$$\begin{aligned} \int \frac{dy}{y^3} &= \int \frac{x}{1+x^2} dx && (6.59) \\ -\frac{1}{2y^2} &= \frac{1}{2}\ln(1+x^2)+c && (6.60) \\ y(0) &= 1 \Rightarrow c = -\frac{1}{2} && (6.61) \\ y^2 &= \frac{1}{1-\ln(1+x^2)} && (6.62) \\ y(x) &= \frac{1}{\sqrt{1-\ln(1+x^2)}} && (6.63) \end{aligned}$$

Determine the interval of validity. Need

$$\ln(1+x^2) < 1 \Rightarrow x^2 < e-1 \qquad (6.64)$$

So the interval of validity is $-\sqrt{e-1} < x < \sqrt{e-1}$. ■

■ **Example 6.15**

$$\frac{dy}{dx} = \frac{y-1}{x^2+1} \qquad (6.65)$$

The equilibrium solution is $y(x) = 1$ and our IC is $y(0) = 1$, so in this case the solution is the constant function $y(s) = 1$. ■

■ **Example 6.16**

$$\textit{(Review IBP)}\frac{dy}{dt} = e^{y-t}\sec(y)(1+t^2), \quad y(0) = 0 \qquad (6.66)$$

Separate by rewriting, and using Integration By Parts (IBP)

$$\begin{aligned} \frac{dy}{dt} &= \frac{e^y e^{-t}}{\cos(y)}(1+t^2) && (6.67) \\ \int e^{-y}\cos(y)dy &= \int e^{-t}(1+t^2)dt && (6.68) \\ \frac{e^{-y}}{2}(\sin(y)-\cos(y)) &= -e^{-t}(t^2+2t+3)+\frac{5}{2} && (6.69) \end{aligned}$$

Won't be able to find an explicit solution so leave in implicit form. In the implicit form it is difficult to find the interval of validity so we will stop here. ■

■ **Example 6.17** Solve the differential equation $xy' = y+1$.

Solution: Separate variables

$$\frac{y'}{y+1} = \frac{1}{x}$$

Write $y' = dy/dx$ and Rearrange $$\frac{dy}{y+1} = \frac{dx}{x}$$

Integrate both sides $$\int \frac{1}{y+1}dy = \int \frac{1}{x}dx$$

Simplify $$\ln(y+1) = \ln(x)+C$$

Let $C = \ln(a)$ for constant a $$\ln(y+1) = \ln(x)+\ln(a) = \ln(ax)$$

put both sides into the exponential function $$y+1 = ax.$$

Thus, we have a *family* of solution $y = ax - 1$. One curve for each value of the constant a. This is more commonly referred to as the **general solution**. Finding a **particular solution** means choosing a value of a so that one curve remains. ■

6.3 Linear First-Order Equations, Method of Integrating Factors

Consider the general equation

$$\frac{dy}{dt} + p(t)y = g(t) \tag{6.70}$$

We said in Chapter 1 if $p(t)$ and $g(t)$ are constants we can solve the equation explicitly. Unfortunately this is not the case when they are not constants. We need the method of **integrating factor** developed by Leibniz, who also invented calculus, where we multiply (6.70) by a certain function $\mu(t)$, chosen so the resulting equation is integrable. $\mu(t)$ is called the **integrating factor**. The challenge of this method is finding it.

Summary of Method:
1. Rewrite the equation as (MUST BE IN THIS FORM)

$$y' + ay = f \tag{6.71}$$

2. Find an integrating factor, which is any function

$$\mu(t) = e^{\int a(t)dt}. \tag{6.72}$$

3. Multiply both sides of (6.71) by the integrating factor.

$$\mu(t)y' + a\mu(t)y = f\mu(t) \tag{6.73}$$

4. Rewrite as a derivative

$$(\mu y)' = \mu f \tag{6.74}$$

5. Integrate both sides to obtain

$$\mu(t)y(t) = \int \mu(t)f(t)dt + C \tag{6.75}$$

and thus

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)f(t)dt + \frac{C}{\mu(t)} \tag{6.76}$$

Now lets see some examples:

■ **Example 6.18** Find the general solution of

$$y' = y + e^{-t} \tag{6.77}$$

Step 1:

$$y' - y = e^t \tag{6.78}$$

Step 2:

$$\mu(t) = e^{-\int 1dt} = e^{-t} \tag{6.79}$$

Step 3:

$$e^{-t}(y' - y) = e^{-2t} \tag{6.80}$$

Step 4:

$$(e^{-t}y)' = e^{-2t} \tag{6.81}$$

Step 5:

$$e^{-t}y = \int e^{-2t}dt = -\frac{1}{2}e^{-2t} + C \tag{6.82}$$

Solve for y

$$y(t) = -\frac{1}{2}e^{-t} + Ce^{t} \tag{6.83}$$

■

■ **Example 6.19** Find the general solution of

$$y' = y\sin t + 2te^{-\cos t} \tag{6.84}$$

and $y(0) = 1$.
Step 1:

$$y' - y\sin t = 2te^{-\cos t} \tag{6.85}$$

Step 2:

$$\mu(t) = e^{-\int \sin t dt} = e^{\cos t} \tag{6.86}$$

Step 3:

$$e^{\cos t}(y' - y\sin t) = 2t \tag{6.87}$$

Step 4:

$$(e^{\cos t}y)' = 2t \tag{6.88}$$

Step 5:

$$e^{\cos t}y = t^2 + C \tag{6.89}$$

So the general solution is:

$$y(t) = (t^2 + C)e^{-\cos t} \tag{6.90}$$

With IC

$$y(t) = (t^2 + e)e^{-\cos t} \tag{6.91}$$

■

■ **Example 6.20** Find General Solution to

$$y' = y\tan t + \sin t \tag{6.92}$$

with $y(0) = 2$. Note Integrating factor

$$\mu(t) = e^{-\int \tan dt} = e^{\ln(\cos t)} = \cos t \tag{6.93}$$

Final Answer

$$y(t) = -\frac{\cos t}{2} + \frac{5}{2\cos t} \tag{6.94}$$

■

■ **Example 6.21** Solve

$$2y' + ty = 2 \tag{6.95}$$

with $y(0) = 1$. Integrating Factor

$$\mu(t) = e^{t^2/4} \tag{6.96}$$

Final Answer

$$y(t) = e^{-t^2/4} \int_0^t e^{s^2/4} ds + e^{-t^2/4}. \tag{6.97}$$

■

6.3.1 REVIEW: Integration By Parts

This is the most important integration technique learned in Calculus 2. We will derive the method. Consider the product rule for two functions of t.

$$\frac{d}{dt}\Big[uv\Big] = u\frac{dv}{dt} + v\frac{du}{dt} \tag{6.98}$$

Integrate both sides from a to b

$$uv\Big|_a^b = \int_a^b u\frac{dv}{dt} + \int_a^b v\frac{du}{dt} \tag{6.99}$$

Rearrange the resulting terms

$$\int_a^b u\frac{dv}{dt} = uv\Big|_a^b - \int_a^b v\frac{du}{dt} \tag{6.100}$$

Practicing this many times will be helpful on the homework. Consider two examples.

■ **Example 6.22** Find the integral $\int_1^9 \ln(t)dt$. First define u, du, dv, and v.

$$u = \ln(t) \qquad dv = dt \tag{6.101}$$

$$du = \frac{1}{t}dt \qquad v = t \tag{6.102}$$

Thus

$$\int_1^9 \ln(t)dt = t\ln(t)\Big|_1^9 - \int_1^9 1dt \tag{6.103}$$
$$= 9\ln(9) - t\Big|_1^9 \tag{6.104}$$
$$= 9\ln(9) - 9 + 1 \tag{6.105}$$
$$= 9\ln(9) - 8 \tag{6.106}$$

■

■ **Example 6.23** Find the integral $\int e^x \cos(x)dx$. First define u, du, dv, and v.

$$u = \cos(x) \qquad dv = e^x dx \tag{6.107}$$
$$du = -\sin(x)dx \qquad v = e^x \tag{6.108}$$

Thus

$$\int e^x \cos(x)dx = e^x\cos(x) - \int e^x \sin(x)dx \tag{6.109}$$

Do Integration By Parts Again

$$u = \sin(x) \qquad dv = e^x dx \tag{6.110}$$
$$du = \cos(x)dx \qquad v = e^x \tag{6.111}$$

So

$$\int e^x \cos(x)dx = e^x\cos(x) - \int e^x \sin(x)dx \tag{6.112}$$
$$= e^x\cos(x) + e^x\sin(x) - \int e^x \cos(x)dx \tag{6.113}$$
$$2\int e^x \cos(x)dx = e^x\big(\cos(x) + \sin(x)\big) \tag{6.114}$$
$$\int e^x \cos(x)dx = \frac{1}{2}e^x\big(\cos(x) + \sin(x)\big) + C \tag{6.115}$$

■

Notice when we do not have limits of integration we need to include the arbitrary constant of integration C.

6.3.2 Modeling With First Order Equations

Last Time: We solved separable ODEs and now we want to look at some applications to real world situations

There are two key questions to keep in mind throughout this section:
1. How do we write a differential equation to model a given situation?
2. What can the solution tell us about that situation?

■ **Example 6.24** *(Radioactive Decay)*

$$\frac{dN}{dt} = -\lambda N(t), \tag{6.116}$$

where $N(t)$ is the number of atoms of a radioactive isotope and $\lambda > 0$ is the decay constant. The equation is separable, and if the initial data is $N(0) = N_0$, the solution is

$$N(t) = N_0 e^{-\lambda t}. \tag{6.117}$$

so we can see that radioactive decay is exponential. ■

■ **Example 6.25** *(Newton's Law of Cooling)* If we immerse a body in an environment with a constant temperature E, then if $B(t)$ is the temperature of the body we have

$$\frac{dB}{dt} = \kappa(E - B), \tag{6.118}$$

where $\kappa > 0$ is a constant related to the material of the body and how it conducts heat. This equation is separable. We solved it before with the initial condition $B(0) = B_0$ to get

$$B(t) = E - \frac{E - B_0}{e^{\kappa t}}. \tag{6.119}$$

■

Approaches to writing down a model describing a situation:
1. Remember the derivative is the **rate of change**. It's possible that the description of the problem tells us directly what the rate of change is. Newton's Law of Cooling tells us the rate of change of the body's temperature was proportional to the difference in temperature between the body and the environment. All we had to do was set the relevant terms equal.

2. There are also cases where we are not explicitly given the formula for the rate of change. But we may be able to use the physical description to define the rate of change and then set the derivative equal to that. Note: The derivative = increase - decrease. This type of thinking is only applicable to first order equations since higher order equations are not formulated as rate of change equals something.

3. We may just be adapting a known differential equation to a particular situation, i.e. Newton's Second Law $F = ma$. It is either a first or second order equation depending on if you define it for position for velocity. Combine all forces and plug in value for F to yield the differential equation. Used for falling bodies, harmonic motion, and pendulums.

4. The last possibility is to determine two different expressions for the same quantity and set the equal to derive a differential equation. Useful when discussing PDEs later in the course.

The first thing one must do when approaching a modeling problem is determining which of the four situations we are in. It is crucial to practice this identification now it will be useful on exams and later sections. Secondly, your differential equation should not depend on the initial condition. The IC only tells the starting position and should not effect how a system evolves.

Type I: (Interest)

Suppose there is a bank account that gives $r\%$ interest per year. If I withdraw a constant w dollars per month, what is the differential equation modeling this?

Ans: Let t be time in years, and denote the balance after t years as $B(t)$. $B'(t)$ is the rate of change of my account balance from year to year, so it will be the difference between the amount added and the amount withdrawn. The amount added is interest and the amount withdrawn is $12w$. Thus

$$B'(t) = \frac{r}{100}B(t) - 12w \tag{6.120}$$

This is a linear equation, so we can solve by integrating factor. Note: UNITS ARE IMPORTANT, w is withdrawn each month, but $12w$ is withdrawn per year.

■ **Example 6.26** Bill wants to take out a 25 year loan to buy a house. He knows that he can afford maximum monthly payments of \$400. If the going interest rate on housing loans is 4%, what is the largest loan Bill can take out so that he will be able to pay it off in time?

Ans: Measure time t in years. The amount Bill owes will be $B(t)$. We want $B(25) = 0$. The 4% interest rate will take the form of $.04B$ added. He can make payments of $12 \times 400 = 4800$ each year. So the IVP will be

$$B'(t) = .04B(t) - 4800, \quad B(25) = 0 \tag{6.121}$$

This is a linear equation in standard form, use integrating factor

$$B'(t) - .04B(t) = -4800 \tag{6.122}$$

$$\mu(t) = e^{\int -.04dt} = e^{-.04t} \tag{6.123}$$

$$(e^{-\frac{4}{100}t}B(t))' = -4800e^{-\frac{4}{100}t} \tag{6.124}$$

$$e^{-\frac{4}{100}t}B(t) = -4800\int e^{-\frac{4}{100}t}dt = 120000e^{-\frac{4}{100}t} + c \tag{6.125}$$

$$B(t) = 120000 + ce^{\frac{4}{100}t} \tag{6.126}$$

$$B(25) = 0 = 120000 + ce \Rightarrow c = -120000e^{-1} \tag{6.127}$$

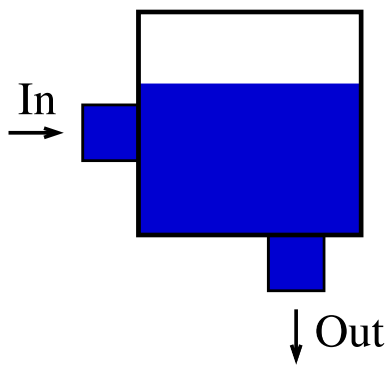
$$B(t) = 120000 - 120000e^{\frac{4}{100}(t-25)} \tag{6.128}$$

We want the size of the loan, which is the amount Bill begins with $B(0)$:

$$B(0) = 120000 - 120000e^{-1} = 120000(1 - e^{-1}) \tag{6.129}$$

■

Type II: (Mixing Problems)



We have a mixing tank containing some liquid inside. Contaminant is being added to the tank at some constant rate and the mixed solution is drained out at a (possibly different) rate. We will want to find the amount of contaminant in the tank at a given time.
How do we write the DE to model this process? Let $P(t)$ be the amount of pollutant (Note: Amount

of pollutant, not the concentration) in the tank at time t. We know the amount of pollutant that is entering and leaving the tank each unit of time. So we can use the second approach

$$\text{Rate of Change of} P(t) = \text{Rate of entry of contaminant} - \text{Rate of exit of contaminant} \quad (6.130)$$

The **rate of entry** can be defined in different ways. 1. Directly adding contaminant i.e. pipe adding food coloring to water. 2. We might be adding solution with a known concentration of contaminant to the tank (amount = concentration x volume).
What is the **rate of exit**? Suppose that we are draining the tank at a rate of r_{out}. The amount of contaminant leaving the tank will be the amount contained in the drained solution, that is given by rate x concentration. We know the rate, and we need the concentration. This will just be the concentration of the solution in the tank, which is in turn given by the amount of contaminant in the tank divided by the volume.

$$\text{Rate of exit of contaminant} = \text{Rate of drained solution} \times \frac{\text{Amount of Contaminant}}{\text{Volume of Tank}} \quad (6.131)$$

or

$$\text{Rate of exit of contaminant} = r_{out}\frac{P(t)}{V(t)}. \quad (6.132)$$

What is $V(t)$? The Volume is decreasing by r_{out} at each t. Is there anything being added to the volume? That depends if we are adding some solution to the tank at a certain rate r_{in}, that will add to the in-tank volume. If we directly add contaminant not in solution, nothing is added. So determine which situation by reading the problem. In the first case if the initial volume is V_0, we'll get $V(t) = V_0 + t(r_{in} - r_{out})$, and in the second, $V(t) = V_0 - tr_{out}$.

■ **Example 6.27** Suppose a 120 gallon well-mixed tank initially contains 90 lbs. of salt mixed with 90 gal. of water. Salt water (with a concentration of 2 lb/gal) comes into the tank at a rate of 4 gal/min. The solution flows out of the tank at a rate of 3 gal/min. How much salt is in the tank when it is full?

Ans: We can immediately write down the expression for volume $V(t)$. How much liquid is entering each minute? 4 gallons. How much is leaving the tank in the same minute? 3 gallons. So each minute the Volume increases by 1 gallon, and we have $V(t) = 90 + (4-3)t = 90 + t$. This tells us the tank will be full at $t = 30$.
We let $P(t)$ be the amount of salt (in pounds) in the tank at time t. Ultimately, we want to determine $P(30)$, since this is when the tank will be full. We need to determine the rates at which salt is entering and leaving the tank. How much salt is entering? 4 gallons of salt water enter the tank each minute, and each of those gallons has 2lb. of salt dissolved in it. Hence we are adding 8 lbs. of salt to the tank each minute. How much is exiting the tank? 3 gallons leave each minute, and the concentration in each of those gallons is $P(t)/V(t)$. Recall

$$\text{Rate of Change of} P(t) = \text{Rate of entry of contaminant} - \text{Rate of exit of contaminant} \quad (6.133)$$

$$\text{Rate of exit of contaminant} = \text{Rate of drained solution} \times \frac{\text{Amount of Contaminant}}{\text{Volume of Tank}} \quad (6.134)$$

$$\frac{dP}{dt} = (4gal/min)(2lb/gal) - (3gal/min)(\frac{P(t)lb}{V(t)gal}) = 8 - \frac{3P(t)}{90+t} \quad (6.135)$$

This is the ODE for the salt in the tank, what is the IC? P(0) = 90 as given by the problem. Now we have an IVP so solve (since linear) using integrating factor

$$\frac{dP}{dt}+\frac{3}{90+t}P(t) = 8 \tag{6.136}$$
$$\mu(t) = e^{\int \frac{3}{90+t}dt} = e^{3\ln(90+t)} = (90+t)^3 \tag{6.137}$$
$$((90+t)^3P(t))' = 8(90+t)^3 \tag{6.138}$$
$$(90+t)^3P(t) = \int 8(90+t)^3dt = 2(90+t)^4+c \tag{6.139}$$
$$P(t) = 2(90+t)+\frac{c}{(90+t)^3} \tag{6.140}$$
$$P(0) = 90 = 2(90)+\frac{c}{90^3} \Rightarrow c = -(90)^4 \tag{6.141}$$
$$P(t) = 2(90+t)-\frac{90^4}{(90+t)^3} \tag{6.142}$$

Remember we wanted $P(30)$ which is the amount of salt when the tank is full. So

$$P(30) = 240-\frac{90^4}{120^3} = 240-90(\frac{3}{4})^3 = 240-90(\frac{27}{64}). \tag{6.143}$$

We could ask for amount of salt at anytime before overflow and all would be the same besides last step where we replace 30 with the time wanted. ■

Exercise: What is the concentration of the tank when the tank is full?

■ **Example 6.28** A full 20 liter tank has 30 grams of yellow food coloring dissolved in it. If a yellow food coloring solution (with concentration of 2 grams/liter) is piped into the tank at a rate of 3 liters/minute while the well mixed solution is drained out of the tank at a rate of 3 liters/minute, what is the limiting concentration of yellow food coloring solution in the tank?

Ans: The ODE would be

$$\frac{dP}{dt} = (3L/min)(2g/L)-(3L/min)\frac{P(t)g}{V(t)L} = 6-\frac{3P}{20} \tag{6.144}$$

Note that volume is constant since we are adding and removing the same amount at each time step. Use the method of integrating factor.

$$\mu(t) = e^{\int \frac{3}{20}dt} = e^{\frac{3}{20}t} \tag{6.145}$$
$$(e^{\frac{3}{20}t}P(t))' = 6e^{\frac{3}{20}t} \tag{6.146}$$
$$e^{\frac{3}{20}t}P(t) = \int 6e^{\frac{3}{20}t}dt = 40e^{\frac{3}{20}t}+c \tag{6.147}$$
$$P(t) = 40+\frac{c}{e^{\frac{3}{20}t}} \tag{6.148}$$
$$P(0) = 20 = 40+c \Rightarrow c = -20 \tag{6.149}$$
$$P(t) = 40-\frac{20}{e^{\frac{3}{20}t}}. \tag{6.150}$$

Now what will happen to the concentration in the limit, or as $t \to \infty$. We know the volume will always be 20 liters.

$$\lim_{t\to\infty}\frac{P(t)}{V(t)} = \lim_{t\to\infty}\frac{40-20e^{-\frac{3}{20}t}}{20} = 2 \tag{6.151}$$

So the limiting concentration is $2g/L$. Why does this make physical sense? After a period of time the concentration of the mixture will be exactly the same as the concentration of the incoming solution. It turns out that the same process will work if the concentration of the incoming solution is variable. ■

■ **Example 6.29** A 150 gallon tank has 60 gallons of water with 5 pounds of salt dissolved in it. Water with a concentration of $2+\cos(t)$ lbs/gal comes into the tank at a rate of 9 gal/hr. If the well mixed solution leaves the tank at a rate of 6 gal/hour, how much salt is in the tank when it overflows?

Ans: The only difference is the incoming concentration is variable. Given the Volume starts at 600 gal and increases at a rate of 3 gal/min

$$\frac{dP}{dt} = 9(2+\cos(t)) - \frac{6P}{60+3t} \tag{6.152}$$

Our IC is $P(0) = 5$ and use the method of integrating factor

$$\begin{aligned}
\mu(t) &= e^{\int \frac{6}{60+3t}dt} = e^{2\ln(20+t)} = (20+t)^2. && (6.153)\\
((20+t)^2 P(t))' &= 9(2+\cos(t))(20+t)^2 && (6.154)\\
(20+t)^2 P(t) &= \int 9(2+\cos(t))(20+t)^2 dt && (6.155)\\
&= 9(\frac{2}{3}(20+t)^3 + (20+t)^2\sin(t) + 2(20+t)\cos(t) - 2\sin(t)) + c && (6.156)\\
P(t) &= 9(\frac{2}{3}(20+t) + \sin(t) + \frac{2\cos(t)}{20+t} - \frac{2\sin(t)}{(20+t)^2}) + \frac{c}{(20+t)^2} && (6.157)\\
P(0) &= 5 = 9(\frac{2}{3}(20) + \frac{2}{20}) + \frac{c}{400} = 120 + \frac{9}{10} + \frac{c}{400} && (6.158)\\
c &= -46360 && (6.159)
\end{aligned}$$

We want to know how much salt is in the tank when it overflows. This happens when the volume hits 150, or at $t = 30$.

$$P(30) = 300 + 9\sin(30) + \frac{18\cos(30)}{50} - \frac{18\sin(30)}{2500} - \frac{46360}{2500} \tag{6.160}$$

So $P(t) \approx 272.63$ pounds. ■

We could make the problem more complicated by assuming that there will be a change in the situation if the solution ever reached a critical concentration. The process would still be the same, we would just need to solve two different but limited IVPs.

Type III: (Falling Bodies)
Lets consider an object falling to the ground. This body will obey Newton's Second Law of Motion,

$$m\frac{dv}{dt} = F(t,v) \tag{6.161}$$

where m is the object's mass and F is the net force acting on the body. We will look at the situation where the only forces are air resistance and gravity. It is crucial to be careful with the signs. Throughout this course **downward displacements and forces are positive**. Hence the force due to gravity is given by $F_G = mg$, where $g \approx 10m/s^2$ is the gravitational constant.
Air Resistance acts against velocity. If the object is moving up air resistance works downward, always in opposite direction. We will assume air resistance is linearly dependant on velocity (ie

$F_A = \alpha v$, where F_A is the force due to air resistance). This is not realistic, but it simplifies the problem. So $F(t,v) = F_G + F_A = 10 - \alpha v$, and our ODE is

$$m\frac{dv}{dt} = 10m - \alpha v \tag{6.162}$$

■ **Example 6.30** A 50 kg object is shot from a cannon straight up with an initial velocity of 10 m/s off the very tip of a bridge. If the air resistance is given by $5v$, determine the velocity of the mass at any time t and compute the rock's terminal velocity.

Ans: Two parts: 1. When the object is moving upwards and 2. When the object is moving downwards. If we look at the forces it turns out we get the same DE

$$50v' = 500 - 5v \tag{6.163}$$

The IC is $v(0) = -10$, since we shot the object upwards. Our DE is linear and we can use integrating factor

$$v' + \frac{1}{10}v = 10 \tag{6.164}$$

$$\mu(t) = e^{\frac{t}{10}} \tag{6.165}$$

$$(e^{\frac{t}{10}}v(t))' = 10e^{\frac{t}{10}} \tag{6.166}$$

$$e^{\frac{t}{10}}v(t) = \int 10e^{\frac{t}{10}}\,dt = 100e^{\frac{t}{10}} + c \tag{6.167}$$

$$v(t) = 100 + \frac{c}{e^{\frac{t}{10}}} \tag{6.168}$$

$$v(0) = -10 = 100 + c \Rightarrow c = -110 \tag{6.169}$$

$$v(t) = 100 - \frac{110}{e^{\frac{t}{10}}}. \tag{6.170}$$

What is the terminal velocity of the rock? The terminal velocity is given by the limit of the velocity as $t \to \infty$, which is 100. We could also have computed the velocity of the rock when it hit the ground if we knew the height of the bridge (integrate to get position). ■

■ **Example 6.31** A 60kg skydiver jumps out of a plane with no initial velocity. Assuming the magnitude of air resistance is given by $0.8|v|$, what is the appropriate initial value problem modeling his velocity?

Ans: Air Resistance is an upward force, while gravity is acting downward. So our force should be

$$F(t,v) = mg - .8v \tag{6.171}$$

thus our IVP is

$$60v' = 60g - .8v, \quad v(0) = 0 \tag{6.172}$$

■

6.4 Existence and Uniqueness

Last Time: We developed 1st Order ODE models for physical systems and solved them using the methods of Integrating Factor and Separable Equations.

In Section 1.3 we noted three common questions we would be concerned with this semester.

1. (Existence) Given an IVP, does a solution exist?
2. (Uniqueness) If a solution exists, is it unique?
3. If a solution exists, how do we find it?

We have spent a lot of time on developing methods, now we will spend time on the first two questions. **Without Solving** an IVP, what information can we derive about the existence and uniqueness of solutions? Also we will note strong differences between linear and nonlinear equations.

6.4.1 Linear Equations

While we will focus on first order linear equations, the same basic ideas work for higher order linear equations.

Theorem 6.4.1 *(Fundamental Theorem of Existence and Uniqueness for Linear Equations)* Consider the IVP

$$y' + p(t)y = q(t), \quad y(t_0) = y_0. \tag{6.173}$$

If $p(t)$ and $q(t)$ are continuous functions on an open interval $\alpha < t_0 < \beta$, then there exists a unique solution to the IVP defined on the interval (α, β).

R The same result holds for general IVPs. If we have the IVP

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \ldots + a_1(t)y' + a_0(t)y = g(t), \quad y(t_0) = y_0, \ldots, y^{(n-1)}(t_0) = y_0^{(n-1)} \tag{6.174}$$

then if $a_i(t)$ (for $i = 0, \ldots, n-1$) and $g(t)$ are continuous on an open interval $\alpha < t_0 < \beta$, there exists a unique solution to the IVP defined on the interval (α, β).

What does Theorem 1 tell us?
(1) If the given linear differential equation is nice, not only do we know **EXACTLY ONE** solution exists. In most applications knowing a solution is unique is more important than knowing a solution exists.
(2) If the interval (α, β) is the largest interval on which $p(t)$ and $q(t)$ are continuous, then (α, β) is the interval of validity to the unique solution guaranteed by the theorem. Thus given a "nice" IVP there is no need to solve the equation to find the interval of validity. The interval only depends on t_0 since the interval must contain it, but does not depend on y_0.

■ **Example 6.32** Without solving, determine the interval of validity for the solution to the following IVP

$$(t^2 - 9)y' + 2y = \ln|20 - 4t|, \quad y(4) = -3 \tag{6.175}$$

Ans: If we look at Theorem 1, we need to write our equation in the form given in Theorem 1 (i.e. coefficient of y' is 1). So rewrite as

$$y' + \frac{2}{t^2 - 9} = \frac{\ln|20 - 4t|}{t^2 - 9} \tag{6.176}$$

Next we identify where either of the two other coefficients are discontinuous. By removing those points we find all intervals of validity. Then the last step is to identify which interval of validity contains t_0.

Using the notation in Theorem 1, $p(t)$ is discontinuous when $t = \pm 3$, since at those points we are dividing by zero. $q(t)$ is discontinuous at $t = 5$, since the natural log of 0 does not exists (only defined on $(0, \infty)$). This yields four intervals of validity where both $p(t)$ and $q(t)$ are continuous

$$(-\infty, -3), \quad (-3, 3), \quad (3, 5), \quad (5, \infty) \tag{6.177}$$

Notice the endpoints are where $p(t)$ and $q(t)$ are discontinuous, guaranteeing within each interval **both** are continuous. Now all that is left is to identify which interval contains $t_0 = 4$. Thus our interval of validity is $(3, 5)$. ■

R The other intervals of validity we found are intervals of validity for the same differential equation, but for different initial conditions. For example, if our IC was $y(2) = 5$ then the interval of validity must contain 2, so the answer would be $(-3, 3)$.

What happens if our IC is at one of the bad points where $p(t)$ and $q(t)$ are discontinuous? Unfortunately we are unable to conclude anything, since the theorem does not apply. On the other hand we cannot say that a solution does not exist just because the hypothesis are not met, so the bottom line is that we cannot conclude anything.

6.4.2 Nonlinear Equations

We saw in the linear case every "nice enough" equation has a unique solution except for if the initial conditions are ill-posed. But even this seemingly simple nonlinear equation

$$(\frac{dt}{dx})^2 + x^2 + 1 = 0 \tag{6.178}$$

has no real solutions.

So we have the following revision of Theorem 1 that applies to nonlinear equations as well. Since this is applied to a broader class the conclusions are expected to be weaker.

Theorem 6.4.2 Consider the IVP

$$y' = f(t, y), \quad y(t_0) = y_0. \tag{6.179}$$

If f and $\frac{\partial f}{\partial y}$ are continuous functions on some rectangle $\alpha < t_0 < \beta, \gamma < y_0 < \delta$ containing the point (t_0, y_0), then there is a unique solution to the IVP defined on some interval (a, b) satisfying $\alpha < a < t_0 < b \leq \beta$.

OBSERVATION:
(1) Unlike Theorem 1, Theorem 2 does not tell us the interval of a unique solution guaranteed by it. Instead, it tells us the largest possible interval that the solution will exist in, we would need to actually solve the IVP to get the interval of validity.
(2) For nonlinear differential equations, the value of y_0 may affect the interval of validity, as we will see in a later example. We want our IC to NOT lie on the boundary of a region where f or its partial derivative are discontinuous. Then we find the largest t-interval on the line $y = y_0$ containing t_0 where everything is continuous.

R Theorem 2 refers to **partial derivative** $\frac{\partial f}{\partial y}$ of the function of two variables $f(t, y)$. We will talk extensively about this later, but for now we treat t as a constant and take a normal derivative with respect to y. For example

$$f(t,y) = t^2 - 2y^3 t, \quad \text{then} \quad \frac{\partial f}{\partial y} = -6y^2 t. \tag{6.180}$$

■ **Example 6.33** Determine the largest possible interval of validity for the IVP

$$y' = x\ln(y), \quad y(2) = e \tag{6.181}$$

We have $f(x,y) = x\ln(y)$, so $\frac{\partial f}{\partial y} = \frac{x}{y}$. f is discontinuous when $y \leq 0$, and f_y (partial derivative with respect to y) is discontinuous when $y = 0$. Since our IC $y(2) = e > 0$ there is no problem since y_0 is never in the discontinuous region. Since there are no discontinuities involving x, then the rectangle is $-\infty < x_0 < \infty, 0 < y_0 < \infty$. Thus the theorem concludes that the unique solution exists somewhere inside $(-\infty, \infty)$. ■

R Note that this basically told us nothing, and nonlinear problems are quite harder to deal with than linear. What can happen if the conditions of Theorem 2 are NOT met?

■ **Example 6.34** Determine all possible solutions to the IVP

$$y' = y^{\frac{1}{3}}, \quad y(0) = 0. \tag{6.182}$$

First note this does not satisfy the conditions of the theorem, since $f_y = \frac{1}{3y^{2/3}}$ is not continuous at $y_0 = y = 0$. Now solve the equation it is separable. Notice the equilibrium solution is $y = 0$. This satisfies the IC, but let's solve the equation.

$$\int y^{-1/3} dy = \int dt \tag{6.183}$$

$$\frac{3}{2} y^{2/3} = t + c \tag{6.184}$$

$$y(0) = 0 \tag{6.185}$$

$$y(t) = \pm\left(\frac{2}{3}t\right)^{\frac{3}{2}} \tag{6.186}$$

(6.187)

The IC does not rule out either of these possibilities, so we end up with three possible solutions (these two and the equilibrium solution $y(t) \equiv 0$). ■

6.5 Other Methods for First-Order Equations

6.5.1 Autonomous Equations with Population Dynamics

First order differential equations relate the slope of a function to the values of the function and the independent variable. We can visualize this using direction fields. This in principle can be very complicated and it might be hard to determine which initial values correspond to which outcomes. However, there is a special class of equations, called **autonomous equations**, where this process is simplified. The first thing to note is autonomous equations do not depend on t

$$y' = f(y) \tag{6.188}$$

R Notice that all autonomous equations are separable.

What we need to know to study the equation qualitatively is which values of y make y' zero, positive, or negative. The values of y making $y' = 0$ are the **equilibrium solutions**. They are constant solutions and are indicated on the ty-plane by horizontal lines.

After we establish the equilibrium solutions we can study the positivity of $f(y)$ on the intermediate intervals, which will tell us whether the equilibrium solutions attract nearby initial conditions (in which case they are called **asymptotically stable**), repel them (**unstable**), or some combination of them (**semi-stable**).

■ **Example 6.35** Consider

$$y' = y^2 - y - 2 \tag{6.189}$$

Start by finding the equilibrium solutions, values of y such that $y' = 0$. In this case we need to solve $y^2 - y - 2 = (y-2)(y+1) = 0$. So the equilibrium solutions are $y = -1$ and $y = 2$. There are constant solutions and indicated by horizontal lines. We want to understand their stability. If we plot $y^2 - y - 2$ versus y, we can see that on the interval $(-\infty, -1), f(y) > 0$. On the interval $(-1, 2), f(y) < 0$ and on $(2, \infty), f(y) > 0$. Now consider the initial condition.

(1) If the IC $y(t_0) = y_0 < -1, y' = f(y) > 0$ and $y(t)$ will increase towards -1.
(2) If the IC $-1 < y_0 < 2, y' = f(y) < 0$, so the solution will decrease towards -1. Since the solutions below -1 go to -1 and the solutions above -1 go to -1, we conclude $y(t) = -1$ is an asymptotically stable equilibrium.
(3) If $y_0 > 2, y' = f(y) > 0$, so the solution increases away from 2. So at $y(t) = 2$ above and below solutions move away so this is an unstable equilibrium. ■

■ **Example 6.36** Consider

$$y' = (y-4)(y+1)^2 \tag{6.190}$$

The equilibrium solutions are $y = -1$ and $y = 4$. To classify them, we graph $f(y) = (y-4)(y+1)^2$.
(1) If $y < -1$, we can see that $f(y) < 0$, so solutions starting below -1 will tend towards $-\infty$.
(2) If $-1 < y_0 < 4, f(y) < 0$, so solutions starting here tend downwards to -1. So $y(t) = 1$ is semistable.
(3) If $y > 4, f(y) > 0$, solutions starting above 4 will asymptotically increase to ∞, so $y(t) = 4$ is unstable since no nearby solutions converge to it. ■

Populations

The best examples of autonomous equations come from population dynamics. The most naive model is the "Population Bomb" since it grows without any deaths

$$P'(t) = rP(t) \tag{6.191}$$

with $r > 0$. The solution to this differential equation is $P(t) = P_0 e^{rt}$, which indicates that the population would increase exponentially to ∞. This is not realistic at all.

A better and more accurate model is the "Logistic Model"

$$P'(t) = rP(1 - \frac{P}{N}) = rP - \frac{r}{N}P^2 \tag{6.192}$$

where $N > 0$ is some constant. With this model we have a birth rate of rP and a mortality rate of

$\frac{r}{N}P^2$. The equation is separable so let's solve it.

$$\frac{dP}{P(1-\frac{P}{N})} = rdt \tag{6.193}$$

$$\int (\frac{1}{P} + \frac{1/N}{1-P/N})dP = \int rdt \tag{6.194}$$

$$\ln|P| - \ln|1-\frac{P}{N}| = rt + c \tag{6.195}$$

$$\frac{P}{1-\frac{P}{N}} = Ae^{rt} \tag{6.196}$$

$$P = Ae^{rt} = \frac{1}{N}Ae^{rt}P \tag{6.197}$$

$$P(t) = \frac{Ae^{rt}}{1+\frac{A}{N}e^{rt}} = \frac{AN}{Ne^{-rt}+A} \tag{6.198}$$

if $P(0) = P_0$, then $A = \frac{P_0 N}{N-P_0}$ to yield

$$P(t) = \frac{P_0 N}{(N-P_0)e^{-rt}+P_0} \tag{6.199}$$

In its present form its hard to analyze what is going on so let's apply the methods from the first section to analyze the stability.

Looking at the logistic equation, we can see that our equilibrium solutions are $P = 0$ and $P = N$. Graphing $f(P) = rP(1-\frac{N}{P})$, we see that
(1) If $P < 0, f(P) < 0$
(2) If $0 < P < N, f(P) > 0$
(3) If $P > N, f(P) < 0$
Thus 0 is unstable while while N is asymptotically stable, so we can conclude for initial population $P_0 > 0$

$$\lim_{t\to\infty} P(t) = N \tag{6.200}$$

So what is N? It is the carrying capacity for the environment. If the population exists, it will grow towards N, but the closer it gets to N the slower the population will grow. If the population starts off greater then the carrying capacity for the environment $P_0 > N$, then the population will die off until it reaches that stable equilibrium position. And if the population starts off at N, the births and deaths will balance out perfectly and the population will remain exactly at $P_0 = N$.

Note: It is possible to construct similar models that have unstable equilibria above 0.

Exercise 6.1 Show that the equilibrium population $P(t) = N$ is unstable for the autonomous equation

$$P'(t) = rP(\frac{P}{N}-1). \tag{6.201}$$

■

6.5.2 Bernoulli Equations

The Bernoulli equation is a special first order equation of the form:

$$y' + Py = Qy^n, \tag{6.202}$$

where P, Q are functions of x. Even though the equation is not linear, we can reduce it to a linear equation with the following transformation: $z = y^{1-n}$ where by Chain Rule $z' = (1-n)y^{-n}y'$. Now multiply (6.202) by $(1-n)y^{-n}$ to find

$$\begin{aligned}(1-n)y^{-n}y' + (1-n)Py^{1-n} &= (1-n)Q \\ z' + (1-n)Pz &= (1-n)Q.\end{aligned}$$

We now have a first order linear equation that could be solved using the method of integrating factor.

6.5.3 Exact Equations

The final category of first order differential equations we will consider are **Exact Equations**. These nonlinear equations have the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{6.203}$$

where $y = y(x)$ is a function of x and find the

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{6.204}$$

where these two derivatives are **partial derivatives**.

Multivariable Differentiation

If we want a partial derivative of $f(x,y)$ with respect to x we treat y as a constant and differentiate normally with respect to x. On the other hand, if we want a partial derivative of $f(x,y)$ with respect to y we treat x as a constant and differentiate normally with respect to y.

■ **Example 6.37** Let $f(x,y) = x^2y = y^2$. Then

$$\frac{\partial f}{\partial x} = 2xy \tag{6.205}$$

$$\frac{\partial f}{\partial y} = x^2 + 2y. \tag{6.206}$$

■

■ **Example 6.38** Let $f(x,y) = y\sin(x)$

$$\frac{\partial f}{\partial x} = y\cos(y) \tag{6.207}$$

$$\frac{\partial f}{\partial y} = \sin(x) \tag{6.208}$$

■

We also need the crucial tool of the multivariable chain rule. If we have a function $\Phi(x, y(x))$ depending on some variable x and a function y depending on x, then

$$\frac{d\Phi}{dx} = \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{dy}{dx} = \Phi_x + \Phi_y y' \tag{6.209}$$

Solving Exact Equations

Start with an example to illustrate the method.

■ **Example 6.39** Consider

$$2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0 \tag{6.210}$$

The first step in solving an exact equation is to find a certain function $\Phi(x,y)$. Finding $\Phi(x,y)$ is most of the work. For this example it turns out

$$\Phi(x,y) = y^2 + (x^2+1)y - 3x^3 \tag{6.211}$$

Notice if we compute the partial derivatives of Φ, we obtain

$$\Phi_x(x,y) = 2xy - 9x^2 \tag{6.212}$$
$$\Phi_y(x,y) = 2y + x^2 + 1. \tag{6.213}$$

Looking back at the differential equation, we can rewrite it as

$$\Phi_x + \Phi_y \frac{dy}{dx} = 0. \tag{6.214}$$

Thinking back to the chain rule we can express as

$$\frac{d\Phi}{dx} = 0 \tag{6.215}$$

Thus if we integrate, $\Phi = c$, where c is a constant. So the general solution is

$$y^2 + (x^2+1)y - 3x^3 = c \tag{6.216}$$

for some constant c. If we had an initial condition, we could use it to find the particular solution to the initial value problem. ■

Let's investigate the last example further. An exact equation has the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{6.217}$$

with $M_y(x,y) = N_x(x,y)$. The key is to construct $\Phi(x,y)$ such that the DE turns into

$$\frac{d\Phi}{dx} = 0 \tag{6.218}$$

by using the multivariable chain rule. Thus we require $\Phi(x,y)$ satisfy

$$\Phi_x(x,y) = M(x,y) \tag{6.219}$$
$$\Phi_y(x,y) = N(x,y) \tag{6.220}$$

R A standard fact from multivariable calculus is that mixed partial derivatives commute. That is why we want $M_y = N_x$, so $M_y = \Phi_{xy}$ and $N_x = \Phi_{yx}$, and so these should be equal for Φ to exist. Make sure you check the function is exact before wasting time on the wrong solution process.

Once we have found Φ, then $\frac{d\Phi}{dx} = 0$, and so

$$\Phi(x,y) = c \tag{6.221}$$

yielding an implicit general solution to the differential equation.

So the majority of the work is computing $\Phi(x,y)$. How can we find this desired function, let's retry Example 3, filling in the details.

■ **Example 6.40** Solve the initial value problem

$$2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0, \quad y(0) = 2 \tag{6.222}$$

Let's begin by checking the equation is in fact exact.

$$M(x,y) = 2xy - 9x^2 \tag{6.223}$$

$$N(x,y) = 2y + x^2 + 1 \tag{6.224}$$

$$\tag{6.225}$$

Then $M_y = 2x = N_y$, so the equation is exact.

Now how do we find $\Phi(x,y)$? We have $\Phi_x = M$ and $\Phi_y = N$. Thus we could compute Φ in one of two ways

$$\Phi(x,y) = \int M dx \quad \text{or} \quad \Phi(x,y) = \int N dy. \tag{6.226}$$

In general it does not usually matter which you choose, one may be easier to integrate than the other. In this case

$$\Phi(x,y) = \int 2xy - 9x^2 dx = x^2y - 3x^3 + h(y). \tag{6.227}$$

Notice since we only integrate with respect to x we can have an arbitrary function only depending on y. If we differentiate $h(y)$ with respect to x we still get 0 like an arbitrary constant c. So in order to have the highest accuracy we take on an arbitrary function of y. Note if we integrated N with respect to y we would get an arbitrary function of x. DO NOT FORGET THIS!

Now all we need is to find $h(y)$. We know if we differentiate Φ with respect to x, then $h(y)$ will vanish which is unhelpful. So instead differentiate with respect to y, since $\Phi_y = N$ in order to be exact. so any terms in N that aren't in Φ_y must be $h'(y)$.

So $\Phi_y = x^2 + h'(y)$ and $N = x^2 + 2y + 1$. Since these are equal we have $h'(y) = 2y + 1$, an so

$$h(y) = \int h'(y)dy = y^2 + y \tag{6.228}$$

R We will drop the constant of integration we get from integrating h since it will combine with the constant c that we get in the solution process.

Thus, we have

$$\Phi(x,y) = x^2y - 3x^3 + y^2 + y = y^2 + (x^2 + 1)y - 3x^3, \tag{6.229}$$

which is precisely the Φ that we used in Example 3. Observe

$$\frac{d\Phi}{dx} = 0 \tag{6.230}$$

and thus $\Phi(x,y) = y^2 + (x^2+1)y - 3x^3 = c$ for some constant c. To compute c, we'll use our initial condition $y(0) = 2$

$$2^2 + 2 = c \Rightarrow c = 6 \tag{6.231}$$

and so we have a particular solution of

$$y^2 + (x^2+1)y - 3x^3 = 6 \tag{6.232}$$

This is a quadratic equation in y, so we can complete the square or use quadratic formula to get an explicit solution, which is the goal when possible.

$$\begin{aligned}
y^2 + (x^2+1)y - 3x^3 &= 6 && (6.233)\\
y^2 + (x^2+1)y + \frac{(x^2+1)^2}{4} &= 6 + 3x^3 + \frac{(x^2+1)^2}{4} && (6.234)\\
(y + \frac{x^2+1}{2})^2 &= \frac{x^4 + 12x^3 + 2x^2 + 25}{4} && (6.235)\\
y(x) &= \frac{-(x^2+1) \pm \sqrt{x^4 + 12x^3 + 2x^2 + 25}}{2} && (6.236)
\end{aligned}$$

Now we use the initial condition to figure out whether we want the $+$ or $-$ solution. Since $y(0) = 2$ we have

$$2 = y(0) = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2} = 2, -3 \tag{6.237}$$

Thus we see we want the $+$ so our particular solution is

$$y(x) = \frac{-(x^2+1) + \sqrt{x^4 + 12x^3 + 2x^2 + 25}}{2} \tag{6.238}$$

■

■ **Example 6.41** Solve the initial value problem

$$2xy^2 + 2 = 2(3 - x^2y)y', \quad y(-1) = 1. \tag{6.239}$$

First we need to put it in the standard form for exact equations

$$2xy^2 + 2 - 2(3 - x^2y)y' = 0. \tag{6.240}$$

Now, $M(x,y) = 2xy^2 + 2$ and $N(x,y) = -2(3 - x^2y)$. So $M_y = 4xy = N_x$ and the equation is exact.

The next step is to compute $\Phi(x,y)$. We choose to integrate N this time

$$\Phi(x,y) = \int N dy = \int 2x^2y - 6dy = x^2y^2 - 6y + h(x). \tag{6.241}$$

To find $h(x)$, we compute $\Phi_x = 2xy^2 + h'(x)$ and notice that for this to be equal to M, $h'(x) = 2$. Hence $h(x) = 2x$ and we have an implicit solution of

$$x^2y^2 - 6y + 2x = c. \tag{6.242}$$

Now, we use the IC $y(-1) = 1$:

$$1 - 6 - 2 = c \Rightarrow c = -7 \tag{6.243}$$

So our implicit solution is

$$x^2y^2 - 6y + 2x + 7 = 0. \tag{6.244}$$

Again complete the square or use quadratic formula

$$y(x) = \frac{6 \pm \sqrt{36 - 4x^2(2x+7)}}{2x^2} \tag{6.245}$$

$$= \frac{3 \pm \sqrt{9 - 2x^3 - 7x^2}}{x^2} \tag{6.246}$$

and using the IC, we see that we want $-$ solution, so the explicit particular solution is

$$y(x) = \frac{3 - \sqrt{9 - 2x^3 - 7x^2}}{x^2} \tag{6.247}$$

■

■ **Example 6.42** Solve the IVP

$$\frac{2ty}{t^2+1} - 2t - (4 - \ln(t^2+1))y' = 0, \quad y(2) = 0 \tag{6.248}$$

and find the solution's interval of validity.

This is already in the right form. Check if it is exact, $M(t,y) = \frac{2ty}{t^2+1} - 2t$ and $N(t,y) = \ln(t^2+1) - 4$, so $M_y = \frac{2t}{t^2+1} = N_t$. Thus the equation is exact. Now compute $\Phi(x,y)$. Integrate M

$$\Phi = \int M dt = \int \frac{2ty}{t^2+1} dt = y\ln(t^2+1) - t^2 + h(y). \tag{6.249}$$

$$\Phi_y = \ln(t^2+1) + h'(y) = \ln(t^2+1) - 4 = N \tag{6.250}$$

so we conclude $h'(y) = -4$ and thus $h(y) = -4y$. So our implicit solution is then

$$y\ln(t^2+1) - t^2 - 4y = c \tag{6.251}$$

and using the IC we find $c = -4$. Thus the particular solution is

$$y\ln(t^2+1) - t^2 - 4y = -4 \tag{6.252}$$

Solve explicitly to obtain

$$y(x) = \frac{t^2 - 4}{\ln(t^2+1) - 4}. \tag{6.253}$$

Now let's find the interval of validity. We do not have to worry about the natural log since $t^2 + 1 > 0$ for all t. Thus we want to avoid division by 0.

$$\ln(t^2+1) - 4 = 0 \tag{6.254}$$

$$\ln(t^2+1) = 4 \tag{6.255}$$

$$t^2 = e^4 - 1 \tag{6.256}$$

$$t = \pm\sqrt{e^4 - 1} \tag{6.257}$$

So there are three possible intervals of validity, we want the one containing $t = 2$, so $(-\sqrt{e^4-1}, \sqrt{e^4-1})$.

■

Example 6.43 Solve

$$3y^3e^{3xy} - 1 + (2ye^{3xy} + 3xy^2e^{3xy})y' = 0, \quad y(1) = 2 \tag{6.258}$$

We have

$$M_y = 9y^2e^{3xy} + 9xy^3e^{3xy} = N_x \tag{6.259}$$

Thus the equation is exact. Integrate M

$$\Phi = \int M dx = \int 3y^3e^{3xy} - 1 = y^2e^{3xy} - x + h(y) \tag{6.260}$$

and

$$\Phi_y = 2ye^{3xy} + 3xy^2e^{3xy} + h'(y) \tag{6.261}$$

Comparing Φ_y to N, we see that they are already identical, so $h'(y) = 0$ and $h(y) = 0$. So

$$y^2e^{3xy} - x = c \tag{6.262}$$

and using the IC gives $c = 4e^6 - 1$. Thus our implicit particular solution is

$$y^2e^{3xy} - x = 4e^6 - 1, \tag{6.263}$$

and we are done because we will not be able to solve this explicitly. ■

6.5.4 Homogeneous Equations

A **homogeneous function** of x and y of degree n means that a function can be written as $x^n f(y/x)$. For example

$$x^3 - xy^2 = x^3[1 - (y/x)^2]$$

is a homogeneous function of degree 3. An equation of the form

$$P(x,y)dx + Q(x,y)dy = 0 \tag{6.264}$$

where P, Q are homogeneous functions of the same degree and the equation can be rewritten as

$$y' = \frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)} = f\left(\frac{y}{x}\right). \tag{6.265}$$

This suggest that homogeneous equations of this form could be solved by a change of variables $v = y/x$ or $y = xv$. After substitution the equation becomes separable! $y' = f(v)$.

6.6 Second-Order Linear Equations with Constant Coefficients and Zero Right-Hand Side

6.6.1 Basic Concepts

The example of a second order equation which we have seen many times before is Newton's Second Law when expressed in terms of position $s(t)$ is

$$m\frac{d^2s}{dt^2} = F(t, s', s) \tag{6.266}$$

One of the most basic 2nd order equations is $y'' = -y$. By inspection, we might notice that this has two obvious nonzero solutions: $y_1(t) = \cos(t)$ and $y_2(t) = \sin(t)$. But consider $9\cos(t) - 2\sin(t)$? This is also a solution. Anything of the form $y(t) = c_1\cos(t) + c_2\sin(t)$, where c_1 and c_2 are arbitrary constants. Every solution if no conditions are present has this form.

■ **Example 6.44** Find all of the solutions to $y'' = 9y$

We need a function whose second derivative is 9 times the original function. What function comes back to itself without a sign change after two derivatives? Always think of the exponential function when situations like this arise. Two possible solutions are $y_1(t) = e^{3t}$ and $y_2(t) = e^{-3t}$. In fact so are any combination of the two. This is the principal of **linear superposition**. So $y(t) = c_1 e^{3t} + c_2 e^{-3t}$ are infinitely many solutions.

EXERCISE: Check that $y_1(t) = e^{3t}$ and $y_2(t) = e^{-3t}$ are solutions to $y'' = 9y$. ■

The general form of a second order linear differential equation is

$$p(t)y'' + q(t)y' + r(t)y = g(t). \tag{6.267}$$

We call the equation **homogeneous** if $g(t) = 0$ and **nonhomogeneous** if $g(t) \neq 0$.

Theorem 6.6.1 *(Principle of Superposition)* If $y_1(t)$ and $y_2(t)$ are solutions to a second order linear homogeneous differential equation, then so is any linear combination

$$y(t) = c_1 y_1(t) + c_2 y_2(t). \tag{6.268}$$

This follows from the homogeneity and the fact that a derivative is a linear operator. So given any two solutions to a homogeneous equation we can find infinitely more by combining them. The main goal is to be able to write down a general solution to a differential equation, so that with some initial conditions we could uniquely solve an IVP. We want to find $y_1(t)$ and $y_2(t)$ so that the general solution to the differential equation is $y(t) = c_1 y_1(t) + c_2 y_2(t)$. By different we mean solutions which are not constant multiples of each other.

Now reconsider $y'' = -y$. We found two different solutions $y_1(t) = \cos(t)$ and $y_2(t) = \sin(t)$ and any solution to this equation can be written as a linear combination of these two solutions, $y(t) = c_1 \cos(t) + c_2 \sin(t)$. Since we have two constants and a 2nd order equation we need two initial conditions to find a particular solution. We are generally given these conditions in the form of y and y' defined at a particular t_0. So a typical problem might look like

$$p(t)y'' + q(t)y' + r(t)y = 0, \quad y'(t_0) = y_0', \quad y(t_0) = y_0 \tag{6.269}$$

■ **Example 6.45** Find a particular solution to the initial value problem

$$y'' + y = 0, \quad y(0) = 2, \quad y'(0) = -1 \tag{6.270}$$

We have established the general solution to this equation is

$$y(t) = c_1 \cos(t) + c_2 \sin(t) \tag{6.271}$$

To apply the initial conditions, we'll need to know the derivative

$$y'(t) = -c_1 \sin(t) + c_2 \cos(t) \tag{6.272}$$

Plugging in the initial conditions yields

$$2 = c_1 \tag{6.273}$$

$$-1 = c_2 \tag{6.274}$$

so the particular solution is

$$y(t) = 2\cos(t) - \sin(t). \tag{6.275}$$

■

Sometimes when applying initial conditions we will have to solve a system of equations, other times it is as easy as the previous example.

6.6.2 Homogeneous Equations With Constant Coefficients

We will start with the easiest class of second order linear homogeneous equations, where the coefficients $p(t)$, $q(t)$, and $r(t)$ are constants. The equation will have the form

$$ay'' + by' + c = 0. \tag{6.276}$$

How do we find solutions to this equation? From calculus we can find a function that is linked to its derivatives by a multiplicative constant, $y(t) = e^{rt}$. Now that we have a candidate plug it into the differential equation. First calculate the derivatives $y'(t) = re^{rt}$ and $y''(t) = r^2e^{rt}$.

$$a(r^2e^{rt}) + b(re^{rt}) + ce^{rt} = 0 \tag{6.277}$$
$$e^{rt}(ar^2 + br + c) = 0 \tag{6.278}$$

What can we conclude? If $y(t) = e^{rt}$ is a solution to the differential equation, then $e^{rt}(ar^2+br+c) = 0$. Since $e^{rt} \neq 0$, then $y(t) = e^{rt}$ will solve the differential equation as long as r is a solution to

$$ar^2 + br + c = 0. \tag{6.279}$$

This equation is called the **characteristic equation** for $ay'' + by' + c = 0$.

Thus, to find a solution to a linear second order homogeneous constant coefficient equation, we begin by writing down the characteristic equation. Then we find the roots r_1 and r_2 (not necessarily distinct or real). So we have the solutions

$$y_1(t) = e^{r_1 t}, \quad y_2(t) = e^{r_2 t}. \tag{6.280}$$

Of course, it is also possible these are the same, since we might have a repeated root. We will see in a future section how to handle these. In fact, we have three cases.

■ **Example 6.46** Find two solutions to the differential equation $y'' - 9y = 0$ (Example 1). The characteristic equation is $r^2 - 9 = 0$, and this has roots $r = \pm 3$. So we have two solutions $y_1(t) = e^{3t}$ and $y_2(t) = e^{-3t}$, which agree with what we found earlier. ■

The three cases are the same as the three possibilities for types of roots of quadratic equations:
(1) Real, distinct roots $r_1 \neq r_2$.
(2) Complex roots $r_1, r_2 = \alpha \pm \beta i$.
(3) A repeated real root $r_1 = r_2 = r$.
We'll look at each case more closely in the lectures to come.

6.7 Complex Roots of the Characteristic Equation

Last Time: We considered the Wronskian and used it to determine when we have solutions to a second order linear equation or if given one solution we can find another which is linearly independent.

6.7.1 Review Real, Distinct Roots

Recall that a second order linear homogeneous differential equation with constant coefficients

$$ay'' + by' + cy = 0 \tag{6.281}$$

is solved by $y(t) = e^{rt}$, where r solves the **characteristic equation**

$$ar^2 + br + c = 0 \tag{6.282}$$

So when there are two distinct roots $r_1 \neq r_2$, we get two solutions $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$. Since they are distinct we can immediately conclude the general solution is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \tag{6.283}$$

Then given initial conditions we can solve c_1 and c_2.

Exercises:
(1) $y'' + 3y' - 18y = 0, \quad y(0) = 0, \quad y'(0) = -1.$
ANS: $y(t) = \frac{1}{9}e^{-6t} - \frac{1}{9}e^{3t}$.
(2) $y'' - 7y' + 10y = 0, \quad y(0) = 3, \quad y(0) = 2$
ANS: $y(t) = -\frac{4}{3}e^{5t} + \frac{13}{3}e^{2t}$
(3) $2y'' - 5y' + 2y = 0, \quad y(0) = -3, \quad y'(0) = 3$
ANS: $y(t) = -6e^{\frac{1}{2}t} + 3e^{2t}$.
(4) $y'' + 5y' = 0, \quad y(0) = 2, \quad y'(0) = -5$
ANS: $y(t) = 1 + e^{-5t}$
(5) $y'' - 2y' - 8 = 0, \quad y(2) = 1, \quad y'(2) = 0$
ANS: $y(t) = \frac{1}{3e^8}e^{4t} + \frac{2e^4}{3}e^{-2t}$
(6) $y'' + y' - 3y = 0$
ANS: $y(t) = c_1 e^{\frac{-1+\sqrt{13}}{2}t} + c_2 e^{\frac{-1-\sqrt{13}}{2}t}$.

6.7.2 Complex Roots

Now suppose the characteristic equation has complex roots of the form $r_{1,2} = \alpha \pm i\beta$. This means we have two solutions to our differential equation

$$y_1(t) = e^{(\alpha + i\beta)t}, \quad y_2(t) = e^{(\alpha - i\beta)t} \tag{6.284}$$

This is a problem since $y_1(t)$ and $y_2(t)$ are complex-valued. Since our original equation was both simple and had real coefficients, it would be ideal to find two real-valued "different" enough solutions so that we can form a real-valued general solution. There is a way to do this.

Theorem 6.7.1 *(Euler's Formula)*

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{6.285}$$

In other words, we can write an imaginary exponential as a sum of sin and cos. How do we establish this fact? There are two ways:

(1) **Differential Equations**: First we want to write $e^{i\theta} = f(\theta) + ig(\theta)$. We also have

$$f' + ig' = \frac{d}{d\theta}[e^{i\theta}] = ie^{i\theta} = if - g. \tag{6.286}$$

Thus $f' = -g$ and $g' = f$, so $f'' = -f$ and $g'' = -g$. Since $e^0 = 1$, we know that $f(0) = 1$ and $g(0) = 0$. We conclude that $f(\theta) = \cos(\theta)$ and $g(\theta) = \sin(\theta)$, so

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{6.287}$$

(2) **Taylor Series**: Recall that the Taylor series for e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \tag{6.288}$$

while the Taylor series for $\sin(x)$ and $\cos(x)$ are

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... \quad (6.289)$$

$$\cos(x) = \sum_{n-0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... \quad (6.290)$$

$$(6.291)$$

If we set $x = i\theta$ in the first series, we get

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \quad (6.292)$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - ... \quad (6.293)$$

$$= (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ...) + i(\theta - \frac{\theta^3}{3!} + \frac{i\theta^5}{5!} - ...) \quad (6.294)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n!)} + i \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} \quad (6.295)$$

$$= \cos(\theta) + i\sin(\theta) \quad (6.296)$$

So we can write our two complex exponentials as

$$e^{(\alpha+i\beta)t} = e^{\alpha t} e^{i\beta t} = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t)) \quad (6.297)$$

$$e^{(\alpha-i\beta)t} = e^{\alpha t} e^{-i\beta t} = e^{\alpha t}(\cos(\beta t) - i\sin(\beta t)) \quad (6.298)$$

where the minus sign pops out of the sign in the second equation since sin is odd and cos is even. Notice our new expression is still complex-valued. However, by the Principle of Superposition, we can obtain the following solutions

$$y_1(t) = \frac{1}{2}(e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))) + \frac{1}{2}(e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))) = e^{\alpha t}\cos(\beta t) \quad (6.299)$$

$$y_2(t) = \frac{1}{2i}(e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))) - \frac{1}{2i}(e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))) = e^{\alpha t}\sin(\beta t) \quad (6.300)$$

EXERCISE: Check that $y_1(t) = e^{\alpha t}\cos(\beta t)$ and $y_2(t) = e^{\alpha t}\sin(\beta t)$ are in fact solutions to the beginning differential equation when the roots are $\alpha \pm i\beta$.

So now we have two real-valued solutions $y_1(t)$ and $y_2(t)$. It turns out they are linearly independent, so if the roots of the characteristic equation are $r_{1,2} = \alpha \pm i\beta$, we have the general solution

$$y(t) = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t) \quad (6.301)$$

Let's consider some examples:

■ **Example 6.47** Solve the IVP

$$y'' - 4y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = -2 \quad (6.302)$$

The characteristic equation is

$$r^2 - 4r + 9 = 0 \quad (6.303)$$

which has roots $r_{1,2} = 2 \pm i\sqrt{5}$. Thus the general solution and its derivatives are

$$y(t) = c_1 e^{2t}\cos(\sqrt{5}t) + c_2 e^{2t}\sin(\sqrt{5}t) \quad (6.304)$$
$$y'(t) = 2c_1 e^{2t}\cos(\sqrt{5}t) - \sqrt{5}c_1 e^{2t}\sin(\sqrt{5}t) + 2c_2 e^{2t}\sin(\sqrt{5}t) + \sqrt{5}c_2 e^{2t}\cos(\sqrt{5}t) \quad (6.305)$$

If we apply the initial conditions, we get

$$0 = c_1 \quad (6.306)$$
$$-2 = 2c_1 + \sqrt{5}c_2 \quad (6.307)$$

which is solved by $c_1 = 0$ and $c_2 = -\frac{2}{\sqrt{5}}$. So the particular solution is

$$y(t) = -\frac{2}{\sqrt{5}} e^{2t}\sin(\sqrt{5}t). \quad (6.308)$$

■

■ **Example 6.48** Solve the IVP

$$y'' - 8y' + 17y = 0, \quad y(0) = 2, \quad y'(0) = 5. \quad (6.309)$$

The characteristic equation is

$$r^2 - 8r + 17 = 0 \quad (6.310)$$

which has roots $r_{1,2} = 4 \pm i$. Hence the general solution and its derivatives are

$$y(t) = c_1 e^{4t}\cos(t) = c_2 e^{4t}\sin(t) \quad (6.311)$$
$$y'(t) = 4c_1 e^{4t}\cos(t) - c_1 e^{4t}\sin(t) + 4c_2 e^{4t}\sin(t) + c_2 e^{4t}\cos(t) \quad (6.312)$$

and plugging in initial conditions yields the system

$$2 = c_1 \quad (6.313)$$
$$5 = 4c_1 + c_2 \quad (6.314)$$

so we conclude $c_1 = 2$ and $c_2 = -3$ and the particular solution is

$$y(t) = 2e^{4t}\cos(t) - 3e^{4t}\sin(t) \quad (6.315)$$

■

■ **Example 6.49** Solve the IVP

$$4y'' + 12y' + 10y = 0, \quad y(0) = -1, \quad y'(0) = 3 \quad (6.316)$$

The characteristic equation is

$$4r^2 + 12r + 10 = 0 \quad (6.317)$$

which has roots $r_{1,2} = -\frac{3}{2} \pm \frac{1}{2}i$. So the general solution and its derivative are

$$y(t) = c_1 e^{\frac{3}{2}t}\cos(\frac{t}{2}) + c_2 e^{\frac{3}{2}t}\sin(\frac{t}{2}) \quad (6.318)$$
$$y'(t) = \frac{3}{2}c_1 e^{\frac{3}{2}t}\cos(\frac{t}{2}) - \frac{1}{2}c_1 e^{\frac{3}{2}t}\sin(\frac{t}{2}) + \frac{3}{2}c_2 e^{\frac{3}{2}t}\sin(\frac{t}{2}) + \frac{1}{2}c_2 e^{\frac{3}{2}t}\cos(\frac{t}{2}) \quad (6.319)$$

Plugging in the initial condition yields

$$-1 = c_1 \tag{6.320}$$
$$3 = \frac{3}{2}c_1 + \frac{1}{2}c_2 \tag{6.321}$$

which has solution $c_1 = -1$ and $c_2 = 9$. The particular solution is

$$y(t) = -e^{\frac{3}{2}t}\cos(\frac{t}{2}) + 9e^{\frac{3}{2}t}\sin(\frac{t}{2}) \tag{6.322}$$

■

■ **Example 6.50** Solve the IVP

$$y'' + 4y = 0, \quad y(\frac{\pi}{4}) = -10, \quad y'(\frac{\pi}{4}) = 4. \tag{6.323}$$

The characteristic equation is

$$r^2 + 4 = 0 \tag{6.324}$$

which has roots $r_{1,2} = \pm 2i$. The general solution and its derivatives are

$$y(t) = c_1\cos(2t) + c_2\sin(2t) \tag{6.325}$$
$$y'(t) = -2c_1\sin(2t) + 2c_2\cos(2t). \tag{6.326}$$

The initial conditions give the system

$$-10 = c_2 \tag{6.327}$$
$$4 = -2c_1 \tag{6.328}$$

so we conclude that $c_1 = -2$ and $c_2 = -10$ and the particular solution is

$$y(t) = -2\cos(2t) - 10\sin(2t). \tag{6.329}$$

■

6.8 Repeated Roots of the Characteristic Equation and Reduction of Order

Last Time: We considered cases of homogeneous second order equations where the roots of the characteristic equation were complex.

6.8.1 Repeated Roots

The last case of the characteristic equation to consider is when the characteristic equation has repeated roots $r_1 = r_2 = r$. This is a problem since our usual solution method produces the same solution twice

$$y_1(t) = e^{r_1 t} = e^{r_2 t} = y_2(t) \tag{6.330}$$

But these are the same and are not linearly independent. So we will need to find a second solution which is "different" from $y_1(t) = e^{rt}$. What should we do?

Start by recalling that if the quadratic equation $ar^2 + br + c = 0$ has a repeated root r, it must be $r = -\frac{b}{2a}$. Thus our solution is $y_1(t) = e^{-\frac{b}{2a}}$. We know any constant multiple of $y_1(t)$ is also a solution. These will still be linearly dependent to $y_1(t)$. Can we find a solution of the form

$$y_2(t) = v(t)y_1(t) = v(t)e^{-\frac{b}{2a}t} \tag{6.331}$$

i.e. y_2 is the product of a function of t and y_1.

Differentiate $y_2(t)$:

$$y_2'(t) = v'(t)e^{-\frac{b}{2a}t} - \frac{b}{2a}v(t)e^{-\frac{b}{2a}t} \quad (6.332)$$

$$y_2''(t) = v''(t)e^{-\frac{b}{2a}} - \frac{b}{2a}v'(t)e^{-\frac{b}{2a}t} - \frac{b}{2a}v'(t)e^{-\frac{b}{2a}t} + \frac{b^2}{4a^2}v(t)e^{-\frac{b}{2a}t} \quad (6.333)$$

$$= v''(t)e^{-\frac{b}{2a}t} - \frac{b}{a}v'(t)e^{-\frac{b}{2a}t} + \frac{b^2}{4a^2}v(t)e^{-\frac{b}{2a}t}. \quad (6.334)$$

Plug in differential equation:

$$a(v''e^{-\frac{b}{2a}t} - \frac{b}{a}v'e^{-\frac{b}{2a}t} + \frac{b^2}{4a^2}ve^{-\frac{b}{2a}t}) + b(v'e^{-\frac{b}{2a}t} - \frac{b}{2a}ve^{-\frac{b}{2a}t}) + c(ve^{-\frac{b}{2a}t}) = 0 \quad (6.335)$$

$$e^{-\frac{b}{2a}t}\left(av'' + (-b+b)v' + (\frac{b^2}{4a} - \frac{b^2}{2a} + c)v\right) = 0 \quad (6.336)$$

$$e^{-\frac{b}{2a}t}\left(av'' - \frac{1}{4a}(b^2 - 4ac)v\right) = 0 \quad (6.337)$$

Since we are in the repeated root case, we know the discriminant $b^2 - 4ac = 0$. Since exponentials are never zero, we have

$$av'' = 0 \Rightarrow v'' = 0 \quad (6.338)$$

We can drop the a since it cannot be zero, if a were zero it would be a first order equation! So what does v look like

$$v(t) = c_1 t + c_2 \quad (6.339)$$

for constants c_1 and c_2. Thus for any such $v(t)$, $y_2(t) = v(t)e^{-\frac{b}{2a}t}$ will be a solution. The most general possible $v(t)$ that will work for us is $c_1 t + c_2$. Take $c_1 = 1$ and $c_2 = 0$ to get a specific $v(t)$ and our second solution is

$$y_2(t) = te^{-\frac{b}{2a}t} \quad (6.340)$$

and the general solution is

$$y(t) = c_1 e^{-\frac{b}{2a}t} + c_2 te^{-\frac{b}{2a}t} \quad (6.341)$$

R Here's another way of looking at the choice of constants. Suppose we do not make a choice. Then we have the general solution

$$y(t) = c_1 e^{-\frac{b}{2a}t} + c_2(ct+k)e^{-\frac{b}{2a}t} \quad (6.342)$$

$$= c_1 e^{-\frac{b}{2a}t} + c_2 cte^{-\frac{b}{2a}t} + c_2 ke^{-\frac{b}{2a}t} \quad (6.343)$$

$$= (c_1 + c_2 k)e^{-\frac{b}{2a}t} + c_2 cte^{-\frac{b}{2a}t} \quad (6.344)$$

since they are all constants we just get

$$y(t) = c_1 e^{-\frac{b}{2a}t} + c_2 te^{-\frac{b}{2a}t} \quad (6.345)$$

To summarize: if the characteristic equation has repeated roots $r_1 = r_2 = r$, the general solution is

$$y(t) = c_1 e^{rt} + c_2 te^{rt} \quad (6.346)$$

Now for examples:

■ **Example 6.51** Solve the IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = -1, \quad y'(0) = 6 \tag{6.347}$$

The characteristic equation is

$$r^2 - 4r + 4 = 0 \tag{6.348}$$
$$(r-2)^2 = 0 \tag{6.349}$$

so we see that we have a repeated root $r = 2$. The general solution and its derivative are

$$y(t) = c_1 e^{2t} + c_2 t e^{2t} \tag{6.350}$$
$$y'(t) = 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t e^{2t} \tag{6.351}$$

and plugging in initial conditions yields

$$-1 = c_1 \tag{6.352}$$
$$6 = 2c_1 + c_2 \tag{6.353}$$

so we have $c_1 = -1$ and $c_2 = 8$. The particular solution is

$$y(t) = -e^{2t} + 6te^{2t} \tag{6.354}$$

■

■ **Example 6.52** Solve the IVP

$$16y'' + 40y' + 25y = 0, \quad y(0) = -1, \quad y'(0) = 2. \tag{6.355}$$

The characteristic equation is

$$16r^2 + 40r + 25 = 0 \tag{6.356}$$
$$(4r+5)^2 = 0 \tag{6.357}$$

and so we conclude that we have a repeated root $r = -\frac{5}{4}$ and the general solution and its derivative are

$$y(t) = c_1 e^{-\frac{5}{4}t} + c_2 t e^{-\frac{5}{4}t} \tag{6.358}$$
$$y'(t) = -\frac{5}{4} c_1 e^{-\frac{5}{4}t} + c_2 e^{-\frac{5}{4}t} - \frac{5}{4} c_2 t e^{-\frac{5}{4}t} \tag{6.359}$$

Plugging in the initial conditions yields

$$-1 = c_1 \tag{6.360}$$
$$2 = -\frac{5}{4} c_1 + c_2 \tag{6.361}$$

so $c_1 = -1$ and $c_2 = \frac{5}{4}$. The particular solution is

$$y(t) = -e^{-\frac{5}{4}t} + \frac{3}{4} t e^{-\frac{5}{4}t} \tag{6.362}$$

■

6.8.2 Reduction of Order

We have spent the last few lectures analyzing second order linear homogeneous equations with constant coefficients, i.e. equations of the form

$$ay'' + by' + cy = 0 \tag{6.363}$$

Let's now consider the case when the coefficients are not constants

$$p(t)y'' + q(t)y' + r(t)y = 0 \tag{6.364}$$

In general this is not easy, but if we can guess a solution, we can use the techniques developed in the repeated roots section to find another solution. This method will be called **Reduction Of Order**. Consider a few examples

■ **Example 6.53** Find the general solution to

$$2t^2y'' + ty' - 3y = 0 \tag{6.365}$$

given that $y_1(t) = t^{-1}$ is a solution.

ANS: Think back to repeated roots. We know we had a solution $y_1(t)$ and needed to find a distinct solution. What did we do? We asked which nonconstant function $v(t)$ make $y_2(t) = v(t)y_1(t)$ is also a solution. The y_2 derivatives are

$$y_2 = vt^{-1} \tag{6.366}$$
$$y_2' = v't^{-1} - vt^{-2} \tag{6.367}$$
$$y_2'' = v''t^{-1} - v't^{-2} + 2vt^{-3} = v''t^{-1} - 2v't^{-2} + 2vt^{-3} \tag{6.368}$$

The next step is to plug into the original equation so we can solve for v:

$$2t^2(v''t^{-1} - 2v't^{-2} + 2vt^{-3}) + t(v't^{-1} - vt^{-2}) - 3vt^{-1} = 0 \tag{6.369}$$
$$2v''t - 4v' + 4vt^{-1} + v' - vt^{-1} - 3vt^{-1} = 0 \tag{6.370}$$
$$2tv'' - 3v' = 0 \tag{6.371}$$

Notice that the only terms left involve v'' and v', not v. This also happened in the repeated root case. The v term should always disappear at this point, so we have a check on our work. If there is a v term left we have done something wrong.

Now we know that if y_2 is a solution, the function v must satisfy

$$2tv'' - 3v' = 0 \tag{6.372}$$

But this is a second order linear homogeneous equation with nonconstant coefficients. Let $w(t) = v'(t)$. By changing variables our equation becomes

$$w' - \frac{3}{2t}w = 0. \tag{6.373}$$

So by Integrating Factor

$$\mu(t) = e^{\int -\frac{3}{2t}dt} = e^{-\frac{3}{2}\ln(t)} = t^{-\frac{3}{2}} \tag{6.374}$$
$$(t^{-\frac{3}{2}}w)' = 0 \tag{6.375}$$
$$t^{-\frac{3}{2}}w = c \tag{6.376}$$
$$w(t) = ct^{\frac{3}{2}} \tag{6.377}$$

So we know what $w(t)$ must solve the equation. But to solve our original differential equation, we do not need $w(t)$, we need $v(t)$. Since $v'(t) = w(t)$, integrating w will give our v

$$v(t) = \int w(t)dt \quad (6.378)$$

$$= \int ct^{\frac{3}{2}t}dt \quad (6.379)$$

$$= \frac{2}{5}ct^{\frac{5}{2}} + k \quad (6.380)$$

Now this is the general form of $v(t)$. Pick $c = 5/2$ and $k = 0$. Then $v(t) = t^{\frac{5}{2}}$, so $y_2(t) = v(t)y_1(t) = t^{\frac{3}{2}}$, and the general solution is

$$y(t) = c_1t^{-1} + c_2t^{\frac{3}{2}} \quad (6.381)$$

■

Reduction of Order is a powerful method for finding a second solution to a differential equation when we do not have any other method, but we need to have a solution to begin with. Sometimes even finding the first solution is difficult.

We have to be careful with these problems sometimes the algebra is tedious and one can make sloppy mistakes. Make sure the v terms disappears when we plug in the derivatives for y_2 and check the solution we obtain in the end in case there was an algebra mistake made in the solution process.

■ **Example 6.54** Find the general solution to

$$t^2y'' + 2ty' - 2y = 0 \quad (6.382)$$

given that

$$y_1(t) = t \quad (6.383)$$

is a solution.

Start by setting $y_2(t) = v(t)y_1(t)$. So we have

$$y_2 = tv \quad (6.384)$$

$$y_2' = tv' + v \quad (6.385)$$

$$y_2'' = tv'' + v' + v' = tv'' + 2v'. \quad (6.386)$$

Next, we plug in and arrange terms

$$t^2(tv'' + 2v') + 2t(tv' + v) - 2tv = 0 \quad (6.387)$$

$$t^3v'' + 2t^2v' + 2t^2v' + 2tv - 2tv = 0 \quad (6.388)$$

$$t^3v'' + 4t^2v' = 0. \quad (6.389)$$

Notice the v drops out as desired. We make the change of variables $w(t) = v'(t)$ to obtain

$$t^3w' + 4t^2w = 0 \quad (6.390)$$

which has integrating factor $\mu(t) = t^4$.

$$(t^4w)' = 0 \quad (6.391)$$

$$t^4w = c \quad (6.392)$$

$$w(t) = ct^{-4} \quad (6.393)$$

So we have

$$v(t) = \int w(t)dt \tag{6.394}$$
$$= \int ct^{-4}dt \tag{6.395}$$
$$= -\frac{c}{3}t^{-3} + k. \tag{6.396}$$

A nice choice for the constants is $c = -3$ and $k = 0$, so $v(t) = t^{-3}$, which gives a second solution of $y_2(t) = v(t)y_1(t) = t^{-2}$. So our general solution is

$$y(t) = c_1 t + c_2 t^{-2} \tag{6.397}$$

■

6.9 Second-Order Linear Equations with Constant Coefficients and Non-zero Right-Hand Side

Last Time: We considered cases of homogeneous second order equations where the roots of the characteristic equation were repeated real roots. Then we looked at the method of reduction of order to produce a second solution to an equation given the first solution.

6.9.1 Nonhomogeneous Equations

A second order nonhomogeneous equation has the form

$$p(t)y'' + q(t)y' + r(t)y = g(t) \tag{6.398}$$

where $g(t) \neq 0$. How do we get the general solution to these?

Suppose we have two solutions $Y_1(t)$ and $Y_2(t)$. The Principle of Superposition no longer holds for nonhomogeneous equations. We cannot just take a linear combination of the two to get another solution. Consider the equation

$$p(t)y'' + q(t)y' + r(t)y = 0 \tag{6.399}$$

which we will call the **associated homogeneous equation**.

Theorem 6.9.1 Suppose that $Y_1(t)$ and $Y_2(t)$ are two solutions to equation (6.398) and that $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions to (6.399). Then $Y_1(t) - Y_2(t)$ is a solution to Equation (6.399) and has the form

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t) \tag{6.400}$$

Notice the notation used, it will be standard. Uppercase letters are solutions to the nonhomogeneous equation and lower case letters to denote solutions to the homogeneous equation.

Let's verify the theorem by plugging in $Y_1 - Y_2$ to (6.399)

$$p(t)(Y_1 - Y_2)'' + q(t)(Y_1 - Y_2)' + r(t)(Y_1 - Y_2) = 0 \tag{6.401}$$
$$\big(p(t)Y_1'' + q(t)Y_1' + r(t)Y_1\big) - \big(p(t)Y_2'' + q(t)Y_2' + r(t)Y_2\big) = 0 \tag{6.402}$$
$$g(t) - g(t) = 0 \tag{6.403}$$
$$0 = 0 \tag{6.404}$$

So we have that $Y_1(t) - Y_2(t)$ solves equation (6.399). We know that $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions to equation (6.399) and so any solution can be written as a linear combination of them. Thus for constants c_1 and c_2

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t) \tag{6.405}$$

So the difference of any two solutions of (6.398) is a solution to (6.399). Suppose we have a solution to (6.398), which we denote by $Y_p(t)$. Let $Y(t)$ denote the general solution. We have seen

$$Y(t) - Y_p(t) = c_1 y_1(t) + c_2 y_2(t) \tag{6.406}$$

or

$$Y(t) = c_1 y_1(t) + c_2 y_2(t) + Y_p(t) \tag{6.407}$$

where y_1 and y_2 are a fundamental set of solutions to $Y(t)$. We will call

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t) \tag{6.408}$$

the **complimentary solution** and $Y_p(t)$ a **particular solution**. So, the general solution can be expressed as

$$Y(t) = y_c(t) + Y_p(t). \tag{6.409}$$

Thus, to find the general solution of (6.398), we'll need to find the general solution to (6.399) and then find some solution to (6.398). Adding these two pieces together give the general solution to (6.398).

If we vary a solution to (6.398) by just adding in some solution to Equation (6.399), it will still solve Equation (6.398). Now the goal of this section is to find some particular solution $Y_p(t)$ to Equation (6.398). We have two methods. The first is the method of **Undetermined Coefficients**, which reduces the problem to an algebraic problem, but only works in a few situations. The other called Variation of Parameters is a much more general method that always works but requires integration which may or may not be tedious.

6.9.2 Undetermined Coefficients

The major disadvantage of this solution method is that it is only useful for constant coefficient differential equations, so we will focus on

$$ay'' + by' + cy = g(t) \tag{6.410}$$

for $g(t) \neq 0$. The other disadvantage is it only works for a small class of $g(t)$'s.

Recall that we are trying to find some particular solution $Y_p(t)$ to Equation (6.410). The idea behind the method is that for certain classes of nonhomogeneous terms, we're able to make a good educated guess as to how $Y_p(t)$ should look, up to some unknown coefficients. Then we plug our guess into the differential equation and try to solve for the coefficients. If we can, our guess was correct and we have determined $Y_p(t)$. If we cannot solve for the coefficients, then we guessed incorrectly and we will need to try again.

6.9.3 The Basic Functions

There are three types of basic types of nonhomogeneous terms $g(t)$ that can be used for this method: exponentials, trig functions (sin and cos), and polynomials. One we know how they work individually and combination will be similar.

Exponentials

Let's walk through an example where $g(t)$ is an exponential and see how to proceed.

■ **Example 6.55** Determine a particular solution to

$$y'' - 4y' - 12y = 2e^{4t}. \tag{6.411}$$

How can we guess the form of $Y_p(t)$? When we plug $Y_p(t)$ into the equation, we should get $g(t) = 2e^{4t}$. We know that exponentials never appear or disappear during differentiation, so try

$$Y_p(t) = Ae^{4t} \tag{6.412}$$

for some coefficient A. Differentiate, plug in, and see if we can determine A. Plugging in we get

$$16Ae^{4t} - 4(4Ae^{4t}) - 12Ae^{4t} = 2e^{4t} \tag{6.413}$$

$$-12Ae^{4t} = 2e^{4t} \tag{6.414}$$

For these to be equal we need A to satisfy

$$-12A = 2 \Rightarrow A = -\frac{1}{6}. \tag{6.415}$$

So with this choice of A, our guess works, and the particular solution is

$$Y_p(t) = -\frac{1}{6}e^{4t}. \tag{6.416}$$

■

Consider the following full problem:

■ **Example 6.56** Solve the IVP

$$y'' - 4y' - 12y = 2e^{4t}, \quad y(0) = -\frac{13}{6}, \quad y'(0) = \frac{7}{3}. \tag{6.417}$$

We know the general solution has the form

$$y(t) = y_c(t) + Y_p(t) \tag{6.418}$$

where the complimentary solution $y_c(t)$ is the general solution to the associated homogeneous equation

$$y'' - 4y' - 12y = 0 \tag{6.419}$$

and $Y_p(t)$ is the particular solution to the original differential equation. From the previous example we know

$$Y_p(t) = -\frac{1}{6}e^{4t}. \tag{6.420}$$

What is the complimentary solution? Our associated homogeneous equation has constant coefficients, so we need to find roots of the characteristic equation.

$$r^2 - 4r - 12 = 0 \tag{6.421}$$

$$(r-6)(r+2) = 0 \tag{6.422}$$

So we conclude that $r_1 = 6$ and $r_2 = -2$. These are distinct roots, so the complimentary solution will be

$$y_c(t) = c_1e^{6t} + c_2e^{-2t} \tag{6.423}$$

We must be careful to remember the initial conditions are for the non homogeneous equation, not the associated homogeneous equation. Do not apply them at this stage to y_c, since that is not a solution to the original equation.

So our general solution is the sum of $y_c(t)$ and $Y_p(t)$. We'll need it and its derivative to apply the initial conditions

$$y(t) = c_1e^{6t} + c_2e^{-2t} - \frac{1}{6}e^{4t} \tag{6.424}$$

$$y'(t) = 6c_1e^{6t} - 2c_2e^{2t} - \frac{2}{3}e^{4t} \tag{6.425}$$

Now apply the initial conditions

$$-\frac{13}{6} = y(0) = c_1 + c_2 - \frac{1}{6} \tag{6.426}$$

$$\frac{7}{3} = y'(0) = 6c_1 - 2c_2 - \frac{2}{3} \tag{6.427}$$

This system is solved by $c_1 = -\frac{1}{8}$ and $c_2 = -\frac{15}{8}$, so our solution is

$$y(t) = -\frac{1}{8}e^{6t} - \frac{15}{8}e^{-2t} - \frac{1}{6}e^{4t}. \tag{6.428}$$

■

Trig Functions

The second class of nonhomogeneous terms for which we can use this method are trig functions, specifically sin and cos.

■ **Example 6.57** Find a particular solution for the following IVP

$$y'' - 4y' - 12y = 6\cos(4t). \tag{6.429}$$

In the first example the nonhomogeneous term was exponential, and we know when we differentiate exponentials they persist. In this case, we've got a cosine function. When we differentiate a cosine, we get sine. So we expect an initial guess to require a sine term in addition to cosine. Try

$$Y_p(t) = A\cos(4t) + B\sin(4t). \tag{6.430}$$

Now differentiate and plug in

$$-16A\cos(4t) - 16B\sin(4t) - 4\big(-4A\sin(4t) + 4B\cos(4t)\big) - 12\big(A\cos(4t) + B\sin(4t)\big) = 13\cos(4t) \tag{6.431}$$

$$(-16A - 16B - 12A)\cos(4t) + (-16B + 16A - 12B)\sin(4t) = 13\cos(4t) \tag{6.432}$$

$$(-28A - 16B)\cos(4t) + (16A - 28B)\sin(4t) = 13\cos(4t) \tag{6.433}$$

To solve for A and B set the coefficients equal. Note that the coefficient for $\sin(4t)$ on the right hand side is 0. So we get the system of equations

$$\cos(4t) \quad : \quad -28A - 16B = 13 \tag{6.434}$$

$$\sin(4t) \quad : \quad 16A - 28B = 0. \tag{6.435}$$

This system is solved by $A = -\frac{7}{20}$ and $B = -\frac{1}{5}$. So a particular solution is

$$Y_p(t) = -\frac{7}{20}\cos(4t) - \frac{1}{5}\sin(4t) \tag{6.436}$$

■

Note that the guess would have been the same if $g(t)$ had been sine instead of cosine.

Polynomials

The third and final class of nonhomogeneous term we can use with this method are polynomials.

■ **Example 6.58** Find a particular solution to

$$y'' - 4y' - 12y = 3t^3 - 5t + 2. \tag{6.437}$$

In this case, $g(t)$ is a cubic polynomial. When differentiating polynomials the order decreases. So if our initial guess is a cubic, we should capture all terms that will arise. Our guess

$$Y_p(t) = At^3 + Bt^2 + Ct + D. \tag{6.438}$$

Note that we have a t^2 term in our equation even though one does not appear in $g(t)$! Now differentiate and plug in

$$6At + 2B - 4(3At^2 + 2Bt + C) - 12(At^3 + Bt^2 + Ct + D) = 3t^2 - 5t + 2 \tag{6.439}$$
$$-12At^3 + (12A - 12B)t^2 + (6A - 8B - 12C)t + (2B - 4C - 12D) = 3t^2 - 5t + 2 \tag{6.440}$$

We obtain a system of equations by setting coefficients equal

$$t^3 \ : \ -12A = 3 \Rightarrow A = -\frac{1}{4} \tag{6.441}$$
$$t^2 \ : \ -12A - 12B = 0 \Rightarrow B = \frac{1}{4} \tag{6.442}$$
$$t \ : \ 6A - 8B - 12C = -5 \Rightarrow C = \frac{1}{8} \tag{6.443}$$
$$1 \ : \ 2B - 4C - 12D = 2 \Rightarrow D = -\frac{1}{6} \tag{6.444}$$

So a particular solution is

$$Y_p(t) = -\frac{1}{4}t^3 + \frac{1}{4}t^2 + \frac{1}{8}t - \frac{1}{6} \tag{6.445}$$

■

Summary

Given each of the basic types, we make the following guess

$$ae^{\alpha t} \ \Rightarrow \ Ae^{\alpha t} \tag{6.446}$$
$$a\cos(\alpha t) \ \Rightarrow \ A\cos(\alpha t) + B\sin(\alpha t) \tag{6.447}$$
$$a\sin(\alpha t) \ \Rightarrow \ A\cos(\alpha t) + B\sin(\alpha t) \tag{6.448}$$
$$a_n t^n + a_{n-1}t^{n-1} + \ldots + a_1 t + a_0 \ \Rightarrow \ A_n t^n + A_{n-1}t^{n-1} + \ldots + A_1 t + A_0 \tag{6.449}$$

6.9.4 Products

The idea for products is to take products of our forms above.

■ **Example 6.59** Find a particular solution to

$$y'' - 4y' - 12y = te^{4t} \tag{6.450}$$

Start by writing the guess for the individual pieces. $g(t)$ is the product of a polynomial and an exponential. Thus guess for the polynomial is $At + B$ while the guess for the exponential is Ce^{4t}. So the guess for the product should be

$$Ce^{4t}(At + B) \tag{6.451}$$

We want to minimize the number of constants, so

$$Ce^{4t}(At+B) = e^{4t}(ACt+BC). \tag{6.452}$$

Rewrite with two constants

$$Y_p(t) = e^{4t}(At+B) \tag{6.453}$$

Notice this is the guess as if it was just t with the exponential multiplied to it. Differentiate and plug in

$$\begin{aligned}
16e^{4t}(At+B)+8Ae^{4t}-4(4e^{4t}(At+B)+Ae^{4t})-12e^{4t}(At+B) &= te^{4t} &(6.454)\\
(16A-16A-12A)t^{4t}+(16B+8A-16B-4A-12B)e^{4t} &= te^{4t} &(6.455)\\
-12Ate^{4t}+(4A-12B)e^{4t} &= te^{4t} &(6.456)
\end{aligned}$$

Then we set the coefficients equal

$$\begin{aligned}
te^{4t}: \quad -12A = 1 \quad &\Rightarrow \quad A = -\frac{1}{12} &(6.457)\\
e^{4t}: \quad (4A-12B) = 0 \quad &\Rightarrow \quad B = -\frac{1}{36} &(6.458)
\end{aligned}$$

So, a particular solution for this differential equation is

$$Y_p(t) = e^{4t}(-\frac{1}{12}t-\frac{1}{36}) = -\frac{e^{4t}}{36}(3t+1). \tag{6.459}$$

■

Basic Rule: If we have a product with an exponential write down the guess for the other piece and multiply by an exponential without any leading coefficient.

■ **Example 6.60** Find a particular solution to

$$y''-4y'-12y = 29e^{5t}\sin(3t). \tag{6.460}$$

We try the following guess

$$Y_p(t) = e^{5t}(A\cos(3t)+B\sin(3t)). \tag{6.461}$$

So differentiate and plug in

$$\begin{aligned}
&25e^{5t}(A\cos(3t)+B\sin(3t))+30e^{5t}(-A\sin(3t)+B\cos(3t)+\\
&9e^{5t}(-A\cos(3t)-B\sin(3t))-4(5e^{5t}(A\cos(3t)+B\sin(3t))+\\
&3e^{5t}(-A\sin(3t)+B\cos(3t)))-12e^{5t}(A\cos(3t)+B\sin(3t)) = 29e^{5t}\sin(3t)
\end{aligned} \tag{6.462}$$

Gather like terms

$$(-16A+18B)e^{5t}\cos(3t)+(-18A-16B)e^{5t}\sin(3t) = 29e^{5t}\sin(3t) \tag{6.463}$$

Set the coefficients equal

$$\begin{aligned}
e^{5t}\cos(3t): \quad -16A+18B &= 0 &(6.464)\\
e^{5t}\sin(3t): \quad -18A-16B &= 29 &(6.465)
\end{aligned}$$

This is solved by $A = -\frac{9}{10}$ and $B = -\frac{4}{5}$. So a particular solution to this differential equation is

$$Y_p(t) = e^{5t}(-\frac{9}{10}t-\frac{4}{5}) = -\frac{e^{5t}}{10}(9t+8). \tag{6.466}$$

■

■ **Example 6.61** Write down the form of the particular solution to

$$y'' - 4y' - 12y = g(t) \tag{6.467}$$

for the following $g(t)$:
(1) $g(t) = (9t^2 - 103t)\cos(t)$

Here we have a product of a quadratic and a cosine. The guess for the quadratic is

$$At^2 + Bt + C \tag{6.468}$$

and the guess for the cosine is

$$D\cos(t) + E\sin(t). \tag{6.469}$$

Multiplying the two guesses gives

$$(At^2 + Bt + C)(D\cos(t)) \quad + \quad (At^2 + Bt + C)(E\sin(t)) \tag{6.470}$$
$$(ADt^2 + BDt + CD)\cos(t) \quad + \quad (AEt^2 + BEt + CE)\sin(t). \tag{6.471}$$

Each of the coefficients is a product of two constants, which is another constant. Simply to get our final guess

$$Y_p(t) = (At^2 + Bt + C)\cos(t) + (Dt^2 + Et + F)\sin(t) \tag{6.472}$$

This is indicative of the general rule for a product of a polynomial and a trig function. Write down the guess for the polynomial, multiply by cosine, then add to that the guess for the polynomial multiplied by a sine.

(2) $g(t) = e^{-2t}(3 - 5t)\cos(9t)$

This homogeneous term has all three types of special functions. So combining the two general rules above, we get

$$Y_p(t) = e^{-2t}(At + B)\cos(9t) + e^{-2t}(Ct + D)\sin(9t). \tag{6.473}$$

■

6.9.5 Sums

We have the following important fact. If Y_1 satisfies

$$p(t)y'' + q(t)y' + r(t)y = g_1(t) \tag{6.474}$$

and Y_2 satisfies

$$p(t)y'' + q(t)y' + r(t)y = g_2(t) \tag{6.475}$$

then $Y_1 + Y_2$ satisfies

$$p(t)y'' + q(t)y' + r(t)y = g_1(t) + g_2(t) \tag{6.476}$$

This means that if our nonhomogeneous term $g(t)$ is a sum of terms we can write down the guesses for each of those terms and add them together for our guess.

■ **Example 6.62** Find a particular solution to

$$y'' - 4y' - 12y = e^{7t} + 12. \tag{6.477}$$

Our nonhomogeneous term $g(t) = e^{7t} + 12$ is the sum of an exponential $g_1(t) = e^{7t}$ and a 0 degree polynomial $g_2(t) = 12$. The guess is

$$Y_p(t) = Ae^{7t} + B \tag{6.478}$$

This cannot be simplified, so this is our guess. Differentiate and plug in

$$49Ae^{7t} - 28Ae^{7t} - 12Ae^{7t} - 12B = e^{7t} + 12 \tag{6.479}$$

$$9Ae^{7t} - 12B = e^{7t} + 12. \tag{6.480}$$

Setting the coefficients equal gives $A = \frac{1}{9}$ and $B = -1$, so our particular solution is

$$Y_p(t) = \frac{1}{9}e^{7t} - 1. \tag{6.481}$$

■

■ **Example 6.63** Write down the form of a particular solution to

$$y'' - 4y' - 12y = g(t) \tag{6.482}$$

for each of the following $g(t)$:
(1) $g(t) = 2\cos(t) - 9\sin(3t)$

Our guess for the cosine is

$$A\cos(3t) + B\sin(3t) \tag{6.483}$$

Additionally, our guess for the sine is

$$C\cos(3t) + D\sin(3t) \tag{6.484}$$

So if we add the two of them together, we obtain

$$A\cos(3t) + B\sin(3t) + C\cos(3t) + D\sin(3t) = (A+C)\cos(3t) + (B+D)\sin(3t) \tag{6.485}$$

But $A+C$ and $B+D$ are just some constants, so we can replace them with the guess

$$Y_p(t) = A\cos(3t) + B\sin(3t). \tag{6.486}$$

(2) $g(t) = \sin(t) - 2\sin(14t) - 5\cos(14t)$

Start with a guess for the $\sin(t)$

$$A\cos(t) + B\sin(t). \tag{6.487}$$

Since they have the same argument, the previous example showed we can combine the guesses for $\cos(14t)$ and $\sin(14t)$ into

$$C\cos(14t) + D\sin(14t) \tag{6.488}$$

So the final guess is

$$Y_p(t) = A\cos(t) + B\sin(t) + C\cos(14t) + D\cos(14t) \tag{6.489}$$

(3) $g(t) = 7\sin(10t) - 5t^2 + 4t$
Here we have the sum of a trig function and a quadratic so the guess will be

$$Y_p(t) = A\cos(10t) + B\sin(10t) + Ct^2 + Dt + E. \tag{6.490}$$

(4) $g(t) = 9e^t + 3te^{-5t} - 5e^{-5t}$
This can be rewritten as $9e^t + (3t-5)e^{-5t}$. So our guess will be

$$Y_p(t) = Ae^t + (Bt + C)e^{-5t} \tag{6.491}$$

(5) $g(t) = t^2\sin(t) + 4\cos(t)$
So our guess will be

$$Y_p(t) = (At^2 + Bt + C)\cos(t) + (Dt^2 + Et + F)\sin(t). \tag{6.492}$$

(6) $g(t) = 3e^{-3t} + e^{-3t}\sin(3t) + \cos(3t)$
Our guess

$$Y_p(t) = Ae^{-3t} + e^{-3t}(B\cos(3t) + C\sin(3t)) + D\cos(3t) + E\sin(3t). \tag{6.493}$$

This seems simple, right? There is one problem which can arise you need to be aware of

■ **Example 6.64** Find a particular solution to

$$y'' - 4y' - 12y = e^{6t} \tag{6.494}$$

This seems straightforward, so try $Y_p(t) = Ae^{6t}$. If we differentiate and plug in

$$36Ae^{6t} - 24Ae^{6t} - 12Ae^{6t} = e^{6t} \tag{6.495}$$

$$0 = e^{6t} \tag{6.496}$$

Exponentials are never zero. So this cannot be possible. Did we make a mistake on our original guess? Yes, if we went through the normal process and found the complimentary solution in this case

$$y_c(t) = c_1e^{6t} + c_2e^{-2t}. \tag{6.497}$$

So our guess for the particular solution was actually part of the complimentary solution. So we need to find a different guess. Think back to repeated root solutions and try $Y_p(t) = Ate^{6t}$. Try it

$$(36Ate^{6t} + 12Ae^{6t}) - 4(6Ate^{6t} + Ae^{6t}) - 12Ate^{6t} = e^{6t} \tag{6.498}$$

$$(36A - 24A - 12A)te^{6t} + (12A - 4A)e^{6t} = e^{6t} \tag{6.499}$$

$$8Ae^{6t} = e^{6t} \tag{6.500}$$

Setting the coefficients equal, we conclude that $A = \frac{1}{8}$, so

$$Y_p(t) = \frac{1}{8}te^{6t}. \tag{6.501}$$

■

NOTE: If this situation arises when the complimentary solution has a repeated root and has the form

$$y_c(t) = c_1e^{rt} + c_2te^{rt} \tag{6.502}$$

then our guess for the particular solution should be

$$Y_p(t) = At^2e^{rt}. \tag{6.503}$$

■

6.9.6 Method of Undetermined Coefficients

Then we want to construct the general solution $y(t) = y_c(t) + Y_p(t)$ by following these steps:
(1) Find the general solution of the corresponding homogeneous equation.
(2) Make sure $g(t)$ belongs to a special set of basic functions we will define shortly.
(3) If $g(t) = g_1(t) + ... + g_n(t)$ is the sum of n terms, then form n subproblems each of which contains only one $g_i(t)$. Where the ith subproblem is

$$ay'' + by' + cy = g_i(t) \tag{6.504}$$

(4) For the ith subproblem assume a particular solution of the appropriate functions (exponential, sine, cosine, polynomial). If there is a duplication in $Y_i(t)$ with a solution to the homogeneous problem then multiply $Y_i(t)$ by t (or if necessary t^2).
(5) Find the particular solution $Y_i(t)$ for each subproblem. Then the sum of the Y_i is a particular solution for the full nonhomogeneous problem.
(6) Form the general solution by summing all the complimentary solutions from the homogeneous equation and the n particular solutions.
(7) Use the initial conditions to determine the values of the arbitrary constants remaining in the general solution.

Now for more examples, write down the guess for the particular solution:
(1) $y'' - 3y' - 28y = 6t + e^{-4t} - 2$
First we find the complimentary solution using the characteristic equation

$$y_c(t) = c_1 e^{7t} + c_2 e^{-4t} \tag{6.505}$$

Now look at the nonhomogeneous term which is a polynomial and exponential, $6t - 2 + e^{-4t}$. So our initial guess should be

$$At + B + Ce^{-4t} \tag{6.506}$$

The first two terms are fine, but the last term is in the complimentary solution. Since Cte^{-4t} does not show up in the complimentary solution our guess should be

$$Y_p(t) = At + B + Cte^{-4t}. \tag{6.507}$$

(2) $y'' - 64y = t^2 e^{8t} + \cos(t)$
The complimentary solution is

$$y_c(t) = c_1 e^{8t} + c_2 e^{-8t}. \tag{6.508}$$

Our initial guess for a particular solution is

$$(At^2 + Bt + C)e^{8t} + D\cos(t) + E\sin(t) \tag{6.509}$$

Again we have a Ce^{8t} term which is also in the complimentary solution. So we need to multiply the entire first term by t, so our final guess is

$$Y_p(t) = (At^3 + Bt^2 + Ct)e^{8t} + D\cos(t) + E\sin(t). \tag{6.510}$$

(3) $y'' + 4y' = e^{-t}\cos(2t) + t\sin(2t)$
The complimentary solution is

$$y_c(t) = c_1\cos(2t) + c_2\sin(2t) \tag{6.511}$$

Our first guess for a particular solution would be

$$e^{-t}(A\cos(2t)+B\sin(2t))+(Ct+D)\cos(2t)+(Et+F)\sin(2t) \tag{6.512}$$

We notice the second and third terms contain parts of the complimentary solution so we need to multiply by t, so we have a our final guess

$$Y_p(t)=e^{-t}(A\cos(2t)+B\sin(2t))+(Ct^2+Dt)\cos(2t)+(Et^2+Ft)\sin(2t). \tag{6.513}$$

(4) $y''+2y'+5=e^{-t}\cos(2t)+t\sin(2t)$

Notice the nonhomogeneous term in this example is the same as in the previous one, but the equation has changed. Now the complimentary solution is

$$y_c(t)=c_1e^{-t}\cos(2t)+c_2e^{-t}\sin(2t) \tag{6.514}$$

So our initial guess for the particular solution is the same as the last example

$$e^{-t}(A\cos(2t)+B\sin(2t))+(Ct+D)\cos(2t)+(Et+F)\sin(2t) \tag{6.515}$$

This time the first term causes the problem, so multiply the first term by t to get the final guess

$$Y_p(t)=te^{-t}(A\cos(2t)+B\sin(2t))+(Ct+D)\cos(2t)+(Et+F)\sin(2t) \tag{6.516}$$

So even though the nonhomogeneous parts are the same the guess also depends critically on the complimentary solution and the differential equation itself.

(5) $y''+4y'+4y=t^2e^{-2t}+2e^{-2t}$

The complimentary solution is

$$y_c(t)=c_1e^{-2t}+c_2te^{-2t} \tag{6.517}$$

Notice that we can factor out a e^{-2t} from out nonhomogeneous term, which becomes $(t^2+2)e^{-2t}$. This is the product of a polynomial and an exponential, so our initial guess is

$$(At^2+Bt+C)e^{-2t} \tag{6.518}$$

But the Ce^{-2t} term is in $y_c(t)$. Also, Cte^{-2t} is in $y_c(t)$. So we must multiply by t^2 to get our final guess

$$Y_p(t)=(At^4+Bt^3+C)e^{-2t}. \tag{6.519}$$

6.10 Mechanical and Electrical Vibrations

Last Time: We studied the method of undetermined coefficients thoroughly, focusing mostly on determining guesses for particular solutions once we have solved for the complimentary solution.

6.10.1 Applications

The first application is mechanical vibrations. Consider an object of a given mass m hanging from a spring of natural length l, but there are a number of applications in engineering with the same general setup as this.

We will establish the convention that **always** the downward displacement and forces are **positive**, while upward displacements and forces are negative. BE CONSISTENT. We also measure all displacements from the equilibrium position. Thus if our displacement is $u(y)$, $u=0$ corresponds to the center of gravity as it hangs at rest from a spring.

We need to develop a differential equation to model the displacement u of the object. Recall Newton's Second Law

$$F = ma \tag{6.520}$$

where m is the mass of the object. We want our equation to be for displacement, so we'll replace a by u'', and Newton's Second Law becomes

$$F(t,u,u') = mu''. \tag{6.521}$$

What are the various forces acting on the object? We will consider four different forces, some of which may or may not be present in a given situation.

(1) **Gravity**, F_g
The gravitational force always acts on an object. It is given by

$$F_g = mg \tag{6.522}$$

where g is the acceleration due to gravity. For simpler computations, you may take $g = 10$ m/s. Notice gravity is always positive since it acts downward.

(2) **Spring**, F_s
We attach an object to a spring, and the spring will exert a force on the object. Hooke's Law governs this force. The spring force is proportional to the displacement of the spring from its natural length. What is the displacement of the spring? When we attach an object to a spring, the spring gets stretched. The length of the stretched spring is L. Then the displacement from its natural length is $L+u$.
So the spring force is

$$F_s = -k(L+u) \tag{6.523}$$

where $k > 0$ is the **spring constant**. Why is it negative? It is to make sure the force is in the correct direction. If $u > -L$, i.e. the spring has been stretched beyond its natural length, then $u+L > 0$ and so $F_s < 0$, which is what we expect because the spring would pull upward on the object in this situation. If $u < -L$, so the spring is compressed, then the spring force would push the object back downwards and we expect to find $F_s > 0$.

(3) **Damping**, F_d
We will consider some situations where the system experiences damping. This will not always be present, but always notice if damping is involved. Dampers work to counteract motion (example: shocks on a car), so this will oppose the direction of the object's velocity.
In other words, if the object has downward velocity $u' > 0$, we would want the damping force to be acting in the upwards direction, so that $F_d < 0$. Similarly, if $u' < 0$, we want $F_d > 0$. Assume all damping is linear.

$$F_d = -\gamma u' \tag{6.524}$$

where $\gamma > 0$ is the **damping constant**.

(4) **External Force**, $F(t)$
This is encompasses all other forces present in a problem. An example is a spring hooked up to a piston that exerts an extra force upon it. We call $F(t)$ the **forcing function**, and it is just the sum of any of the external forces we have in a particular problem.

The most important part of any problem is identifying all the forces involved in the problem. Some may not be present. The forces will change depending on the particular situation. Let's consider the general form of our differential equation modeling a spring system. We have

$$F(t,u,u') = F_g + F_s + F_d + F(t) \tag{6.525}$$

so that Newton's Second Law becomes

$$mu'' = mg - k(L+u) - \gamma u' + F(t), \tag{6.526}$$

or upon reordering it becomes

$$mu'' + \gamma u' + ku = mg - kL + F(t). \tag{6.527}$$

What happens when the object is at rest. Equilibrium is $u = 0$, there are only two forces acting on the object: gravity and the spring force. Since the object is at rest, these two forces must balance to 0. So $F_g + F_s = 0$. In other words,

$$mg = kL. \tag{6.528}$$

So our equation simplifies to

$$mu'' + \gamma u' + ku = F(t), \tag{6.529}$$

and this is the most general form of our equation, with all forces present. We have the corresponding initial conditions

$$u(0) = u_0 \quad \text{Initial displacement from equilibrium position} \tag{6.530}$$
$$u'(0) = u'_0 \quad \text{Initial Velocity} \tag{6.531}$$

Before we discuss individual examples, we need to touch on how we might figure out the constants k and γ if they are not explicitly given. Consider the spring constant k. We know if the spring is attached to some object with mass m, the object stretches the spring by some length L when it is at rest. We know at equilibrium $mg = kL$. Thus, if we know how much some object with a known mass stretches the spring when it is at rest, we can compute

$$k = \frac{mg}{L}. \tag{6.532}$$

How do we compute γ? If we do not know the damping coefficient from the beginning, we may know how much force a damper exerts to oppose motion of a given speed. Then set $|F_d| = \gamma|u'|$, where $|F_d|$ is the magnitude of the damping force and $|u'|$ is the speed of motion. So we have $\gamma = \frac{F_d}{u'}$. We will see how to compute in examples on damped motion. Let's consider specific spring mass systems.

6.10.2 Free, Undamped Motion

Start with free systems with no damping or external forces. This is the simplest situation since $\gamma = 0$. Our differential equation is

$$mu'' + ku = 0, \tag{6.533}$$

where $m, k > 0$. Solve by considering the characteristic equation

$$mr^2 + k = 0, \tag{6.534}$$

which has roots

$$r_{1,2} = \pm i\sqrt{\frac{k}{m}}. \tag{6.535}$$

We'll write

$$r_{1,2} = \pm i\omega_0, \tag{6.536}$$

where we've substituted

$$\omega_0 = \sqrt{\frac{k}{m}}. \tag{6.537}$$

ω_0 is called the **natural frequency** of the system, for reasons that will be clear shortly.

Since the roots of our characteristic equation are imaginary, the form of our general solution is

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \tag{6.538}$$

This is why we called ω_0 the natural frequency of the system: it is the frequency of motion when the spring-mass system has no interference from dampers or external forces.

Given initial conditions we can solve for c_1 and c_2. This is not the ideal form of the solution though since it is not easy to read off critical information. After we solve for the constants rewrite as

$$u(t) = R\cos(\omega_0 t - \delta), \tag{6.539}$$

where $R > 0$ is the **amplitude of displacement** and δ is the **phase angle of displacement**, sometimes called the **phase shift**.

Before determining how to rewrite the general solution in this desired form lets compare the two forms. When we keep it as the general solution is it easier to find the constants c_1 and c_2. But the new form is easier to work with since we can immediately see the amplitude making it much easier to graph. So ideally we will find the general solution, solve for c_1 and c_2, and then convert to the final form.

Assume we have c_1 and c_2 how do we find R and δ? Consider Equation (6.539) we can use a trig identity to write it as

$$u(t) = R\cos(\delta)\cos(\omega_0 t) + R\sin(\delta)\sin(\omega_0 t). \tag{6.540}$$

Comparing this to the general solution, we see that

$$c_1 = R\cos(\delta), \quad c_2 = R\sin(\delta). \tag{6.541}$$

Notice

$$c_1^2 + c_2^2 = R^2(\cos^2(\delta) + \sin^2(\delta)) = R^2, \tag{6.542}$$

so that, assuming $R > 0$,

$$R = \sqrt{c_1^2 + c_2^2}. \tag{6.543}$$

Also,

$$\frac{c_2}{c_1} = \frac{\sin(\delta)}{\cos(\delta)} = \tan(\delta). \tag{6.544}$$

to find δ.

■ **Example 6.65** A 2kg object is attached to a spring, which it stretches by $\frac{5}{8}m$. The object is given an initial displacement of $1m$ upwards and given an initial downwards velocity of $4m/sec$. Assuming there are no other forces acting on the spring-mass system, find the displacement of the object at time t and express it as a single cosine.

The first step is to write down the initial value problem for this setup. We'll need to find an m and k. m is easy since we know the mass of the object is $2kg$. How about k? We know

$$k = \frac{mg}{L} = \frac{(2)(10)}{\frac{5}{8}} = 32. \tag{6.545}$$

So our differential equation is

$$2u'' + 32u = 0. \tag{6.546}$$

The initial conditions are given by

$$u(0) = -1, \quad u'(0) = 4. \tag{6.547}$$

The characteristic equation is

$$2r^2 + 32 = 0, \tag{6.548}$$

and this has roots $r_{1,2} = \pm 4i$. Hence $\omega_0 = 4$. Check: $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{32/2} = 4$. So our general solution is

$$u(t) = c_1 \cos(4t) + c_2 \sin(4t). \tag{6.549}$$

Using our initial conditions, we see

$$-1 = u(0) = c_1 \tag{6.550}$$

$$4 = u'(0) = 4c_2 \Rightarrow c_2 = 1. \tag{6.551}$$

So the solution is

$$u(t) = -\cos(4t) + \sin(4t). \tag{6.552}$$

We want to write this as a single cosine. Compute R

$$R = \sqrt{c_1^2 + c_2^2} = \sqrt{2}. \tag{6.553}$$

Now consider δ

$$\tan(\delta) = \frac{c_2}{c_1} = -1. \tag{6.554}$$

So δ is in Quadrants II or IV. To decide which look at the values of $\cos(\delta)$ and $\sin(\delta)$. We have

$$\sin(\delta) = c_2 > 0 \tag{6.555}$$

$$\cos(\delta) = c_1 < 0. \tag{6.556}$$

So δ must be in Quadrant II, since there $\sin > 0$ and $\cos < 0$. If we take $\arctan(-1) = -\frac{\pi}{4}$, this has a value in Quadrant IV. Since tan is π-periodic, however, $-\frac{\pi}{4} + \pi = \frac{3\pi}{4}$ is in Quadrant II and also has a tangent of -1 Thus our desired phase angle is

$$\delta = \arctan(\frac{c_2}{c_1}) + \pi = \arctan(-1) + \pi = \frac{3\pi}{4} \tag{6.557}$$

and our solution has the final form

$$u(t) = \sqrt{2}\cos(4t - \frac{3\pi}{4}). \tag{6.558}$$

■

6.10.3 Free, Damped Motion

Now, let's consider what happens if we add a damper into the system with damping coefficient γ. We still consider free motion so $F(t) = 0$, and our differential equation becomes

$$mu'' + \gamma u' + ku = 0. \tag{6.559}$$

The characteristic equation is

$$mr^2 + \gamma r + k = 0, \tag{6.560}$$

and has solution

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}. \tag{6.561}$$

There are three different cases we need to consider, corresponding to the discriminant being positive, zero, or negative.

(1) $\gamma^2 - 4mk = 0$
This case gives a double root of $r = -\frac{\gamma}{2m}$, and so the general solution to our equation is

$$u(t) = c_1 e^{\frac{\gamma}{2m}} + c_2 t e^{-\frac{\gamma}{2m}} \tag{6.562}$$

Notice that $\lim_{t\to\infty} u(t) = 0$, which is good, since this signifies damping. This is called **critical damping** and occurs when

$$\gamma^2 - 4mk = 0 \tag{6.563}$$

$$\gamma = \sqrt{4mk} = 2\sqrt{mk} \tag{6.564}$$

This value of $\gamma - 2\sqrt{mk}$ is denoted by γ_{CR} and is called the **critical damping coefficient**. Since this case separates the other two it is generally useful to be able to calculate this coefficient for a given spring-mass system, which we can do using this formula. Critically damped systems may cross $u = 0$ once but will never cross more than that. No oscillation

(2) $\gamma^2 - 4mk > 0$
In this case, the discriminant is positive and so we will get two distinct real roots r_1 and r_2. Hence our general solution is

$$u(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \tag{6.565}$$

But what is the behavior of this solution? The solution should die out since we have damping. We need to check $\lim_{t\to\infty} u(t) = 0$. Rewrite the roots

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} \tag{6.566}$$

$$= \frac{-\gamma \pm \gamma(\sqrt{1 - \frac{4mk}{\gamma^2}})}{2m} \tag{6.567}$$

$$= -\frac{\gamma}{2m}(1 \pm \sqrt{1 - \frac{4mk}{\gamma^2}}) \tag{6.568}$$

By assumption, we have $\gamma^2 > 4mk$. Hence

$$1 - \frac{4mk}{\gamma^2} < 1 \tag{6.569}$$

and so

$$\sqrt{1-\frac{4mk}{\gamma^2}} < 1. \tag{6.570}$$

so the quantity in parenthesis above is guaranteed to be positive, which means both of our roots are negative.

Thus the damping in this case has the desired effect, and the vibration will die out in the limit. This case, which occurs when $\gamma > \gamma_{CR}$, is called **overdamping**. The solution won't oscillate around equilibrium, but settles back into place. The overdamping kills all oscillation

(3) $\gamma^2 < 4mk$

The final case is when $\gamma < \gamma_{CR}$. In this case, the characteristic equation has complex roots

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = \alpha + i\beta. \tag{6.571}$$

The displacement is

$$\begin{aligned} u(t) &= c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t) & (6.572)\\ &= e^{\alpha t}(c_1\cos(\beta t) + c_2\sin(\beta t)). & (6.573) \end{aligned}$$

In analogy to the free undamped case we can rewrite as

$$u(t) = Re^{\alpha t}\cos(\beta t - \delta). \tag{6.574}$$

We know $\alpha < 0$. Hence the displacement will settle back to equilibrium. The difference is that solutions will oscillate even as the oscillations have smaller and smaller amplitude. This is called **overdamped**.

Notice that the solution $u(t)$ is not quite periodic. It has the form of a cosine, but the amplitude is not constant. A function $u(t)$ is called **quasi-periodic**, since it oscillates with a constant frequency but a varying amplitude. β is called the **quasi-frequency** of the oscillation.

So when we have free, damped vibrations we have one of these three cases. A good example to keep in mind when considering damping is car shocks. If the shocks are new its overdamping, when you hit a bump in the road the car settles back into place. As the shocks wear there is more of an initial bump but the car still settles does not bounce around. Eventually when your shocks where and you hit a bump, the car bounces up and down for a few minutes and then settles like underdamping. The critical point where the car goes from overdamped to underdamped is the critically damped case.

Another example is a washing machine. A new washing machine does not vibrate significantly due to the presence of good dampers. Old washing machines vibrate a lot.

In practice we want to avoid underdamping. We do not want cars to bounce around on the road or buildings to sway in the wind. With critical damping we have the right behavior, but its too hard to achieve this. If the dampers wear a little we are then underdamped. In practice we want to stay overdamped.

■ **Example 6.66** A $2kg$ object stretches a spring by $\frac{5}{8}m$. A damper is attached that exerts a resistive force of $48N$ when the speed is $3m/sec$. If the initial displacement is $1m$ upwards and the initial velocity is $2m/sec$ downwards, find the displacement $u(t)$ at any time t.

This is actually the example from the last class with damping added and different initial conditions. We already know $k = 32$. What is the damping coefficient γ? We know $|F_d| = 48$ when the speed is $|u'| = 3$. So the damping coefficients is given by

$$\gamma = \frac{|F_d|}{|u'|} = \frac{48}{3} = 16. \tag{6.575}$$

Thus the initial value problem is

$$2u'' + 16u' + 32u = 0, \quad u(0) = -1, \quad u'(0) = 2. \tag{6.576}$$

Before we solve it, see which case we're in. To do so, let's calculate the critical damping coefficient.

$$\gamma_{CR} = 2\sqrt{mk} = 2\sqrt{64} = 16. \tag{6.577}$$

So we are critically damped, since $\gamma = \gamma_{CR}$. This means we will get a double root. Solving the characteristic equation we get $r_1 = r_2 = -4$ and the general solution is

$$u(t) = c_1 e^{-4t} + c_2 t e^{-4t}. \tag{6.578}$$

The initial conditions give coefficients $c_1 = -1$ and $c_2 = -2$. So the solution is

$$u(t) = -e^{-4t} - 2te^{-4t} \tag{6.579}$$

Notice there is no oscillations in this case. ∎

∎ **Example 6.67** For the same spring-mass system as in the previous example, attach a damper that exerts a force of $40N$ when the speed is $2m/s$. Find the displacement at any time t.

the only difference from the previous example is the damping force. Lets compute γ

$$\gamma = \frac{|F_d|}{|u'|} = \frac{40}{2} = 20. \tag{6.580}$$

Since we computed $\gamma_{CR} = 16$, this means we are overdamped and the characteristic equation should give us distinct real roots. The IVP is

$$2u'' + 20u' + 32u = 0, \quad u(0) = -1, \quad u(0) = 2. \tag{6.581}$$

The characteristic equation has roots $r_1 = -8$ and $r_2 = -2$. So the general solution is

$$u(t) = c_1 e^{-8t} + c_2 e^{-2t} \tag{6.582}$$

The initial conditions give $c_1 = 0$ and $c_2 = -1$, so the displacement is

$$u(t) = -e^{-2t} \tag{6.583}$$

Notice here we do not actually have a "vibration" as we normally think of them. The damper is strong enough to force the vibrations to die out so quickly that we do not notice much if any of them. ∎

∎ **Example 6.68** For the same spring-mass system as in the previous two examples, add a damper that exerts a force of $16N$ when the speed is $2m/s$.

In this case, the damping coefficient is

$$\gamma = \frac{16}{2} = 8, \tag{6.584}$$

which tells us that this case is underdamped as $\gamma < \gamma_{CR} = 16$. We should expect complex roots of the characteristic equation. The IVP is

$$2u'' + 8u' + 32u = 0, \quad u(0) = -1, \quad u'(0) = 3. \tag{6.585}$$

The characteristic equation has roots

$$r_{1,2} = \frac{-8 \pm \sqrt{192}}{4} = -2 \pm i\sqrt{12}. \tag{6.586}$$

Thus our general solution is

$$u(t) = c_1 e^{-2t} \cos(\sqrt{12}t) + c_2 e^{2t} \sin(\sqrt{12}t) \tag{6.587}$$

The initial conditions give the constants $c_1 = 1$ and $c_2 = \frac{1}{\sqrt{12}}$, so we have

$$u(t) = -e^{-2t} \cos(\sqrt{12}t) + \frac{1}{\sqrt{12}} e^{2t} \sin(\sqrt{12}t). \tag{6.588}$$

Let's write this as a single cosine

$$R = \sqrt{(-1)^2 + (\frac{1}{\sqrt{12}})^2} = \sqrt{\frac{13}{12}} \tag{6.589}$$

$$\tan(\delta) = -\frac{1}{\sqrt{12}} \tag{6.590}$$

As in the undamped case, we look at the signs of c_1 and c_2 to figure out what quadrant δ is in. By doing so, we see that δ has negative cosine and positive sine, so it is in Quadrant II. Hence we need to take the arctangent and add π to it

$$\delta = \arctan(-\frac{1}{\sqrt{12}}) + \pi. \tag{6.591}$$

Thus our displacement is

$$u(t) = \sqrt{\frac{13}{12}} e^{-2t} \cos(\sqrt{12}t - \arctan(-\frac{1}{\sqrt{12}} - \pi). \tag{6.592}$$

In this case, we actually get a vibration, even though its amplitude steadily decreases until it is negligible. The vibration has quasi-frequency $\sqrt{12}$. ■

6.10.4 Forced Vibrations

Last Time: We studied non-forced vibrations with and without damping. We studied the four forces acting on an object gravity, spring force, damping, and external forces.

Forced, Undamped Motion

What happens when the external force $F(t)$ is allowed to act on our system. The function $F(t)$ is called the **forcing function**. We will consider the undamped case

$$mu'' + ku = F(t). \tag{6.593}$$

This is a nonhomogeneous equation, so we will need to find both the complimentary and particular solution.

$$u(t) = u_c(t) + U_p(t), \tag{6.594}$$

Recall that $u_c(t)$ is the solution to the associated homogeneous equation. We will use undetermined coefficients to find the particular solution $U_p(t)$ (if $F(t)$ has an appropriate form) or variation of parameters.

We restrict our attention to the case which appears frequently in applications

$$F(t) = F_0 \cos(\omega t) \quad \text{or} \quad F(t) = F_0 \sin(\omega t) \tag{6.595}$$

The force we are applying to our spring-mass system is a simple periodic function with frequency ω. For now we assume $F(t) = F_0 \cos(\omega t)$, but everything is analogous if it is a sine function. So consider

$$mu'' + ku' = F_0 \cos(\omega t). \tag{6.596}$$

Where the complimentary solution to the analogous free undamped equation is

$$u_c(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t), \tag{6.597}$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural frequency.

We can use the method of undetermined coefficients for this nonhomogeneous term $F(t)$. The initial guess for the particular solution is

$$U_p(t) = A\cos(\omega t) + B\sin(\omega t). \tag{6.598}$$

We need to be careful, note that we are okay since $\omega_0 \neq \omega$, but if the frequency of the forcing function is the same as the natural frequency, then this guess is the complimentary solution $u_c(t)$. Thus, if $\omega_0 = \omega$, we need to multiply by a factor of t. So there are two cases.

(1) $\omega \neq \omega_0$

In this case, our initial guess is not the complimentary solution, so the particular solution will be

$$U_p(t) = A\cos(\omega t) + B\sin(\omega t). \tag{6.599}$$

Differentiating and plugging in we get

$$m\omega^2(-A\cos(\omega t) - B\sin(\omega t)) + k(A\cos(\omega t) + B\sin(\omega t)) = F_0\cos(\omega t) \tag{6.600}$$

$$(-m\omega^2 A + kA)\cos(\omega t) + (-m\omega^2 B + kB)\sin(\omega t) = F_0\cos(\omega t). \tag{6.601}$$

Setting the coefficients equal, we get

$$\cos(\omega t): \quad (-m\omega^2 + k)A = F_0 \quad \Rightarrow \quad A = \frac{F_0}{k - m\omega^2} \tag{6.602}$$

$$\sin(\omega t): \quad (-m\omega^2 + k)B = 0 \quad \Rightarrow \quad B = 0. \tag{6.603}$$

So our particular solution is

$$U_p(t) = \frac{F_0}{k - m\omega^2}\cos(\omega t) \tag{6.604}$$

$$= \frac{F_0}{m(\frac{k}{m} - \omega^2)}\cos(\omega t) \tag{6.605}$$

$$= \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t). \tag{6.606}$$

Notice that the amplitude of the particular solution is dependent on the amplitude of the forcing function F_0 and the difference between the natural frequency and the forcing frequency.

We can write our displacement function in two forms, depending on which form we use for complimentary solution.

$$u(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t) \quad (6.607)$$

$$u(t) = R\cos(\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t) \quad (6.608)$$

Again, we get an analagous solution if the forcing function were $F(t) = F_0\sin(\omega t)$.

The key feature of this case can be seen in the second form. We have two cosine functions with different frequencies. These will interfere with each other causing the net oscillation to vary between great and small amplitude. This phenomena has a name "beats" derived from musical terminology. Thing of hitting a tuning fork after it has already been struck, the volume will increase and decrease randomly. One hears the waves created here in the exact form of our solution.

(2) $\omega = \omega_0$

If the frequency of the forcing function is the same as the natural frequency, so the guess for the particular solution is

$$U_p(t) = At\cos(\omega_0 t) + Bt\sin(\omega_0 t) \quad (6.609)$$

Differentiate and plug in

$$\begin{aligned}(-m\omega_0^2 + k)At\cos(\omega_0 t) + (-m\omega_0^2 + k)Bt\sin(\omega_0 t)\\ + 2m\omega_0 B\cos(\omega_0 t) - 2m\omega_0 A\sin(\omega_0 t) = F_0\cos(\omega_0 t).\end{aligned} \quad (6.610)$$

To begin simplification recall that $\omega_0^2 = \frac{k}{m}$, so $m\omega_0^2 = k$. this means the first two terms will vanish (expected since no analogous terms on right side), and we get

$$2m\omega_0 B\cos(\omega_0 t) - 2m\omega_0 A\sin(\omega_0 t) = F_0\cos(\omega_0 t). \quad (6.611)$$

Now set the coefficients equal

$$\cos(\omega_0 t):\quad 2m\omega_0 B = F_0 \quad B = \frac{F_0}{2m\omega_0} \quad (6.612)$$

$$\sin(\omega_0 t):\quad -2m\omega_0 A = 0 \quad A = 0 \quad (6.613)$$

Thus the particular solution is

$$U_p(t) = \frac{F_0}{2m\omega_0}t\sin(\omega_0 t) \quad (6.614)$$

and the displacement is

$$u(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{2m\omega_0}t\sin(\omega_0 t) \quad (6.615)$$

or

$$u(t) = R\cos(\omega_0 t - \delta) + \frac{F_0}{2m\omega_0}t\sin(\omega_0 t). \quad (6.616)$$

What stands out most about this equation? Notice that as $t \to \infty$, $u(t) \to \infty$ due to the form of the particular solution. Thus, in the case where the forcing frequency is the same as the natural frequency, the oscillation will have an amplitude that continues to increase for all time since the external force adds energy to the system in a way that reinforces the natural motion of the system.

This phenomenon is called **resonance**. Resonance is the phenomenon behind microwave ovens. The microwave radiation strikes the water molecules in what's being heated at their natural frequency, causing them to vibrate faster and faster, which generates heat. A similar trait is noticed in the Bay of Fundy, where tidal forces cause the ocean to resonate, yielding larger and larger tides. Resonance in the ear causes us to be able to distinguish between tones in sound.

A common example is the Tacoma Narrows Bridge. This is incorrect because the oscillation that led to the collapse of the bridge was from a far more complicated phenomenon than the simple resonance we're considering now. In general, for engineering purposes, resonance is something we would like to avoid unless we understand the situation and the effect on the system.

In summary when we drive our system at a different frequency than the natural frequency, the two frequencies interfere and we observe beats in motion. When the system is driven at a natural frequency, the natural motion of the system is reinforced, causing the amplitude of the motion to increase to infinity.

■ **Example 6.69** A $3kg$ object is attached to a spring, which it stretches by $40cm$. There is no damping, but the system is forced with the forcing function

$$F(t) = 10\cos(\omega t) \tag{6.617}$$

such that the system will experience resonance. If the object is initially displaced $20cm$ downward and given an initial upward velocity of $10cm/s$, find the displacement at any time t.

We need to be aware of the units, convert all lengths to meters. Find k

$$k = \frac{mg}{L} = \frac{(3)(10)}{.4} = 75 \tag{6.618}$$

Next, we are told the system experiences resonance. Thus the forcing frequency ω must be the natural frequency ω_0.

$$\omega = \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{75}{3}} = 5 \tag{6.619}$$

Thus our initial value problem is

$$3u'' + 75u = 10\cos(5t) \quad u(0) = .2, \quad u'(0) = -.1 \tag{6.620}$$

The complimentary solution is the general solution of the associated free, undamped case. Since we have computed the natural frequency already, the complimentary solution is just

$$u_c(t) = c_1\cos(5t) + c_2\sin(5t). \tag{6.621}$$

The particular solution (using formula derived above) is

$$\frac{1}{3}t\sin(5t), \tag{6.622}$$

and so the general solution is

$$u(t) = c_1\cos(5t) + c_2\sin(5t) + \frac{1}{3}t\sin(5t). \tag{6.623}$$

The initial conditions give $c_1 = \frac{1}{5}$ and $c_2 = -\frac{1}{50}$, so the displacement can be given as

$$u(t) = \frac{1}{5}\cos(5t) - \frac{1}{50}\sin(5t) + \frac{1}{3}t\sin(5t) \tag{6.624}$$

Let's convert the first two terms to a single cosine.

$$R = \sqrt{(\frac{1}{5})^2 + (-\frac{1}{50})^2} = \sqrt{\frac{101}{2500}} \quad (6.625)$$

$$\tan(\delta) = \frac{-\frac{1}{50}}{\frac{1}{5}} = -\frac{1}{10} \quad (6.626)$$

Looking at the signs of c_1 and c_2, we see that $\cos(\delta) > 0$ and $\sin(\delta) < 0$. Thus δ is in Quadrant IV, and so we can just take the arctangent.

$$\delta = \arctan(-\frac{1}{10}) \quad (6.627)$$

The displacement is then

$$u(t) = \sqrt{\frac{101}{2500}}\cos\left(5t - \arctan(-\frac{1}{10})\right) + \frac{1}{3}t\sin(5t) \quad (6.628)$$

■

6.11 Two-Point Boundary Value Problems and Eigenfunctions

6.11.1 Boundary Conditions

Up until now, we have studied ordinary differential equations and initial value problems. Now we shift to partial differential equations and boundary value problems. Partial differential equations are much more complicated, but are essential in modeling many complex systems found in nature. We need to specify how the solution should behave on the boundary of the region our equation is defined on. The data we prescribe are the **boundary values** or **boundary conditions**, and a combination of a differential equation and boundary conditions is called a **boundary value problem**.

Boundary Conditions depend on the domain of the problem. For an ordinary differential equation our domain was usually some interval on the real line. With a partial differential equation our domain might be an interval or it might be a square in the two-dimensional plane. To see how boundary conditions effect an equation let's examine how they affect the solution of an ordinary differential equation.

■ **Example 6.70** Let's consider the second order differential equation $y'' + y = 0$. Specifying boundary conditions for this equation involves specifying the values of the solution (or its derivatives) at two points, recall this is because the equation is second order. Consider the interval $(0, 2\pi)$ and specify the boundary conditions $y(0) = 0$ and $y(2\pi) = 0$. We know the solutions to the equation have the form

$$y(x) = A\cos(x) + B\sin(x). \quad (6.629)$$

by the method of characteristics. Applying the first boundary condition we see $0 = y(0) = A$. Applying the second condition gives $0 = y(2\pi) = B\sin(2\pi)$, but $\sin(2\pi)$ is already zero so B can be any number. So the solutions to this boundary value problem are any functions of the form

$$y(x) = B\sin(x). \quad (6.630)$$

■

■ **Example 6.71** Consider $y'' + y = 0$ with boundary conditions $y(0) = y(6) = 0$. this seems similar to the previous problem, the solutions still have the general form

$$y(x) = A\cos(x) + B\sin(x) \quad (6.631)$$

and the first condition still tells us $A = 0$. The second condition tells us that $0 = y(6) = B\sin(6)$. Now since $\sin(6) \neq 0$, so we must have $B = 0$ and the entire solution is $y(x) = 0$. ■

Boundary value problems occur in nature all the time. Examine the examples physically. We know from previous chapters $y'' + y = 0$ models an oscillator such as a rock hanging from a spring. The rock will oscillate with frequency $\frac{1}{2\pi}$. The condition $y(0) = 0$ just means that when we start observing, we want the rock to be at the equilibrium spot. If we specify $y(2\pi) = 0$, this will automatically happen, since the motion is 2π periodic. On the other hand, it is impossible for the rock to return to the equilibrium point after 6 seconds. It will come back in 2π seconds, which is more than 6. So the only possible way the rock can be at equilibrium after 6 seconds is if it does not leave, which is why the only solution is the zero solution.

The previous examples are **homogeneous boundary value problems**. We say that a boundary problem is homogeneous if the equation is homogeneous and the two boundary conditions involve zero. That is, homogeneous boundary conditions might be one of these types

$$y(x_1) = 0 \qquad y(x_2) = 0 \tag{6.632}$$
$$y'(x_1) = 0 \qquad y(x_2) = 0 \tag{6.633}$$
$$y(x_1) = 0 \qquad y'(x_2) = 0 \tag{6.634}$$
$$y'(x_1) = 0 \qquad y'(x_2) = 0. \tag{6.635}$$

On the other hand, if the equation is nonhomogeneous or any of the boundary conditions do not equal zero, then the boundary value problem is **nonhomogenous** or **inhomogeneous**. Let's look at some examples of nonhomogeneous boundary value problems.

■ **Example 6.72** Take $y'' + 9y = 0$ with boundary conditions $y(0) = 2$ and $y(\frac{\pi}{6}) = 1$. The general solution to the differential equation is

$$y(x) = A\cos(3x) + B\sin(3x). \tag{6.636}$$

The two conditions give

$$2 = y(0) = A \tag{6.637}$$
$$1 = y(\frac{\pi}{6}) = B \tag{6.638}$$

so that the solution is

$$y(x) = 2\cos(3x) + \sin(3x) \tag{6.639}$$

■

■ **Example 6.73** Take $y'' + 9y = 0$ with boundary conditions $y(0) = 2$ and $y(2\pi) = 2$. The general solution to the differential equation is

$$y(x) = A\cos(3x) + B\sin(3x). \tag{6.640}$$

The two conditions give

$$2 = y(0) = A \tag{6.641}$$
$$2 = y(2\pi) = A. \tag{6.642}$$

This time the second condition did not give and new information, like in Example 1 and B does not affect whether or not the solution satisfies the boundary conditions or not. We then have infinitely many solutions of the form

$$y(x) = 2\cos(3x) + B\sin(3x) \tag{6.643}$$

■

■ **Example 6.74** Take $y'' + 9y = 0$ with boundary conditions $y(0) = 2$ and $y(2\pi) = 4$. The general solution to the differential equation is

$$y(x) = A\cos(3x) + B\sin(3x). \tag{6.644}$$

The two conditions give

$$2 = y(0) = A \tag{6.645}$$
$$4 = y(2\pi) = A. \tag{6.646}$$

On one hand, $A = 2$ and by the second equation $A = 4$. This is impossible and this boundary value problem has no solutions. ■

These examples illustrate that a small change to the boundary conditions can dramatically change the problem, unlike small changes in the initial data for initial value problems.

6.11.2 Eigenvalue Problems

Recall the system studied extensively in previous chapters

$$Ax = \lambda x \tag{6.647}$$

where for certain values of λ, called eigenvalues, there are nonzero solutions called eigenvectors. We have a similar situation with boundary value problems.

Consider the problem

$$y'' + \lambda y = 0 \tag{6.648}$$

with boundary conditions $y(0) = 0$ and $y(\pi) = 0$. The values of λ where we get nontrivial (nonzero) solutions will be **eigenvalues**. The nontrivial solutions themselves are called **eigenfunctions**.

We need to consider three cases separately.

(1) If $\lambda > 0$, then it is convenient to let $\lambda = \mu^2$ and rewrite the equation as

$$y'' + \mu^2 y = 0 \tag{6.649}$$

The characteristic polynomial is $r^2 + \mu^2 = 0$ with roots $r = \pm i\mu$. So the general solution is

$$y(x) = A\cos(\mu x) + B\sin(\mu x) \tag{6.650}$$

Note that $\mu \neq 0$ since $\lambda > 0$. Recall the boundary conditions are $y(0) = 0$ and $y(\pi) = 0$. So the first boundary condition gives $A = 0$. The second boundary condition reduces to

$$B\sin(\mu\pi) = 0 \tag{6.651}$$

For nontrivial solutions $B \neq 0$. So $\sin(\mu\pi) = 0$. Thus $\mu = 1, 2, 3, \ldots$ and thus the eigenvalues λ_n are $1, 4, 9, \ldots, n^2$. The eigenfunctions are only determined up to arbitrary constant, so convention is to choose the arbitrary constant to be 1. Thus the eigenfunctions are

$$y_1(x) = \sin(x) \quad y_2(x) = \sin(2x), \ldots, y_n(x) = \sin(nx) \tag{6.652}$$

(2) If $\lambda < 0$, let $\lambda = -\mu^2$. So the above equation becomes

$$y'' - \mu^2 y = 0 \tag{6.653}$$

The characteristic equation is $r^2 - \mu^2 = 0$ with roots $r = \pm\mu$, so its general solution can be written as

$$y(x) = A\cosh(\mu x) + B\sinh(\mu x) = Ce^{\mu x} + De^{-\mu x} \tag{6.654}$$

The first boundary condition, if considering the first form, gives $A = 0$. The second boundary condition gives $B\sinh(\mu\pi) = 0$. Since $\mu \neq 0$, then $\sinh(\mu\pi) \neq 0$, and therefore $B = 0$. So for $\lambda < 0$ the only solution is $y = 0$, there are no nontrivial solutions and thus no eigenvalues.

(3) If $\lambda = 0$, then the equation above becomes

$$y'' = 0 \tag{6.655}$$

and the general solution if we integrate twice is

$$y(x) = Ax + B \tag{6.656}$$

The boundary conditions are only satisfied when $A = 0$ and $B = 0$. So there is only the trivial solution $y = 0$ and $\lambda = 0$ is not an eigenvalue.

To summarize we only get **real** eigenvalues and eigenvectors when $\lambda > 0$. There may be complex eigenvalues. A basic problem studied later in the chapter is

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0 \tag{6.657}$$

Hence the eigenvalues and eigenvectors are

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad y_n(x) = \sin(\frac{n\pi x}{L}) \quad \text{for } n = 1, 2, 3, \ldots \tag{6.658}$$

This is the classical **Euler Buckling Problem**.

Review Euler's Equations:

Example 6.75 Consider equation of the form

$$t^2 y'' + ty' + y = 0 \tag{6.659}$$

and let $x = \ln(t)$. Then

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{1}{t}\frac{dy}{dx} \tag{6.660}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}(\frac{dy}{dx})\frac{1}{t} + \frac{dy}{dx}(\frac{1}{t})\frac{dy}{dx} \tag{6.661}$$

$$= \frac{d^2y}{dx^2}\frac{1}{t^2} + \frac{dy}{dx}(-\frac{1}{t^2}) \tag{6.662}$$

$$\tag{6.663}$$

Plug these back into the original equation

$$t^2y'' + ty + y = \frac{d^2y}{dx^2} - \frac{dy}{dx} + \frac{dy}{dx} + y = 0 \tag{6.664}$$

$$= y'' + y = 0 \tag{6.665}$$

Thus the characteristic equation is $r^2 + 1 = 0$, which has roots $r = \pm i$. So the general solution is

$$\hat{y}(x) = c_1\cos(x) + c_2\sin(x) \tag{6.666}$$

Recalling that $x = \ln(t)$ our final solution is

$$y(x) = c_1\cos(\ln(t)) + c_2\sin(\ln(t)) \tag{6.667}$$

■

6.12 Systems of Differential Equations

To this point we have focused on solving a single equation, but may real world systems are given as a system of differential equations. An example is Population Dynamics, Normally the death rate of a species is not a constant but depends on the population of predators. An example of a system of first order linear equations is

$$x_1' = 3x_1 + x_2 \tag{6.668}$$
$$x_2' = 2x_1 - 4x_2 \tag{6.669}$$

We call a system like this **coupled** because we need to know what x_1 is to know what x_2 is and vice versa. It is important to note that there will be a lot of similarities between our discussion and the previous sections on second and higher order linear equations. This is because any higher order linear equation can be written as a system of first order linear differential equations.

■ **Example 6.76** Write the following second order differential equation as a system of first order linear differential equations

$$y'' + 4y' - y = 0, \quad y(0) = 2, \quad y'(0) = -2 \tag{6.670}$$

All that is required to rewrite this equation as a first order system is a very simple change of variables. In fact, this is **ALWAYS** the change of variables to use for a problem like this. We set

$$x_1(t) = y(t) \tag{6.671}$$
$$x_2(t) = y'(t) \tag{6.672}$$

Then we have

$$x_1' = y' = x_2 \tag{6.673}$$
$$x_2' = y'' = y - 4y' = x_1 - 4x_2 \tag{6.674}$$

Notice how we used the original differential equation to obtain the second equation. The first equation, $x_1' = x_2$, is always something you should expect to see when doing this. All we have left to do is to convert the initial conditions.

$$x_1(0) = y(0) = 2 \tag{6.675}$$
$$x_2(0) = y'(0) = -2 \tag{6.676}$$

Thus our original initial value problem has been transformed into the system

$$x_1' = x_2, \quad x_1(0) = 2 \tag{6.677}$$
$$x_2' = x_1 - 4x_2, \quad x_2(0) = -2 \tag{6.678}$$

■

Let's do an example for higher order linear equations.

■ **Example 6.77**

$$y^{(4)} + ty''' - 2y'' - 3y' - y = 0 \tag{6.679}$$

as a system of first order differential equations.

We want to use a similar change of variables as the previous example. The only difference is that since our equation in this example is fourth order we will need four new variables instead of two.

$$x_1 = y \tag{6.680}$$
$$x_2 = y' \tag{6.681}$$
$$x_3 = y'' \tag{6.682}$$
$$x_4 = y''' \tag{6.683}$$

Then we have

$$x_1' = y' = x_2 \tag{6.684}$$
$$x_2' = y'' = x_3 \tag{6.685}$$
$$x_3' = y''' = x_4 \tag{6.686}$$
$$x_4' = y^{(4)} = y + 3y' + 2y'' - ty''' = x_1 + 3x_2 + 2x_3 - tx_4 \tag{6.687}$$

as our system of equations. To be able to solve these, we need to review some facts about systems of equations and linear algebra. ■

6.13 Homogeneous Linear Systems with Constant Coefficients

A two-dimensional equation has the form

$$x' = ax + by \tag{6.688}$$
$$y' = cx + dy \tag{6.689}$$

Suppose we have got our system written in matrix form

$$x' = Ax \tag{6.690}$$

How do we solve this equation? If A were a 1×1 matrix, i.e. a constant, and x were a vector with 1 component, the differential equation would be the separable equation

$$x' = ax \tag{6.691}$$

We know this is solved by

$$x(t) = ce^{at}. \tag{6.692}$$

One might guess, then, that in the $n \times n$ case, instead of a we have some other constant in the exponential, and instead of the constant of integration c we have some constant vector η. So our guess for the solution will be

$$x(t) = \eta e^{rt}. \tag{6.693}$$

Plugging the guess into the differential equation gives

$$r\eta e^{rt} = A\eta e^{rt} \tag{6.694}$$
$$(A\eta - r\eta)e^{rt} = 0 \tag{6.695}$$
$$(A - rI)\eta e^{rt} = 0. \tag{6.696}$$

Since $e^{rt} \neq 0$, we end up with the requirement that

$$(A - rI)\eta = 0 \tag{6.697}$$

This should seem familiar, it is the condition for η to be an eigenvector of A with eigenvalue r. Thus, we conclude that for (6.693) to be a solution of the original differential equation, we must have η an eigenvalue of A with eigenvalue r.

That tells us how to get some solutions to systems of differential equations, we find the eigenvalues and vectors of the coefficient matrix A, then form solutions using (6.693). But how will we form the general solution?

Thinking back to the second/higher order linear case, we need enough linearly independent solutions to form a fundamental set. As we noticed last lecture, if we have all simple eigenvalues, then all the eigenvectors are linearly independent, and so the solutions formed will be as well. We will handle the case of repeated eigenvalues later.

So we will find the fundamental solutions of the form (6.693), then take their linear combinations to get our general solution.

6.13.1 The Phase Plane

We are going to rely on qualitatively understanding what solutions to a linear system of differential equations look like, this will be important when considering nonlinear equations. We know the trivial solution $x = 0$ is always a solution to our homogeneous system $x' = Ax$. $x = 0$ is an example of an **equilibrium solution**, i.e. it satisfies

$$x' = Ax = 0 \tag{6.698}$$

and is a constant solution. We will assume our coefficient matrix A is nonsingular ($\det(A) \neq 0$), thus $x = 0$ is the only equilibrium solution.

The question we want to ask is whether other solutions move towards or away from this constant solution as $t \to \pm\infty$, so that we can understand the long term behavior of the system. This is no different than what we did when we classified equilibrium solutions for first order autonomous equations, we will generalize the ideas to systems of equations.

When we drew solution spaces then, we did so on the ty-plane. To do something analogous we would require three dimensions, since we would have to sketch both x_1 and x_2 vs. t. Instead, what we do is ignore t and think of our solutions as trajectories on the x_1x_2-plane. Then our equilibrium solution is the origin. The x_1x_2-plane is called the **phase plane**. We will see examples where we sketch solutions, called **phase portraits**.

6.13.2 Real, Distinct Eigenvalues

Lets get back to the equation $x' = Ax$. We know if λ_1 and λ_2 are real and distinct eigenvalues of the 2×2 coefficient matrix A associated with eigenvectors $\eta^{(1)}$ and $\eta^{(2)}$, respectively. We know from above $\eta^{(1)}$ and $\eta^{(2)}$ are linearly independent, as λ_1 and λ_2 are simple. Thus the solutions obtained from them using (6.693) will also be linearly independent, and in fact will form a fundamental set of solutions. The general solution is

$$x(t) = c_1 e^{\lambda_1 t}\eta^{(1)} + c_2 e^{\lambda_2 t}\eta^{(2)} \tag{6.699}$$

So if we have real, distinct eigenvalues, all that we have to do is find the eigenvectors, form the general solution as above, and use any initial conditions that may exist.

■ **Example 6.78** Solve the following initial value problem

$$x' = \begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix} x \quad x(0) = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{6.700}$$

The first thing we need to do is to find the eigenvalues of the coefficient matrix.

$$0 = \det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} \tag{6.701}$$
$$= \lambda^2 + \lambda - 6 \tag{6.702}$$
$$= (\lambda - 2)(\lambda + 3) \tag{6.703}$$

So the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -3$. Next we need the eigenvectors.

(1) $\lambda_1 = 2$

$$(A - 2I)\eta = 0 \tag{6.704}$$
$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6.705}$$

So we will want to find solutions to the system

$$-4\eta_1 + 2\eta_2 = 0 \tag{6.706}$$
$$2\eta_1 - \eta_2 = 0. \tag{6.707}$$

Using either equation we find $\eta_2 = 2\eta_1$, and so any eigenvector has the form

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ 2\eta_1 \end{pmatrix} \tag{6.708}$$

Choosing $\eta_1 = 1$ we obtain the first eigenvector

$$\eta^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{6.709}$$

(2) $\lambda_2 = -3$

$$(A + 3I)\eta = 0 \tag{6.710}$$
$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6.711}$$

So we will want to find solutions to the system

$$\eta_1 + 2\eta_2 = 0 \tag{6.712}$$
$$2\eta_1 + 4\eta_2 = 0. \tag{6.713}$$

Using either equation we find $\eta_1 = -2\eta_2$, and so any eigenvector has the form

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -2\eta_2 \\ \eta_2 \end{pmatrix}. \tag{6.714}$$

Choosing $\eta_2 = 1$ we obtain the second eigenvector

$$\eta^{(2)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}. \tag{6.715}$$

Thus our general solution is

$$x(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}. \tag{6.716}$$

Now let's use the initial condition to solve for c_1 and c_2. The condition says

$$\begin{pmatrix} 5 \\ 0 \end{pmatrix} = x(0) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}. \tag{6.717}$$

All that's left is to write out is the matrix equation as a system of equations and then solve.

$$c_1 - 2c_2 = 5 \tag{6.718}$$
$$2c_1 + c_2 = 0 \quad \Rightarrow c_1 = 1, c_2 = -2 \tag{6.719}$$

Thus the particular solution is

$$x(t) = e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2e^{-3t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}. \tag{6.720}$$

■

■ **Example 6.79** Sketch the phase portrait of the system from Example 1.

In the last example we saw that the eigenvalue/eigenvector pairs for the coefficient matrix were

$$\lambda_1 = 2 \qquad \eta^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{6.721}$$
$$\lambda_2 = -3 \qquad \eta^{(2)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}. \tag{6.722}$$

The starting point for the phase portrait involves sketching solutions corresponding to the eigenvectors (i.e. with c_1 or $c_2 = 0$). We know that if $x(t)$ is one of these solutions

$$x'(t) = Ac_i e^{\lambda_i t} \eta^{(i)} = c_i \lambda_i e^{\lambda_i t} \eta^{(i)}. \tag{6.723}$$

This is just, for any t, a constant times the eigenvector, which indicates that lines in the direction of the eigenvector are these solutions to the system. There are called **eigensolutions** of the system.

Next, we need to consider the direction that these solutions move in. Let's start with the first eigensolution, which corresponds to the solution with $c_2 = 0$. The first eigenvalue is $\lambda_1 = 2 > 0$. This indicates that this eigensolution will grow exponentially, as the exponential in the solution has a positive exponent. The second eigensolution corresponds to $\lambda_2 = -3 < 0$, so the exponential in the appropriate solution is negative. Hence this solution will decay and move towards the origin.

What does the typical trajectory do (i.e. a trajectory where both $c_1, c_2 \neq 0$)? The general solution is

$$x(t) = c_1 e^{2t} \eta^{(1)} + c_2 e^{-3t} \eta^{(2)}. \tag{6.724}$$

Thus as $t \to \infty$, this solution will approach the positive eigensolution, as the component corresponding to the negative eigensolution will decay away. On the other hand, as $t \to -\infty$, the trajectory will asymptotically reach the negative eigensolution, as the positive eigensolution component will be tiny. The end result is the phase portrait as in Figure 1. When the phase portrait looks like this (which happens in all cases with eigenvalues of mixed signs), the equilibrium solution at the origin is classified as a **saddle point** and is **unstable**.

■

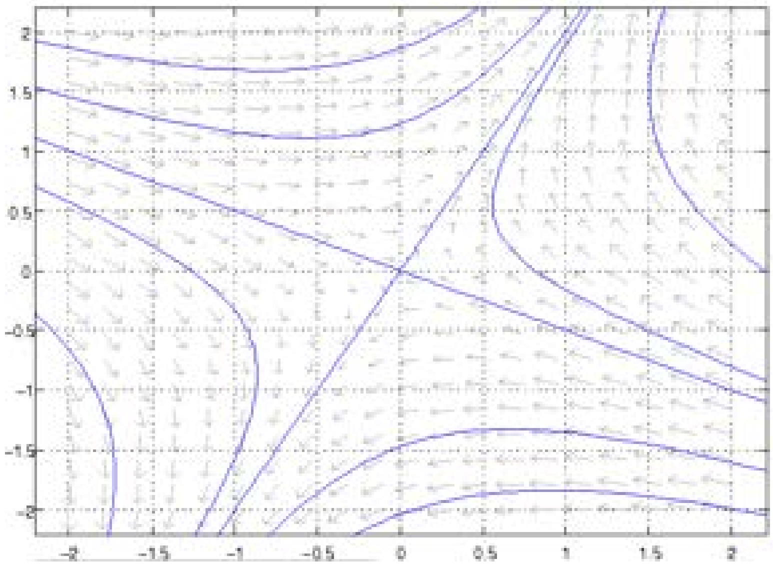


Figure 6.4: Phase Portrait of the saddle point in Example 1

■ **Example 6.80** Solve the following initial value problem.

$$x_1' = 4x_1 + x_2 \quad x_1(0) = 6 \tag{6.725}$$
$$x_2' = 3x_1 + 2x_2 \quad x_2(0) = 2 \tag{6.726}$$

Before we can solve anything, we need to convert this system into matrix form. Doing so converts the initial value problem to

$$x' = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} x \quad x(0) = \begin{pmatrix} 6 \\ 2 \end{pmatrix}. \tag{6.727}$$

To solve, the first thing we need to do is to find the eigenvalues of the coefficient matrix.

$$0 = \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} \tag{6.728}$$
$$= \lambda^2 - 6\lambda + 5 \tag{6.729}$$
$$= (\lambda - 1)(\lambda - 5) \tag{6.730}$$

So the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 5$. Next, we find the eigenvectors.

(1) $\lambda_1 = 1$

$$(A - I)\eta = 0 \tag{6.731}$$
$$\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6.732}$$

So we will want to find solutions to the system

$$3\eta_1 + \eta_2 = 0 \tag{6.733}$$
$$3\eta_1 + \eta_2 = 0. \tag{6.734}$$

Using either equation we find $\eta_2 = -3\eta_1$, and so any eigenvector has the form

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ -3\eta_1 \end{pmatrix} \tag{6.735}$$

Choosing $\eta_1 = 1$ we obtain the first eigenvector

$$\eta^{(1)} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}. \tag{6.736}$$

(2) $\lambda_2 = 5$

$$(A - 5I)\eta = 0 \tag{6.737}$$

$$\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6.738}$$

So we will want to find solutions to the system

$$-\eta_1 + \eta_2 = 0 \tag{6.739}$$

$$3\eta_1 - 3\eta_2 = 0. \tag{6.740}$$

Using either equation we find $\eta_1 = \eta_2$, and so any eigenvector has the form

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_2 \\ \eta_2 \end{pmatrix}. \tag{6.741}$$

Choosing $\eta_2 = 1$ we obtain the second eigenvector

$$\eta^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{6.742}$$

Thus our general solution is

$$x(t) = c_1 e^t \begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{6.743}$$

Now using our initial conditions we solve for c_1 and c_2. The condition gives

$$\begin{pmatrix} 6 \\ 2 \end{pmatrix} = x(0) = c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{6.744}$$

All that is left is to write out this matrix equation as a system of equations and then solve

$$c_1 + c_2 = 6 \tag{6.745}$$

$$-3c_1 + c_2 = 2 \Rightarrow c_1 = 1, c_2 = 5 \tag{6.746}$$

Thus the particular solution is

$$x(t) = e^t \begin{pmatrix} 1 \\ -3 \end{pmatrix} + 5e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{6.747}$$

■

■ **Example 6.81** Sketch the phase portrait of the system from Example 3.

In the last example, we saw that the eigenvalue/eigenvector pairs for the coefficient matrix were

$$\lambda_1 = 1 \qquad \eta^{(1)} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}. \tag{6.748}$$

$$\lambda_2 = 5 \qquad \eta^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{6.749}$$

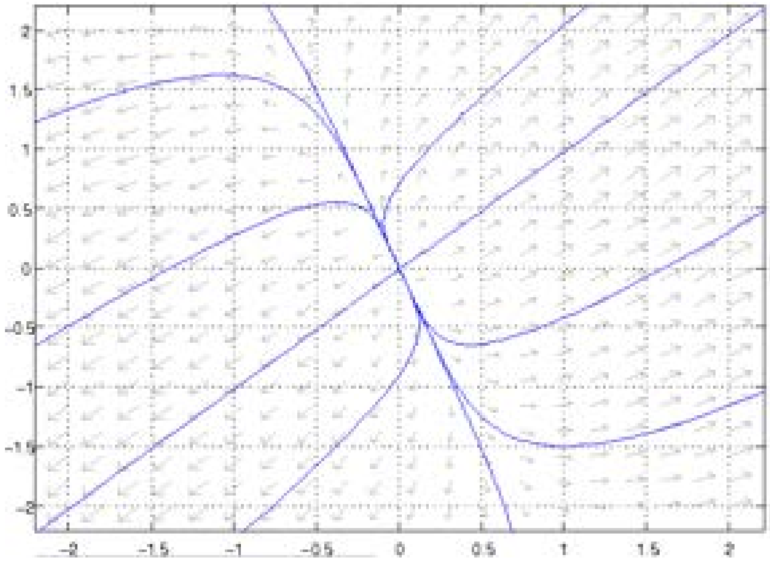


Figure 6.5: Phase Portrait of the unstable node in Example 2

We begin by sketching the eigensolutions (these are straight lines in the directions of the eigenvectors). Both of these trajectories move away from the origin, though, as the eigenvalues are both positive.

Since $|\lambda_2| > |\lambda_1|$, we call the second eigensolution the **fast eigensolution** and the first one the **slow eigensolution**. The term comes from the fact that the eigensolution corresponds to the eigenvalue with larger magnitude will either grow or decay more quickly than the other one.

As both grow in forward time, asymptotically, as $t \to \infty$, the fast eigensolution will dominate the typical trajectory, as it gets larger much more quickly than the slow eigensolution does. So in forward time, other trajectories will get closer and closer to the eigensolution corresponding to $\eta^{(2)}$. On the other hand, as $t \to -\infty$, the fast eigensolution will decay more quickly than the slow one, and so the eigensolution corresponding to $\eta^{(1)}$ will dominate in backwards time.

Thus the phase portrait will look like Figure 2. Whenever we have two positive eigenvalues, every solution moves away from the origin. We call the equilibrium solution at the origin, in this case, a **node** and classify it as being **unstable**.

■

■ **Example 6.82** Solve the following initial value problem.

$$x_1' = -5x_1 + x_2 \quad x_1(0) = 2 \tag{6.750}$$

$$x_2' = 2x_1 - 4x_2 \quad x_2(0) = -1 \tag{6.751}$$

We convert this system into matrix form.

$$x' = \begin{pmatrix} -5 & 1 \\ 2 & -4 \end{pmatrix} x \quad x(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}. \tag{6.752}$$

To solve, the first thing we need to do is to find the eigenvalues of the coefficient matrix.

$$0 = \det(A - \lambda I) = \begin{vmatrix} -5-\lambda & 1 \\ 2 & -4-\lambda \end{vmatrix} \tag{6.753}$$

$$= \lambda^2 + 9\lambda + 18 \tag{6.754}$$

$$= (\lambda + 3)(\lambda + 6) \tag{6.755}$$

So the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -6$. Next, we find the eigenvectors.

(1) $\lambda_1 = -3$

$$(A+3I)\eta = 0 \tag{6.756}$$

$$\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6.757}$$

So we will want to find solutions to the system

$$-2\eta_1 + \eta_2 = 0 \tag{6.758}$$

$$2\eta_1 - \eta_2 = 0. \tag{6.759}$$

Using either equation we find $\eta_2 = 2\eta_1$, and so any eigenvector has the form

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ 2\eta_1 \end{pmatrix} \tag{6.760}$$

Choosing $\eta_1 = 1$ we obtain the first eigenvector

$$\eta^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{6.761}$$

(2) $\lambda_2 = -6$

$$(A+6I)\eta = 0 \tag{6.762}$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6.763}$$

So we will want to find solutions to the system

$$\eta_1 + \eta_2 = 0 \tag{6.764}$$

$$2\eta_1 + 2\eta_2 = 0. \tag{6.765}$$

Using either equation we find $\eta_1 = -\eta_2$, and so any eigenvector has the form

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\eta_2 \\ \eta_2 \end{pmatrix}. \tag{6.766}$$

Choosing $\eta_2 = 1$ we obtain the second eigenvector

$$\eta^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{6.767}$$

Thus our general solution is

$$x(t) = c_1 e^{-3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-6t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{6.768}$$

Now using our initial conditions we solve for c_1 and c_2. The condition gives

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} = x(0) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{6.769}$$

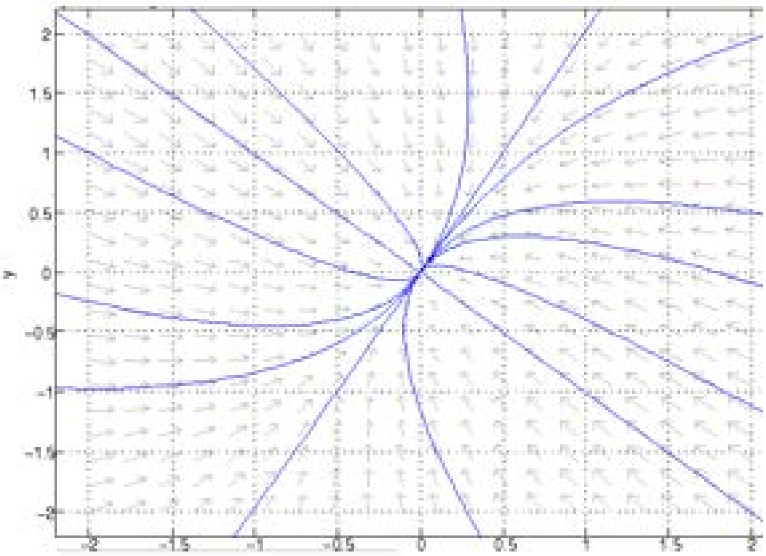


Figure 6.6: Phase Portrait of the Stable Node in Example 3

All that is left is to write out this matrix equation as a system of equations and then solve

$$c_1 - c_2 = 2 \tag{6.770}$$

$$2c_1 + c_2 = -1 \Rightarrow c_1 = \frac{1}{3}, c_2 = -\frac{5}{3} \tag{6.771}$$

Thus the particular solution is

$$x(t) = \frac{1}{3}e^{-3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{5}{3}e^{-6t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{6.772}$$

■

■ **Example 6.83** Sketch the phase portrait of the system from Example 5.

In the last example, we saw that the eigenvalue/eigenvector pairs for the coefficient matrix were

$$\lambda_1 = -3 \qquad \eta^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{6.773}$$

$$\lambda_2 = -6 \qquad \eta^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{6.774}$$

We begin by sketching the eigensolutions. Both of these trajectories decay towards the origin, since both eigenvalues are negative. Since $|\lambda_2| > |\lambda_1|$, the second eigensolution is the fast eigensolution and the first one the slow eigensolution. In the general solution, both exponentials are negative and so every solution will decay and move towards the origin. Asymptotically, as $t \to \infty$ the trajectory gets closer and closer to the origin, the slow eigensolution will dominate the typical trajectory, as it dies out less quickly than the fast eigensolution. So in forward time, other trajectories will get closer and closer to the eigensolution corresponding to $\eta^{(1)}$. On the other hand, as $t \to -\infty$, the fast solution will grow more quickly than the slow one, and so the eigensolution corresponding to $\eta^{(2)}$ will dominate in backwards time.

Thus the phase portrait will look like Figure 3. Whenever we have two negative eigenvalues, every solution moves toward the origin. We call the equilibrium solution at the origin, in this case, a **node** and classify it as being **asymptotically stable**.

■

V Part Five: PDEs and Fourier Series

7. Fourier Series and Transforms

7.1 Introduction to Fourier Series

Fourier series have many applications in the physical sciences. For example, in vibrations and oscillations $\rightarrow$ Think Frequency! The key idea behind Fourier series is to provide an alternative way of expressing a function to the traditional power series, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}$. Instead let us express a function as a sum of sines and cosines

$$f(x) = \sum_{m=0}^{\infty} \left[a_n \cos(m\pi x) + b_n \sin(m\pi x)\right]. \tag{7.1}$$

This does remarkably well at approximating functions. Notice though that the resulting function is periodic. Fourier series are also a key tool in solving PDEs (more later).

7.1.1 Simple Harmonic Motion

Imagine a particle at point p moves around the circle at a constant speed. Let the mass be the projection of p onto a vertical line (spring-mass system). Also, let ω be the angular velocity of p. Thus, $\theta = \omega t$ and the position of the mass is $c = \sin(\theta) = \omega t$. The motion of the point c traces out a sine curve and this motion is called *Simple Harmonic Motion*.

Definition 7.1.1 *(Simple Harmonic Motion)* Simple harmonic motion can take the forms:

$$\sin(\omega t), \quad \cos(\omega t), \quad \sin(\omega t + \phi), \tag{7.2}$$

where ω is the angular velocity and ϕ is the phase (horizontal displacement). Traditional examples include a hanging mass from a spring a pendulum, and a tuning fork.

The location of the point $p = (A\cos(\omega t), A\sin(\omega t)) = Ae^{i\omega t}$ if $P = x + iy$. Here A is the amplitude or the maximum displacement of the object. We also can write down the equation for the motion of the point c

$$\frac{dy}{dt} = \frac{d}{dt}(A\sin(\omega t)) = A\omega\cos(\omega t) = B\cos(\omega t),$$

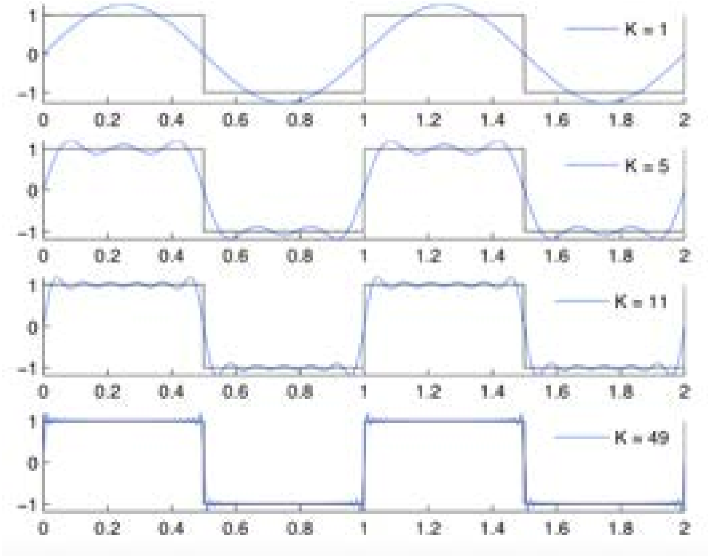
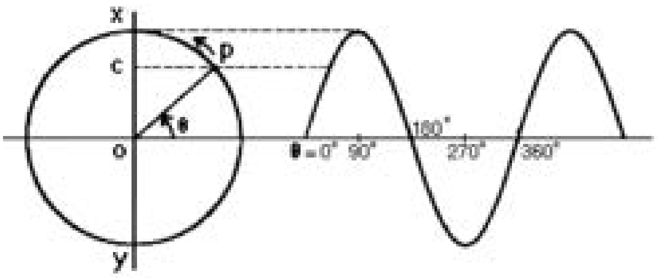
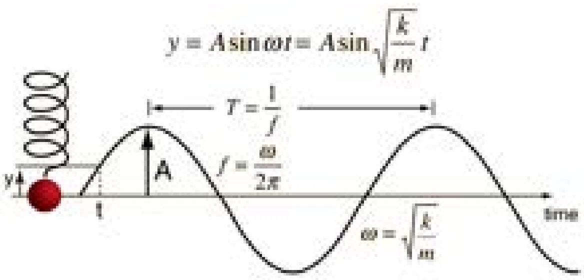


Figure 7.1: Convergence of Fourier Series to a step function (square wave). K is the number of terms.



a)



b)

Figure 7.2: Depiction of simple harmonic motion using a point rotating around the unit circle or a spring mass system.

where $B = A\omega$ is the maximum velocity achieved by the object. In physics we can define the kinetic energy:

$$\text{KE} = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 = \frac{1}{2}mB^2\cos^2(\omega t) \leq \frac{1}{2}mB^2.$$

Therefore, the maximum energy of the system, $\frac{1}{2}mB^2$, is proportional to the maximum velocity B (and therefore the amplitude A).

Since sine/cosine are periodic, one we know the value on the interval $[0, 2\pi)$ (or $[0, L)$), then we know the value everywhere.

■ **Example 7.1** Consider the general function representing simple harmonic motion

$$y(x,t) := A\cos\left[2\pi\left(\frac{x}{\lambda} - ft\right)\right] = A\cos\left[\frac{2\pi}{\lambda}(x - vt)\right]. \tag{7.3}$$

Here A is the amplitude, λ is the wavelength, $f = \frac{\omega}{2\pi}$ is the frequency, $T = \frac{1}{f} = \frac{2\pi}{\omega}$ is the period, and $v = f\lambda$ is the velocity. ■

From an ODE perspective (think last chapter) we have an equation for simple harmonic motion:

$$F_{net} = m\frac{d^2x}{dt^2} = -kx \qquad \text{(Hooke's Law)} \quad \Rightarrow \quad x'' + \frac{k}{m}x = 0. \tag{7.4}$$

Solving this equation with the characteristic polynomial gives $r^2 + \frac{k}{m} = 0$, which implies that $r = \sqrt{\frac{k}{m}}i$. Thus, the general solution is

$$x(t) = C_1\cos(\omega t) + C_2\sin(\omega t) = A\cos(\omega t - \phi), \tag{7.5}$$

where angular velocity $\omega = \sqrt{\frac{k}{m}}$, the amplitude $A = \sqrt{C_1^2 + C_2^2}$ (depends on initial/boundary conditions), and the phase $\phi = \tan^{-1}\left(\frac{C_2}{C_1}\right)$. The frequency $f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ and the period $T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}}$.

Many common periodic functions are not continuous or differentiable such as the square wave, the sawtooth, or the rectified half wave (semi-circle wave). **Problem:** Given a function $f(x)$ how an we expand it into a series of sines and cosines.

7.2 Fourier Coefficients

7.3 Fourier Coefficients

This section follows the outline and layout of Kreyzig Chapter 11. Recall that **Fourier series** are infinite series designed to represent general periodic functions in terms of simpler ones (sines/cosines). The immense theory behind Fourier series can seem complicated, but the application of Fourier series to real problems is much simpler. Fourier series have a distinct advantage over Taylor series in that even discontinuous functions have a Fourier series representation while they do not possess a Taylor series.

The main use of Fourier series are in representing periodic functions.

Definition 7.3.1 *(Periodic Functions)* A function $f(x)$ is called a **periodic function** if $f(x)$ is defined for all real x and if there is some positive number p called the *period* of $f(x)$, such that

$$f(x+p) = f(x) \tag{7.6}$$

for all x. The graph of such a function is obtained from periodic repetition of its graph over any interval of length p.

Examples of periodic functions are $f(x) = \sin(x), \cos(x)$ and examples that are not periodic are $f(x) = x^n, e^x, \cosh(x), \ln(x)$.

R If $f(x)$ has a period p, then it also has a period of $2p$ (and np for any integer $n > 0$). Since

$$f(x+2p) = f([x+p]+p) = f(x+p) = f(x), \qquad \Rightarrow \qquad f(x+np) = f(x).$$

Furthermore, if functions $f(x)$ and $g(x)$ have period p, then $af(x) + bg(x)$ with any constants a and b also has period p.

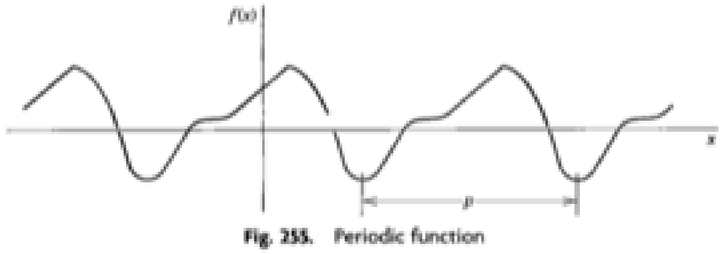


Figure 7.3: Periodic Function. Image from Kreyzig Adv. Engineering Math

Problem: We want to find a representation of a 2π-periodic function in terms of simple functions

$$1, \cos(x), \sin(x), \cos(2x), \sin(2x), ..., \cos(nx), \sin(nx), ..., \tag{7.7}$$

which are all 2π-periodic (see Figure!7.4). The associated series made up of these terms is called a **trigonometric series** and has the form

$$a_0 + a_1 \cos(x) + b_1 \sin(x) + a_2 \cos(2x) + b_2 \sin(2x) + ... = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]. \tag{7.8}$$

where $a_0, a_1, b_1, a_2, b_2, ...$ are all constants called the **coefficients** of the series. Since each term has a period of 2π, then if the series converges the result must also have a period of 2π! Given a function $f(x)$ of period 2π and such that it can be represented by a series of this form, that series converges, and has the sum $f(x)$, then we can use equality to write

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]. \tag{7.9}$$

Thus the righthand side is called the **Fourier series** of $f(x)$.

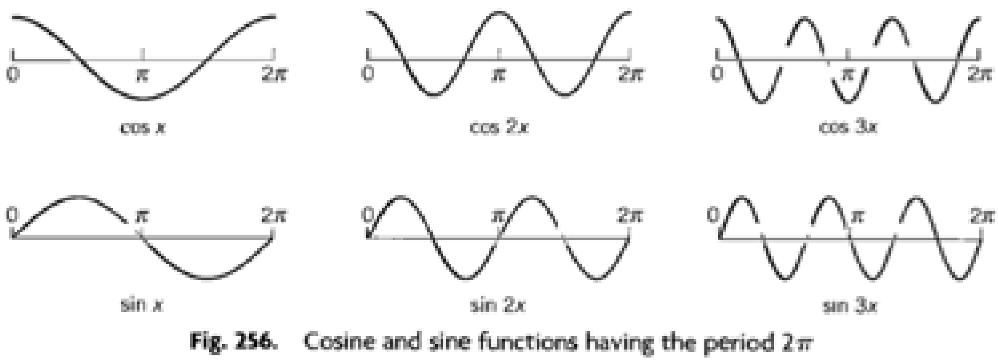


Figure 7.4: Periodic Trig Function. Image from Kreyzig Adv. Engineering Math

Question: How do we find the coefficients a_i, b_i for $i = 0, ..., n$?

Solution: These constants are the so-called **Fourier coefficients** of $f(x)$, given by **Euler formulas**

Theorem 7.3.1 *(Euler Formulas for Fourier Coefficients)*

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \qquad n = 1,2,\ldots$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \qquad n = 1,2,\ldots$$

R The Fourier coefficient a_0 is actually the average value of the function over its period! In other words, if we only take the first term in the Fourier series the function $f(x)$ is approximated by its average value over its period. We can think of this as a "First Approximation".

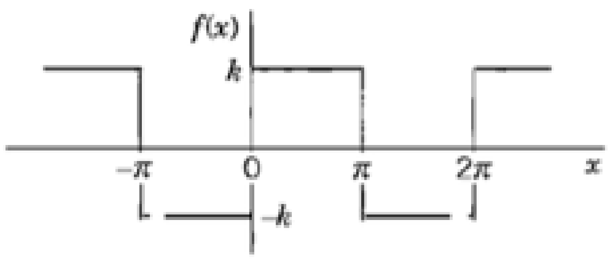
7.3.1 A Basic Example

Start by considering some basic questions when it comes to Fourier series:
1. How are continuous functions able to represent a given discontinuous function?
2. How does the quality of the approximation increase as one takes more and more terms in the Fourier series?

■ **Example 7.2** *(Periodic Rectangular Wave)* Find the Fourier coefficients of the periodic function

$$f(x) = \begin{cases} -k & \text{if} \quad -\pi < x < 0 \\ k & \text{if} \quad 0 < x < \pi \end{cases} .$$

This is a typical function representing an external force acting on a mechanical system or electric circuit (see Fig. 7.5).



(a) The given function $f(x)$ (Periodic rectangular wave)

Figure 7.5: Square wave periodic function. Image from Kreyzig Adv. Engineering Math

Solution: Start by using the formulas for the Fourier coefficients:

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{0} -kdx + \frac{1}{2\pi}\int_{0}^{\pi} kdx \\
&= \frac{1}{2\pi}\left[-kx\Big|_{-\pi}^{0} + kx\Big|_{0}^{\pi}\right] = \frac{1}{2\pi}[-k\pi + k\pi] = 0 \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\int_{-\pi}^{0} -k\cos(nx)dx + \frac{1}{\pi}\int_{0}^{\pi} k\cos(nx)dx \\
&= \frac{1}{\pi}\left[-\frac{k}{n}\sin(nx)\Big|_{-\pi}^{0} + \frac{k}{n}\sin(nx)\Big|_{0}^{\pi}\right] = \frac{1}{\pi}[0+0] = 0 \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\int_{-\pi}^{0} -k\sin(nx)dx + \frac{1}{\pi}\int_{0}^{\pi} k\sin(nx)dx \\
&= \frac{1}{\pi}\left[\frac{k}{n}\cos(nx)\Big|_{-\pi}^{0} - \frac{k}{n}\cos(nx)\Big|_{0}^{\pi}\right] = \frac{k}{n\pi}[1-\cos(-n\pi)-\cos(n\pi)+1] = \frac{2k}{n\pi}(1-\cos(n\pi)).
\end{aligned}$$

Hence, the Fourier sine coefficients are $b_{2n} = 0$ and $b_{2n+1} = \frac{4k}{(2n+1)\pi}$. Thus, the Fourier series is

$$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)] = \frac{4k}{\pi}\left(\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \dots\right).$$

A picture depicting the approximation with the partial sums is shown in Fig. 7.6.

■

7.3.2 Derivation of Euler Formulas

In order to derive the formulas for the Fourier coefficients we heavily rely on the following result about the orthogonality of trigonometric functions.

Theorem 7.3.2 *(Orthogonality of Trigonometric Functions)* On the interval $(-\pi \le x \le \pi)$ the following relations hold:

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = 0 \qquad (n \neq m) \tag{7.10}$$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = 0 \qquad (n \neq m) \tag{7.11}$$

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)dx = 0 \qquad (n \neq m \text{ or } n = m). \tag{7.12}$$

This is proved by transforming each product into a single trig function using the sum/difference formulas.

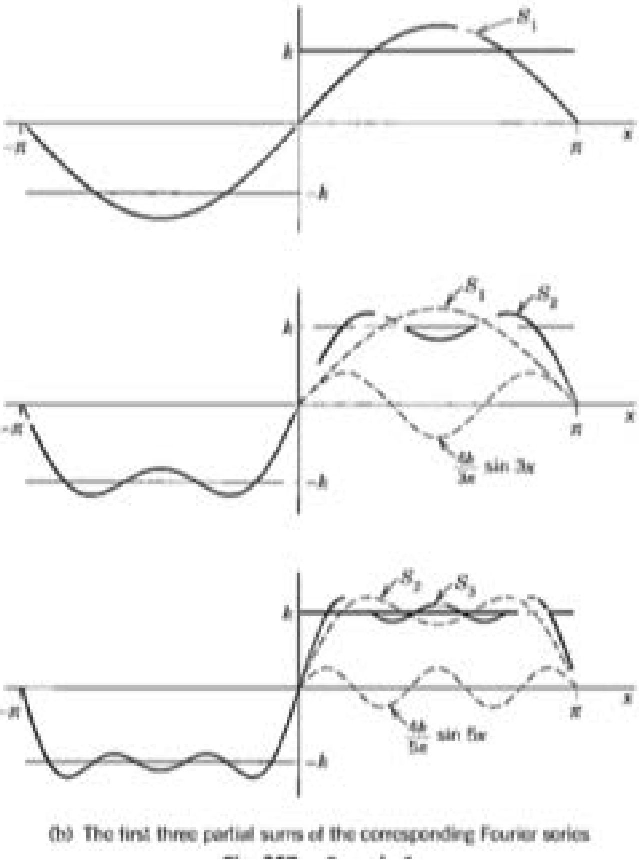


Figure 7.6: Convergence of Fourier series to a periodic function. Image from Kreyzig Adv. Engineering Math

Now apply Theorem 7.3.2 to Fourier series.

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

Integrate both sides
$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} \left(a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \right) dx$$
$$\int_{-\pi}^{\pi} f(x)dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \cos(nx)dx + b_n \int_{-\pi}^{\pi} \sin(nx) \right] dx$$
$$\int_{-\pi}^{\pi} f(x)dx = 2\pi a_0 + 0.$$

This gives the formula for $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx$ since all the integrals in the sum are zero.

Now to get the formula for a_n we repeat this process, but multiply both sides of the expression for a Fourier series by $\cos(mx)$ before integrating.

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

$$\int_{-\pi}^{\pi} f(x)\cos(mx)dx = \int_{-\pi}^{\pi} \left(a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \right) \cos(mx)dx$$

$$\int_{-\pi}^{\pi} f(x)\cos(mx)dx = a_0 \int_{-\pi}^{\pi} \cos(mx)dx + \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx + b_n \int_{-\pi}^{\pi} \sin(nx)\cos(mx) \right] dx$$

$$\int_{-\pi}^{\pi} f(x)\cos(nx)dx = a_n \int_{-\pi}^{\pi} \cos^2(nx)dx = a_n \pi.$$

This gives the formula for $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)dx$.

Now to get the formula for b_n we repeat this process, but multiply both sides of the expression for a Fourier series by $\sin(mx)$ before integrating.

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

$$\int_{-\pi}^{\pi} f(x)\sin(mx)dx = \int_{-\pi}^{\pi} \left(a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \right) \sin(mx)dx$$

$$\int_{-\pi}^{\pi} f(x)\sin(mx)dx = a_0 \int_{-\pi}^{\pi} \sin(mx)dx + \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \cos(nx)\sin(mx)dx + b_n \int_{-\pi}^{\pi} \sin(nx)\sin(mx) \right] dx$$

$$\int_{-\pi}^{\pi} f(x)\sin(nx)dx = b_n \int_{-\pi}^{\pi} \sin^2(nx)dx = b_n \pi.$$

This gives the formula for $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)dx$.

■ **Example 7.3** The *fundamental period* is the smallest positive period. Find it for: $\cos(x), \cos(2x)$, $\sin(\pi x), \sin(2\pi x)$.

Solution: For $\cos(x)$ we know the period is 2π. To find the fundamental period for $\cos(kx)$ we need to find when $kx = 2\pi$ or $x = \frac{2\pi}{k}$. So the fundamental period for $\cos(2x)$ is $\frac{2\pi}{k} = \pi$. The fundamental period for $\cos(\pi x)$ is $\frac{2\pi}{\pi} = 2$ and for $\cos(2\pi x)$ is $\frac{2\pi}{2\pi} = 1$. ■

■ **Example 7.4** Sketch three periods of the 2π-periodic function defined on the interval $-\pi \le x \le \pi$ as

$$a) f(x) = x, \quad b) f(x) = \pi - |x|, \quad c) \begin{cases} 1 & -\pi < x < 0 \\ \cos(\frac{1}{2}x) & 0 < x < \pi \end{cases}.$$

■

■ **Example 7.5** Find the Fourier coefficients of the periodic function

$$f(x) = \begin{cases} 0 & \text{if} \quad -\pi < x < 0 \\ 1 & \text{if} \quad 0 < x < \pi \end{cases}.$$

Solution: Start by using the formulas for the Fourier coefficients:

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{0} 0dx + \frac{1}{2\pi}\int_{0}^{\pi} 1dx \\
&= \frac{1}{2\pi}\left[x\Big|_0^{\pi}\right] = \frac{1}{2\pi}[\pi] = \frac{1}{2} \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\int_{-\pi}^{0} 0dx + \frac{1}{\pi}\int_{0}^{\pi} \cos(nx)dx \\
&= \frac{1}{\pi}\left[\frac{1}{n}\sin(nx)\Big|_0^{\pi}\right] = \frac{1}{\pi}[0+0] = 0 \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\int_{-\pi}^{0} 0dx + \frac{1}{\pi}\int_{0}^{\pi} \sin(nx)dx \\
&= \frac{1}{\pi}\left[-\frac{1}{n}\cos(nx)\Big|_0^{\pi}\right] = \frac{1}{n\pi}[-\cos(-n\pi)] = -\frac{2}{n\pi}\cos(n\pi).
\end{aligned}$$

Hence, the Fourier sine coefficients are $b_{2n} = -\frac{2}{2n\pi}$ and $b_{2n+1} = \frac{2}{(2n+1)\pi}$. Thus, the Fourier series is

$$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)] = \frac{1}{2} + \frac{2}{3\pi}\sin(x) - \frac{1}{2\pi}\sin(2x) + \frac{2}{5\pi}\sin(3x) - \frac{1}{4\pi}\sin(4x) + \dots.$$

■

■ **Example 7.6** Find the Fourier coefficients of the periodic function

$$f(x) = \begin{cases} -x & \text{if} \quad -\pi < x < 0 \\ x & \text{if} \quad 0 < x < \pi \end{cases}.$$

Solution: Start by using the formulas for the Fourier coefficients: (In class with IBP) ■

7.4 Dirichlet Conditions

7.5 Convergence and Sum of a Fourier series

This section follows the outline and layout of Kreyzig Chapter 11. There are a wide range of functions that can be represented by a Fourier series (unlike a Taylor Series).

Theorem 7.5.1 *(Representation by Fourier series)* Let $f(x)$ be a periodic function with period 2π and piecewise continuous on the interval $-\pi \le x \le \pi$. Furthermore, let $f(x)$ have a left-hand and right-hand derivative at each point of that interval. Then the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)] \tag{7.13}$$

converges. It's sum is $f(x)$ except at points x_0 where $f(x)$ is discontinuous. There the sum of the series is the average of the left and right hand limits of $f(x)$ at x_0.

■ **Example 7.7** What does the Fourier Series of $f(x) = \begin{cases} 1 & -3 \le x \le 0 \\ 2x & 0 < x \le 3 \end{cases}$ will converge to at $x = -2, 0, 3, 5, 6$?

Solution: The first two points are inside the original interval of definition of $f(x)$, so we can

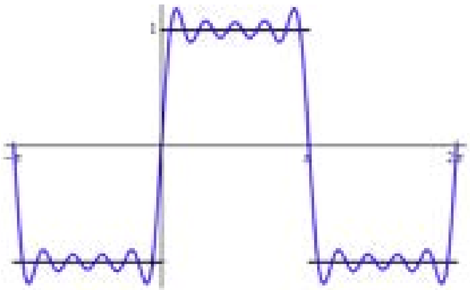


Figure 7.7: Illustration of sum of the first few terms of a Fourier series for a function with a jump discontinuity.

just read the function value directly. The only discontinuity of $f(x)$ occurs at $x=0$. So at $x=-2$, $f(x)$ is nice and continuous. The Fourier Series will converge to $f(-2)=1$. On the other hand, at $x=0$ we have a jump discontinuity, so the Fourier Series will converge to the average of the one-sided limits. $f(0^+)=\lim_{x\to 0^+} f(x)=0$ and $f(0^-)=\lim_{x\to 0^-} f(x)=1$, so the Fourier Series will converge to $\frac{1}{2}[f(0^+)+f(0^-)]=\frac{1}{2}$.

What happens at the other points? Here we consider where $f(x)$ or its periodic extension, $f_{per}(x)$, have jump discontinuities. These can only occur either at $x=x_0+2Lm$ where $-L<x_0<L$ is a jump discontinuity of $f(x)$ or at endpoints $x=\pm L+2Lm$, since the periodic extension might not "sync up" at these points, producing a jump discontinuity.

At $x=3$, we are at one of these "boundary points" and the left-sided limit is 6 while the right-sided limit is 1. Thus the Fourier Series will converge here to $\frac{6+1}{2}=\frac{7}{2}$. $x=5$ is a point of continuity for $f_{per}(x)$ and so the Fourier Series will converge to $f_{per}(5)=f(-1)=1$. $x=6$, is a jump discontinuity (corresponding to $x=0$), so the Fourier Series will converge to $\frac{1}{2}$. ■

■ **Example 7.8** Where does the Fourier Series for $f(x)=\begin{cases} 2 & -2\le x<-1 \\ 1-x & -1\le x\le 2 \end{cases}$ converge at $x=-7,-1,6$?

Solution: None of the points are inside $(-2,2)$ where $f(x)$ is discontinuous. The only points where the periodic extension might be discontinuous are the "boundary points" $x=\pm 2+4k$. In fact, since $f(-2)\neq f(2)$, these will be points of discontinuity. So $f_{per}(x)$ is continuous at $x=-7$, since it is not a boundary point and we have $f_{per}(-7)=f(1)=0$, which is what the Fourier Series will converge to. The same for $x=-1$, the Fourier Series will converge to $f(-1)=\frac{2+2}{2}=2$.

For $x=6$ we are at an endpoint. The left-sided limit is -1, while the right-sided limit is 2, so the Fourier Series will converge to their average $\frac{1}{2}$. ■

■ **Example 7.9** Plot the function the Fourier series will converge to for each of the following functions defined on the interval $-\pi \le x \le \pi$: (Plots in class!)

$$a) f(x) = \begin{cases} 1, & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0, & \text{othewise} \end{cases}.$$

$$b) f(x) = \begin{cases} x+\pi, & \text{if } -\pi < x < 0 \\ -x+\pi, & \text{if } 0 < x < \pi \end{cases}.$$

$$c) f(x) = \begin{cases} 0, & \text{if } -\pi < x < 0 \\ x, & \text{if } 0 < x < \pi \end{cases}.$$

$$d) f(x) = x, \text{ if } -\pi < x < \pi.$$

$$e) f(x) = \begin{cases} -1, & \text{if } -\pi < x < -\frac{\pi}{2} \\ x, & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 1, & \text{if } \frac{\pi}{2} < x < \pi \end{cases}.$$

$$f) f(x) = x^2, \text{ if } -\pi < x < \pi.$$

■

7.5.1 Gibbs Phenomenon

The Gibbs phenomenon, discovered by J. Willard Gibbs (1899) describes how a Fourier series of a piecewise continuously differentiable periodic function behaves at the location of a jump discontinuity. The nth partial sum of the Fourier series has large oscillations near the jump due to the presence of the sines/cosines. This may increase/decrease the maximum of the partial sum above/below that of the function itself.

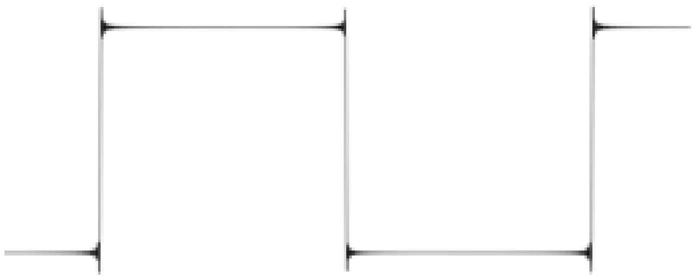


Figure 7.8: Illustration of Gibbs Phenomenon showing an “overshoot" near the jump discontinuities.

7.6 Complex Form of Fourier Series

7.7 Complex Fourier Series

This section follows the outline and layout of Kreyzig Chapter 11. We want to use the knowledge derived from our first chapter on Complex Numbers to write the Fourier series in complex form. The complex form is very useful for physical applications and can be easier to use when solving some differential equations.

In order to write the Fourier series in complex form we must first recall the Euler identity:

Definition 7.7.1 *(Euler Identity)* $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and similarly $e^{-i\theta} = \cos(\theta) - i\sin(\theta)$.

Also, recall by adding and subtracting these functions we get the complex form of the sine and cosine functions

$$\cos(t) = \frac{1}{2}(e^{it} + e^{-it}), \qquad \sin(t) = \frac{1}{2i}(e^{it} - e^{-it}). \tag{7.14}$$

Recall $1/i = -i$ and let $t = nx$ to find

$$\begin{aligned} a_n\cos(nx) + b_n\sin(nx) &= \frac{1}{2}a_n(e^{inx} + e^{-inx}) + \frac{1}{2i}b_n(e^{inx} - e^{-inx}) \\ &= \frac{1}{2}(a_n - ib_n)e^{inx} + \frac{1}{2}(a_n + ib_n)e^{-inx}. \end{aligned}$$

Inputting this expression into the Fourier series while writing $a_0 = c_0, \frac{1}{2}(a_n - ib_n) = c_n, \frac{1}{2}(a_n + ib_n) = k_n$ to find

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[c_n e^{inx} + k_n e^{-inx}\right]. \tag{7.15}$$

where the coefficients c_i, k_i are defined as

$$\begin{aligned} c_n &= \frac{1}{2}(a_n - ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left[\cos(nx) - i\sin(nx)\right]dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \\ k_n &= \frac{1}{2}(a_n + ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left[\cos(nx) + i\sin(nx)\right]dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx. \end{aligned}$$

We simply the formula further by setting $c_{-n} = k_n$, then the Fourier series can be written as:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx, \quad n = 0, \pm 1, \pm 2, ... \tag{7.16}$$

This is the *complex form of the Fourier series* of the **Complex Fourier series** of $f(x)$. The c_n are the complex Fourier coefficients of $f(x)$.

■ Example 7.10 Find the complex Fourier series of $f(x) = e^x$ if $-\pi < x < \pi$ and $f(x+2\pi) = f(x)$ and then from it obtain the usual Fourier series as a check.

Solution: Start by computing the complex Fourier coefficients:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx}dx = \frac{1}{2\pi}\frac{1}{1-in}e^{x-inx}\bigg|_{-\pi}^{\pi} = \frac{1}{2\pi}\frac{1}{1-in}(e^{\pi} - e^{-\pi})(-1)^n.$$

On the right side, $\frac{1}{1-in} = \frac{1+in}{1+n^2}$ and $e^{\pi} - e^{-\pi} = 2\sinh(\pi)$. Hence the complex Fourier series is

$$e^x = \frac{\sinh(\pi)}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\frac{1+in}{1+n^2}e^{inx}. \tag{7.17}$$

Now, derive the traditional Fourier series. Notice that

$$\begin{aligned} (1+in)e^{inx} &= (1+in)\left[\cos(nx) + i\sin(nx)\right] = \left[\cos(nx) - n\sin(nx)\right] + i\left[n\cos(nx) + \sin(nx)\right] \\ (1-in)e^{-inx} &= (1-in)\left[\cos(nx) - i\sin(nx)\right] = \left[\cos(nx) - n\sin(nx)\right] - i\left[n\cos(nx) + \sin(nx)\right] \end{aligned}$$

Adding these to equations results in the imaginary parts canceling

$$(1+in)e^{inx} + (1-in)e^{-inx} = 2\left[\cos(nx) - n\sin(nx)\right], \quad n = 1, 2, ...$$

for $n = 0$ we get 1. Thus, the real Fourier series is

$$e^x = \frac{2\sinh(\pi)}{\pi}\left[\frac{1}{2} - \frac{1}{1+1^2}\left[\cos(x) - \sin(x)\right] + \frac{1}{1+2^2}\left[\cos(2x) - 2\sin(2x)\right] - ...\right].$$

■

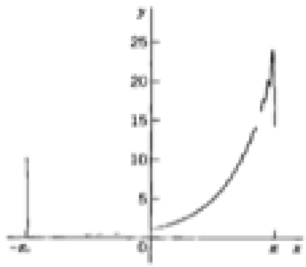


Figure 7.9: Plot of the partial sum for the Fourier series of e^x.

■ **Example 7.11** Find the complex Fourier series of $f(x) = \sin(x)$ if $-\pi < x < \pi$ and $f(x+2\pi) = f(x)$.

Solution: Start by computing the complex Fourier coefficients:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(x)e^{-inx}dx = \frac{1}{2\pi}\left[\frac{e^{in\pi}-e^{-in\pi}}{n^2-1}\right].$$

The coefficients are zero unless $n = \pm 1$ (since $e^{\pm in\pi} = \cos(n\pi) \pm i\sin(n\pi) = \cos(n\pi) = (-1)^n$. Using L'Hospital rule twice gives $c_1 = \frac{1}{2i}$ and $c_{-1} = -\frac{1}{2i}$. Thus, the complex Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = \frac{e^{ix}-e^{-ix}}{2i},$$

which is exactly the complex representation of the sine function. ■

■ **Example 7.12** Find the complex Fourier series of $f(x) = 1$ if $0 < x < T$, $f(x) = 0$ otherwise, and $f(x+2\pi) = f(x)$.

Solution: Start by computing the complex Fourier coefficients:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_0^T e^{-inx}dx = \frac{i}{2\pi n}e^{-inx}\Big|_0^T = \frac{i}{2\pi n}\left[e^{inT}-1\right],$$

when $n \neq 0$. Also, $c_0 = \frac{1}{2\pi}\int_0^T dx = \frac{T}{2\pi}$. Thus, the complex Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = \frac{1}{2\pi}\left[T + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{i}{n}\left[e^{-inT}-1\right]e^{inx}\right].$$

■

7.7.1 General Complex Fourier Series for Intervals $(0, L)$

Definition 7.7.2 The complex Fourier series for an interval from $(0, L)$ has the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx\frac{2\pi}{L}}, \qquad c_n = \frac{1}{L}\int_0^L e^{-inx\frac{2\pi}{L}} f(x)dx. \tag{7.18}$$

The fraction $\frac{2\pi}{L}$ is often written as ω_0 and called the *fundamental angular frequency*.

■ **Example 7.13** Find the complex Fourier series of $f(x) = 1$ if $0 < x < L/2$, $f(x) = -1$ if $L/2 < x < L$, and $f(x+2\pi) = f(x)$.

Solution: Start by computing the complex Fourier coefficients:

$$c_n = \frac{1}{L}\int_0^L e^{-inx\frac{2\pi}{L}} f(x)dx = \frac{1}{L}\left[\int_0^{L/2} e^{-inx\frac{2\pi}{L}}dx - \int_{L/2}^{L} e^{-inx\frac{2\pi}{L}}dx\right] = \frac{i}{2\pi in}\left[e^{-2in\pi}+1-2^{-in\pi}\right],$$

when $n \neq 0$. Also, $c_0 = 0$ since the mean of the function is zero. Thus, the complex Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = \sum_{n=-\infty, n\neq 0}^{\infty} \frac{[1-e^{-in\pi}]}{in\pi} e^{inx\frac{2\pi}{L}}.$$

■

■ **Example 7.14** Find the complex Fourier series for the following functions a) $f(x) = 1$ for $-\pi < x < \pi$, b) $f(x) = -2x$ for $-\pi < x < \pi$.

Solution: For a)

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}dx = -\frac{1}{2\pi in}e^{-inx}\Big|_{-\pi}^{\pi} = -\frac{1}{2\pi in}\left[e^{-in\pi}-e^{in\pi}\right],$$

which is 0 when $n \neq 0$. Thus the only term in the Fourier series is $c_0 = 1$.

For b)

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}dx = \frac{1}{2\pi}\left[\frac{-x}{in}e^{-inx}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{-1}{in}e^{-inx}dx\right] \\ &= \frac{1}{2\pi}\left[\frac{-\pi}{in}[2e^{in\pi}] + \frac{1}{n^2}e^{-inx}\Big|_{-\pi}^{\pi}\right] \\ &= \frac{1}{2\pi}\left[\frac{-\pi}{in}[2e^{in\pi}] + \frac{1}{n^2}[e^{-in\pi}-e^{-in\pi}]\right] \end{aligned}$$

If $n \neq 0$, then $c_n = -\frac{1}{2in}2e^{in\pi} = \frac{i}{n}e^{in\pi}$. If $n = 0$, then $c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} xdx = 0$. Thus, the complex Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = \sum_{n=-\infty, n\neq 0}^{\infty} \frac{i}{n}e^{in\pi}e^{inx} = \sum_{n=-\infty, n\neq 0}^{\infty} \frac{i}{n}e^{in(\pi+x)}.$$

■

7.8 General Fourier Series for Functions of Any Period $p = 2L$

This section follows the outline and layout of Kreyzig Chapter 11. So far all the functions considered have had period $p = 2\pi$ simplifying the formulas for the Fourier coefficients and the Fourier series. In general for applications, most functions do not have period 2π, but rather possess a period of arbitrary length we will call $p = 2L$. The good news is that the Fourier series and formulas for the coefficients for a general function of period $2L$ have a very similar form. We use the notation $p = 2L$ where L is the length of the object under consideration (such as a spring or rod). We will see this application in the Chapter on PDEs.

Key Idea: Find and use a *change of scale* that transforms a 2π periodic function into a function of period $2L$. Recall, the form of the for a function of period 2π

$$g(v) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nv) + b_n \sin(nv)] \tag{7.19}$$

with coefficients

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(v)dv \tag{7.20}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(v)\cos(nv)dv \quad n = 1,2,3,... \tag{7.21}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(v)\sin(nv)dv \quad n = 1,2,3,.... \tag{7.22}$$

Using a change of scale, let $v = kx$ with k such that the old period $v = 2\pi$ and gives a new period where $x = 2L$. Thus, $2\pi = k2L \Rightarrow k = \pi/L$. Therefore, $v = kx = \pi x/L$ and $dv = \frac{\pi}{L}dx$. Now writing $g(v) = f(x)$, the Fourier series and coefficients become

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right] \tag{7.23}$$

with coefficients

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)dx \tag{7.24}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad n = 1,2,3,... \tag{7.25}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad n = 1,2,3,.... \tag{7.26}$$

Also, the complex Fourier series can be expressed for arbitrary intervals

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-\frac{n\pi x}{L}}, \quad c_n = \frac{1}{2L} \int_{-L}^{L} f(x)e^{-\frac{in\pi}{x}}dx. \tag{7.27}$$

■ **Example 7.15** *(Periodic Rectangular Wave)* Find the Fourier series of the function

$$f(x) = \begin{cases} 0 & \text{if } -2 < x < -1 \\ k & \text{if } -1 < x < 1 \\ 0 & \text{if } 1 < x < 2 \end{cases},$$

where $p = 2L = 4$ and $L = 2$.

Solution: Using the formulas just derived we can find the Fourier coefficients

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)dx = \frac{1}{4} \int_{-1}^{1} kdx = \frac{k}{4}x\Big|_{-1}^{1} = \frac{k}{2}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2} \int_{-1}^{1} k\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{k}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_{-1}^{1} = \frac{2k}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2} \int_{-1}^{1} k\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= -\frac{k}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_{-1}^{1} = -\frac{k}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - \cos\left(\frac{-n\pi}{2}\right)\right] = 0$$

Observe that $a_n = 0$ if n is even. Hence the Fourier series is

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left[\cos\left(\frac{\pi}{2}x\right) - \frac{1}{3}\cos\left(\frac{3\pi}{2}x\right) + \frac{1}{5}\cos\left(\frac{5\pi}{2}x\right) + ...\right].$$

■

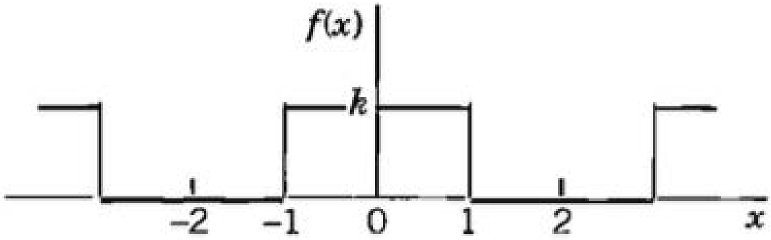


Figure 7.10: Period Rectangular Wave from first example.

■ **Example 7.16** *(Periodic Rectangular Wave)* Find the Fourier series of the function

$$f(x) = \begin{cases} -k & \text{if } -2 < x < 0 \\ k & \text{if } 0 < x < 2 \end{cases},$$

where $p = 2L = 4$ and $L = 2$.

Solution: Using the formulas just derived we can find the Fourier coefficients

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{4}\left[\int_{-2}^{0} -kdx + \int_{0}^{2} kdx\right] = \frac{k}{4}x\Big|_{-2}^{0} + \frac{k}{4}x\Big|_{0}^{2} = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\left[\int_{-2}^{0} -k\cos\left(\frac{n\pi x}{2}\right)dx + \int_{0}^{2} k\cos\left(\frac{n\pi x}{2}\right)dx\right]$$

$$= \frac{1}{2}\left[-\frac{2k}{n\pi}\sin\left(\frac{nx\pi}{2}\right)\Big|_{-2}^{0} + \frac{2k}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_{0}^{2}\right] = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\left[\int_{-2}^{0} -k\sin\left(\frac{n\pi x}{2}\right)dx + \int_{0}^{2} k\sin\left(\frac{n\pi x}{2}\right)dx\right]$$

$$= \frac{1}{2}\left[\frac{2k}{n\pi}\cos\left(\frac{nx\pi}{2}\right)\Big|_{-2}^{0} - \frac{2k}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_{0}^{2}\right]$$

$$= \frac{k}{n\pi}\left[1 - \cos(n\pi) - \cos(n\pi) + 1\right] = \begin{cases} 4k/n\pi & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}.$$

Hence the Fourier series is

$$f(x) = \frac{4k}{\pi}\left[\sin\left(\frac{\pi}{2}x\right) + \frac{1}{3}\sin\left(\frac{3\pi}{2}x\right) + \frac{1}{5}\sin\left(\frac{5\pi}{2}x\right) + ...\right].$$

■

■ **Example 7.17** *(Half-wave Rectifier)* Find the Fourier series of the function

$$f(x) = \begin{cases} 0 & \text{if } -L < t < 0 \\ E\sin(\omega t) & \text{if } 0 < x < L \end{cases},$$

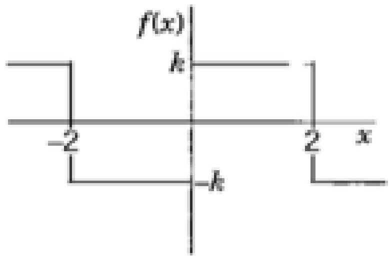


Figure 7.11: Period Rectangular Wave from second example.

where $p = 2L = \frac{2\pi}{\omega}$ and $L = \frac{\pi}{\omega}$.

Solution: Using the formulas just derived we can find the Fourier coefficients

$$
\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(t)dt = \frac{\omega}{2\pi}\int_0^{\pi/\omega} E\sin(\omega t)dt = \frac{E}{\pi} \\
a_n &= \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt = \frac{\omega}{\pi}\int_0^{\pi/\omega} E\sin(\omega t)\cos(n\omega t)dt \\
&= \frac{\omega E}{2\pi}\int_0^{\pi/\omega}[\sin((1+n)\omega t) + \sin((1-n)\omega t)]dt \\
&= \frac{\omega E}{2\pi}\left[-\frac{\cos((1+n)\omega t)}{(1+n)\omega} - \frac{\cos((1-n)\omega t)}{(1-n)\omega}\right]_0^{\pi/\omega} = \frac{E}{2\pi}\left[\frac{-\cos((1+n)\pi)+1}{1+n} + \frac{-\cos((1-n)\pi)+1}{1-n}\right] \\
b_n &= \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt = \frac{\omega}{\pi}\int_0^{\pi/\omega} E\sin(\omega t)\sin(n\omega t)dt.
\end{aligned}
$$

Observe that $a_1 = 0$ and $a_n = 0$ is n is odd. For n even, $a_n = \frac{E}{2\pi}\left(\frac{2}{1+n} + \frac{2}{1-n}\right) = -\frac{2E}{(n-1)(n+1)\pi}$. Also, $b_1 = E/2$ and all other $b_n = 0$. Hence the Fourier series is

$$
f(t) = \frac{E}{\pi} + \frac{E}{2}\sin(\omega t) - \frac{2E}{\pi}\left[\frac{1}{1\cdot 3}\cos(2\omega t) + \frac{1}{3\cdot 5}\cos(4\omega t) + \ldots\right].
$$

■

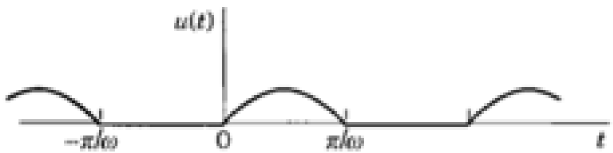


Figure 7.12: Half-wave Rectifier from second example.

■ **Example 7.18** Compute the Fourier Series of $f(x) = 1 + x$ on the interval $(-L, L)$.

Solution: Using the above formulas we have

$$\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L}(1+x)dx = 1 \\
a_m &= \frac{1}{L}\int_{-L}^{L}(1+x)\cos(\frac{m\pi x}{L})dx \\
&= \frac{1+x}{m\pi}\sin(\frac{m\pi x}{L}) + \frac{1}{m^2\pi^2}\cos(\frac{m\pi x}{L})|_{-L}^{L} \\
&= \frac{1}{m^2\pi^2}(\cos(m\pi) - \cos(-m\pi)) = 0 \quad m \neq 0 \\
b_m &= \frac{1}{L}\int_{-L}^{L}(1+x)\sin(\frac{m\pi x}{L})dx \\
&= -\frac{1+x}{m\pi}\cos(\frac{m\pi x}{L}) + \frac{1}{m^2\pi^2}\sin(\frac{m\pi x}{L})|_{-L}^{L} \\
&= -\frac{2L}{m\pi}\cos(m\pi) = (-1)^{m+1}\frac{2L}{m\pi}.
\end{aligned}$$

So the full Fourier series of $f(x)$ is

$$1+x = 1+\frac{2L}{\pi}(\sin(\frac{\pi x}{L}) - \frac{1}{2}\sin(\frac{2\pi x}{L}) + \frac{1}{3}\sin(\frac{3\pi x}{L}) - ...) \quad (7.28)$$

$$= 1+\frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(\frac{(2n-1)\pi x}{L}) - \frac{1}{2n}\sin(\frac{2n\pi x}{L}). \quad (7.29)$$

■

■ **Example 7.19** Compute the Fourier Series for $f(x) = \begin{cases} 2 & -2 \leq x < -1 \\ 1-x & -1 \leq x < 2 \end{cases}$ on the interval $(-2,2)$.

Solution: We start by using the Euler-Fourier Formulas. For the Cosine terms we find

$$\begin{aligned}
a_0 &= \frac{1}{4}\int_{-2}^{2} f(x)dx \\
&= \frac{1}{4}\left(\int_{-2}^{-1} 2dx + \int_{-1}^{2} 1-xdx\right) \\
&= \frac{1}{4}(2+\frac{3}{2}) = \frac{7}{8}
\end{aligned}$$

and

$$\begin{aligned}
a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos(\frac{n\pi x}{2})dx \\
&= \frac{1}{2}\left(\int_{-2}^{-1} 2\cos(\frac{n\pi x}{2})dx + \int_{-1}^{2}(1-x)\cos(\frac{n\pi x}{2})dx\right) \\
&= \frac{1}{2}\left(\frac{4}{n\pi}\sin(\frac{n\pi x}{2})|_{-2}^{-1} + \frac{2(1-x)}{n\pi}\sin(\frac{n\pi x}{2})|_{-1}^{2} - \frac{4}{n^2\pi^2}\left(\cos(\frac{n\pi x}{2})|_{-1}^{2}\right)\right) \\
&= \frac{1}{2}\left(-\frac{4}{n\pi}\sin(\frac{n\pi}{2}) + \frac{4}{n\pi}\sin(\frac{n\pi}{2}) - \frac{4}{n^2\pi^2}(\cos(n\pi) - \cos(\frac{n\pi}{2}))\right) \\
&= \begin{cases} \frac{2}{n^2\pi^2} & n \text{ odd} \\ 0 & n = 4m \\ -\frac{4}{n^2\pi^2} & n = 4m+2 \end{cases}.
\end{aligned}$$

Also, for the sine terms

$$\begin{aligned}
b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin(\frac{n\pi x}{2})dx \\
&= \frac{1}{2}\left(\int_{-2}^{-1} 2\sin(\frac{n\pi x}{2})dx + \int_{-1}^{2}(1-x)\sin(\frac{n\pi x}{2})dx\right) \\
&= \frac{1}{2}\left(-\frac{4}{n\pi}\cos(\frac{n\pi x}{2})|_{-2}^{-1} - \frac{2(1-x)}{n\pi}\cos(\frac{n\pi x}{2})|_{-1}^{2} - \frac{4}{n^2\pi^2}(\sin(\frac{n\pi x}{2})|_{-1}^{2}\right) \\
&= \frac{1}{2}\left(\frac{6}{n\pi}\cos(n\pi) - \frac{4}{n^2\pi^2}\sin(\frac{n\pi}{2})\right) \\
&= \begin{cases} \frac{3}{n\pi} & n \text{even} \\ -\frac{3}{n\pi} - \frac{2}{n^2\pi^2} & n = 4m+1 \\ -\frac{3}{n\pi} + \frac{2}{n^2\pi^2} & n = 4m+3 \end{cases} .
\end{aligned}$$

So we have

$$\begin{aligned}
f(x) = \frac{7}{8} &+ \sum_{m=1}^{\infty} \frac{2}{(4m+1)^2\pi^2}\cos\left(\frac{(4m+1)\pi x}{2}\right) + \left(-\frac{3}{(4m+1)\pi} - \frac{2}{(4m+1)^2\pi^2}\right)\sin\left(\frac{(4m+1)\pi x}{2}\right) \\
&- \frac{4}{(4m+2)^2\pi^2}\cos\left(\frac{(4m+2)\pi x}{2}\right) + \frac{3}{(4m+2)\pi}\sin\left(\frac{(4m+2)\pi x}{2}\right) \\
&+ \frac{2}{(4m+3)^2\pi^2}\cos\left(\frac{(4m+3)\pi x}{2}\right) + \left(-\frac{3}{(4m+3)\pi} + \frac{2}{(4m+3)^2\pi^2}\right)\sin\left(\frac{(4m+3)\pi x}{2}\right) \\
&+ \frac{3}{4m\pi}\sin\left(\frac{4m\pi x}{2}\right).
\end{aligned}$$

■

This example represents a worst case scenario. There are a lot of Fourier coefficients to keep track of. Notice that for each value of m, the summand specifies four different Fourier terms (for $4m, 4m+1, 4m+2, 4m+3$). This can often happen and depending on L, even more terms maybe required.

7.9 Even and Odd Functions

7.10 Even and Odd Functions, Half-Range Expansions

Recall that an **even** function is a function satisfying

$$g(-x) = g(x). \tag{7.30}$$

This means that the graph $y = g(x)$ is symmetric with respect to the y-axis. An **odd** function satisfies

$$g(-x) = -g(x) \tag{7.31}$$

meaning that its graph $y = g(x)$ is symmetric with respect to the origin.

■ **Example 7.20** A monomial x^n is even if n is even and odd if n is odd. $\cos(x)$ is even and $\sin(x)$ is odd. Note $\tan(x)$ is odd. ■

There are some general rules for how products and sums behave:
(1) If $g(x)$ is odd and $h(x)$ is even, their product $g(x)h(x)$ is odd.
(2) If $g(x)$ and $h(x)$ are either both even or both odd, $g(x)h(x)$ is even.

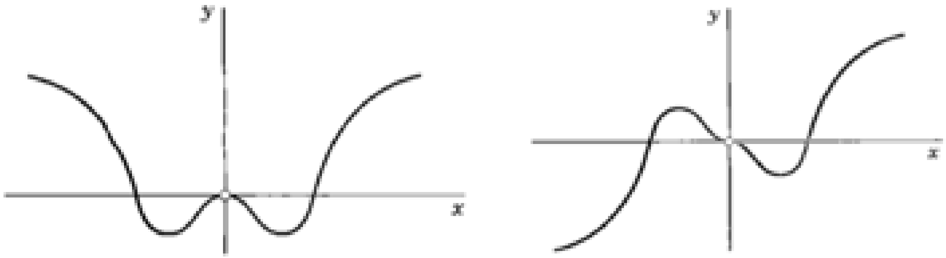


Figure 7.13: Even function (Left) and Odd function (Right).

(3) The sum of two even functions or two odd functions is even or odd, respectively.
To remember the rules consider how many negative signs come out of the arguments.
(4) The sum of an even and an odd function can be anything. In fact, any function on $(-L,L)$ can be written as a sum of an even function, called the **even part**, and an odd function, called the **odd part**.
(5) Differentiation and Integration can change the parity of a function. If $f(x)$ is even, $\frac{df}{dx}$ and $\int_0^x f(s)ds$ are both odd, and vice versa.

The graph of an odd function $g(x)$ must pass through the origin by definition. This also tells us that if $g(x)$ is even, as long as $g'(0)$ exists, then $g'(0)=0$.

Theorem 7.10.1 Definite Integrals on symmetric intervals of odd and even functions have useful properties

$$\int_{-L}^{L}(odd)dx = 0 \quad \text{and} \quad \int_{-L}^{L}(even)dx = 2\int_{0}^{L}(even)dx \tag{7.32}$$

Given a function $f(x)$ defined on $(0,L)$, there is only one way to extend it to $(-L,L)$ to an even or odd function. The **even extension** of $f(x)$ is

$$f_{even}(x) = \begin{cases} f(x) & \text{for } \ 0<x<L \\ f(-x) & \text{for } \ -L<x<0. \end{cases} \tag{7.33}$$

This is just its reflection across the y-axis. Notice that the even extension is not necessarily defined at the origin.

The **odd extension** of $f(x)$ is

$$f_{odd}(x) = \begin{cases} f(x) & \text{for } \ 0<x<L \\ -f(-x) & \text{for } \ -L<x<0 \\ 0 & \text{for } \ x=0 \end{cases} . \tag{7.34}$$

This is just its reflection through the origin.

R Since the cosine terms in a Fourier series are even and the sine terms are odd, then it should not be surprising that an even function is given by a series of cosine terms and an odd function by a series of sine terms.

Theorem 7.10.2 *(Fourier Cosine Series, Fourier Sine Series)* The Fourier series of an **even** function of period $2L$ is a **Fourier Cosine Series**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) \tag{7.35}$$

with coefficients (NOTE: Integration only on the half-interval $(0, L)$!)

$$a_0 = \frac{1}{L}\int_0^L f(x)dx$$
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, 3, \ldots$$

The Fourier series of an **odd** function of period $2L$ is a **Fourier Sine Series**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \tag{7.36}$$

with coefficients

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Theorem 7.10.3 *(Sum and Scalar Multiple)* The Fourier coefficients of a sum $f_1 + f_2$ are the sums of the corresponding Fourier coefficients of f_1 and f_2. The Fourier coefficients of cf are c times the Fourier coefficients of f.

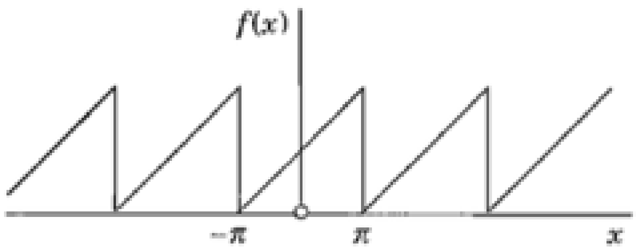


Figure 7.14: Sawtooth function.

■ **Example 7.21** *(Sawtooth Wave)* Find the Fourier series of the function $f(x) = x + \pi$ if $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$.

Solution: Here we have $f = f_1 + f_2$ where $f_1 = x$ and $f_2 = \pi$. The Fourier coefficients of f_2 are zero except for the first one (the constant term, $a_0 = \pi$). Thus, the Fourier coefficients a_n, b_n are those for f_1 except for a_0, which is π. Since f_1 is odd, then $a_n = 0$ for $n = 1, 2, \ldots$ and

$$b_n = \frac{2}{\pi}\int_0^{\pi} f_1(x)\sin(nx)dx = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)dx$$
$$= \frac{2}{\pi}\left[\frac{-x\cos(nx)}{n}\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)dx\right] = -\frac{2}{n}\cos(n\pi).$$

Thus, $b_1 = 2, b_2 = -2/2 = -1, b_3 = 2/3, b_4 = -2/4 = -1/2, ...$ and the Fourier series of $f(x)$ is

$$f(x) = \pi + 2\left(\sin(x) - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - ...\right).$$

■

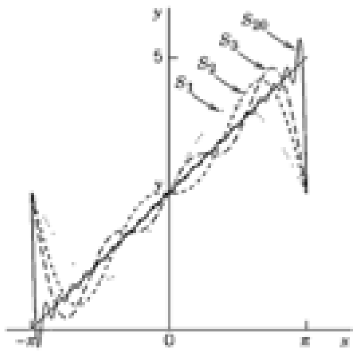


Figure 7.15: First few partial sums.

7.10.1 Half-Range Expansions

Half-range expansions are Fourier series. The basic idea is to represent a function $f(x)$ by a Fourier series where the function is only defined on the interval $(0, L)$. This could represent a violin string or the temperature distribution in a metal bar. We could extend $f(x)$ as a function of period L and develop the extended function into a Fourier Series. This series would in general contain both sine and cosine terms requiring many computations. We know that if the function were even on $(-L, L)$ we would only have to compute the cosine terms. Likewise if the function were odd on the interval $(-L, L)$ then we would only need to compute the cosine terms.

Definition 7.10.1 An **even periodic extension** is a function of period $2L$ which is even, but coincides with the given function $f(x)$ on the interval $(0, L)$ (see f_1 in figure).

An **odd periodic extension** is a function of period $2L$ which is odd, but coincides with the given function $f(x)$ on the interval $(0, L)$ (see f_2 in figure).

R Both extensions have period $2L$. This is where the term **half-range expansion** comes from: f is given on half the range $(0, L)$ giving only half the periodicity of the length $2L$.

■ **Example 7.22** Find the two half-range expansions for the function

$$f(x) = \begin{cases} \frac{2k}{L}x & \text{if } 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L-x) & \text{if } \frac{L}{2} < xL \end{cases}.$$

Solution: a) Even Periodic Extension: Find the Fourier Cosine series, which converges to the even

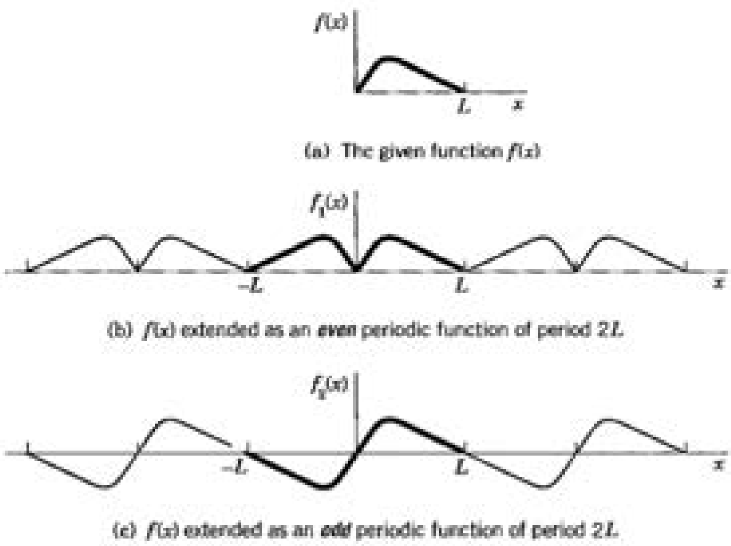


Figure 7.16: Extensions to even and odd functions. $f_1(x)$ is the even periodic extension and $f_2(x)$ is the odd periodic extension.

periodic extension. Start by finding the Fourier coefficients:

$$a_0 = \frac{1}{L}\left[\frac{2k}{L}\int_0^{L/2} x dx + \frac{2k}{L}\int_{L/2}^{L}(L-x)dx\right] = \frac{k}{2}$$

$$a_n = \frac{2}{L}\left[\frac{2k}{L}\int_0^{L/2} x\cos\left(\frac{n\pi}{L}x\right)dx + \frac{2k}{L}\int_{L/2}^{L}(L-x)\cos\left(\frac{n\pi}{L}x\right)dx\right]$$

$$=_{IBP} \frac{L^2}{2n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{L^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2}-1\right) - \frac{L}{n\pi}\left(L-\frac{L}{2}\right)\sin\left(\frac{n\pi}{2}\right) - \frac{L^2}{n^2\pi^2}\left(\cos(n\pi)-\cos\frac{n\pi}{2}\right).$$

Combining the terms in a_n gives $a_n = \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos(n\pi) - 1\right)$ and $a_n = 0$ if $n = 2, 6, 10, ..., 4n - 2, ...$ Hence, the first half-range expansion of $f(x)$ is

$$f(x) = \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi}{L}x + \frac{1}{6^2}\cos\frac{6\pi}{L}x + ...\right).$$

b) Odd Period Extension: Find the Fourier Sine series, which converges to the odd periodic extension. Start by finding the Fourier coefficients:

$$b_n = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

Thus, the other half-range expansion of $f(x)$ is

$$f(x) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}x - \frac{1}{3^2}\sin\frac{3\pi}{L}x + \frac{1}{5^2}\sin\frac{5\pi}{L}x - ...\right).$$

■

7.10.2 Fourier Sine Series

Each of terms in the Fourier Sine Series for $f(x)$, $\sin(\frac{n\pi x}{L})$, is odd. As with the full Fourier Series, each of these terms also has period $2L$. So we can think of the Fourier Sine Series as the expansion of an odd function with period $2L$ defined on the entire line which coincides with $f(x)$ on $(0, L)$.

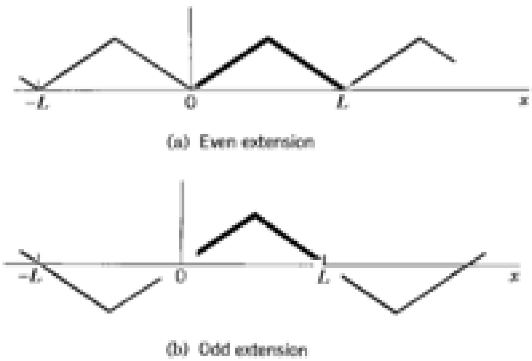


Figure 7.17: Even and odd extensions.

One can show that the full Fourier Series of f_{odd} is the same as the Fourier Sine Series of $f(x)$. Let

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L}) \tag{7.37}$$

be the Fourier Series for $f_{odd}(x)$, with coefficients given in Section 10.3

$$a_n = \frac{1}{L}\int_{-L}^{L} f_{odd}(x)\cos(\frac{n\pi x}{L})dx = 0 \tag{7.38}$$

But f_{odd} is odd and cos is even, so their product is again odd.

$$b_n = \frac{1}{L}\int_{-L}^{L} f_{odd}(x)\sin(\frac{n\pi x}{L})dx \tag{7.39}$$

But both f_{odd} and sin are odd, so their product is even.

$$\begin{aligned} b_n &= \frac{2}{L}\int_{0}^{L} f_{odd}(x)\sin(\frac{n\pi x}{L})dx & (7.40)\\ &= \frac{2}{L}\int_{0}^{L} f(x)\sin(\frac{n\pi x}{L})dx, & (7.41) \end{aligned}$$

which are just the Fourier Sine coefficients of $f(x)$. Thus, as the Fourier Sine Series of $f(x)$ is the full Fourier Series of $f_{odd}(x)$, the $2L$-periodic odd function that the Fourier Sine Series expands is just the periodic extension f_{odd}.

This goes both ways. If we want to compute a Fourier Series for an odd function on $(-L,L)$ we can just compute the Fourier Sine Series of the function restricted to $(0,L)$. It will **almost** converge to the original function on $(-L,L)$ with the only issues occurring at any jump discontinuities. The **only works for odd functions**. Do not use the formula for the coefficients of the Sine Series, unless you are working with an odd function.

■ **Example 7.23** Write down the odd extension of $f(x) = L - x$ on $(0,L)$ and compute its Fourier Series.

Solution: To get the odd extension of $f(x)$ we will need to see how to reflect f across the origin. What we end up with is the function

$$f_{odd}(x) = \begin{cases} L-x & 0<x<L \\ -L-x & -L<x<0 \end{cases} . \tag{7.42}$$

Now, what is the Fourier Series of $f_{odd}(x)$? By the previous discussion, we know that is will be identical to the Fourier Sine Series of $f(x)$, as this will converge on $(-L,0)$ to f_{odd}. So we have

$$f_{odd}(x) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L}), \tag{7.43}$$

where

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L (L-x)\sin(\frac{n\pi x}{L})dx & (7.44)\\ &= \frac{2}{L}[-\frac{L(L-x)}{n\pi}\cos(\frac{n\pi x}{L}) - \frac{L^2}{n^2\pi^2}\sin(\frac{n\pi x}{L})]_0^L & (7.45)\\ &= \frac{2L}{n\pi}. & (7.46)\end{aligned}$$

Thus the desired Fourier Series is

$$f_{odd}(x) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(\frac{n\pi x}{L}). \tag{7.47}$$

■

Can we compute the Fourier Sine Series of a constant function like $f(x) = 1$ which is even? It is important to remember that if we are computing the Fourier Sine Series for $f(x)$, it only needs to converge to $f(x)$ on $(0,L)$, where issues of evenness and oddness do not occur. The Fourier Sine Series will converge to the odd extension of $f(x)$ on $(-L,L)$.

■ **Example 7.24** Find the Fourier Series for the odd extension of

$$f(x) = \begin{cases} \frac{3}{2} & 0 < x < \frac{3}{2} \\ x - \frac{3}{2} & \frac{3}{2} < x < 3. \end{cases} \tag{7.48}$$

on $(-3,3)$.

Solution: The Fourier Series for $f_{odd}(x)$ on $(-3,3)$ will just be the Fourier Sine Series for $f(x)$ on $(0,3)$. The Fourier Sine coefficients for $f(x)$ are

$$\begin{aligned} b_n &= \frac{2}{3}\int_0^3 f(x)\sin(\frac{n\pi x}{3})dx & (7.49)\\ &= \frac{2}{3}\left(\int_0^{\frac{3}{2}}\frac{3}{2}\sin(\frac{n\pi x}{3})dx + \int_{\frac{3}{2}}^3 (x-\frac{3}{2})\sin(\frac{n\pi x}{3})\right) & (7.50)\\ &= \frac{2}{3}\left(-\frac{9}{2n\pi}\cos(\frac{n\pi x}{3})|_0^{\frac{3}{2}} + \frac{3(x-\frac{3}{2})}{n\pi}\cos(\frac{n\pi x}{3})|_{\frac{3}{2}}^3 + \frac{9}{n^2\pi^2}\sin(\frac{n\pi x}{3})|_{\frac{3}{2}}^3\right) & (7.51)\\ &= \frac{2}{3}\left(-\frac{9}{2n\pi}(\cos(\frac{n\pi}{2})-1) - \frac{9}{2n\pi}\cos(n\pi) - \frac{9}{n^2\pi^2}\sin(\frac{n\pi x}{2})\right) & (7.52)\\ &= \frac{2}{3}(\frac{9}{2n\pi}(1-\cos(\frac{n\pi}{2}) + (-1)^{n+1}) - \frac{9}{n^2\pi^2}\sin(\frac{n\pi}{2})\Big) & (7.53)\\ &= \frac{3}{n\pi}\left(1-\cos(\frac{n\pi}{2}) + (-1)^{n+1} - \frac{2}{n\pi}\sin(\frac{n\pi}{2})\right) & (7.54)\end{aligned}$$

and the Fourier Series is

$$f_{odd}(x) = \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}[1-\cos(\frac{n\pi}{2}) + (-1)^{n+1} - \frac{2}{n\pi}\sin(\frac{n\pi}{2})]\sin(\frac{n\pi x}{3}). \tag{7.55}$$

■

7.10.3 Fourier Cosine Series

Now consider what happens for the Fourier Cosine Series of $f(x)$ on $(0,L)$. This is analogous to the Sine Series case. Every term in the Cosine Series has the form

$$a_n \cos(\frac{n\pi x}{L}) \tag{7.56}$$

and hence is even, so the entire Cosine Series is even. Thus, the Cosine Series must converge on $(-L,L)$ to an even function which coincides on $(0,L)$ with $f(x)$. This must be the even extension

$$f_{even}(x) = \begin{cases} f(x) & 0 < x < L \\ f(-x) & -L < x < 0 \end{cases} . \tag{7.57}$$

Notice that this definition does not specify the value of the function at zero, the only restriction on an even function at zero is that, if it exists, the derivative should be zero.

It is straight forward enough to show that the Fourier coefficients of $f_{even}(x)$ coincide with the Fourier Cosine coefficients of $f(x)$. The Euler-Fourier formulas give

$$a_n = \frac{1}{L}\int_{-L}^{L} f_{even}(x)\cos(\frac{n\pi x}{L})dx \tag{7.58}$$

$$= \frac{2}{L}\int_{0}^{L} f_{even}(x)\cos(\frac{n\pi x}{L})dx \quad \text{since } f_{even}(x)\cos(\frac{n\pi x}{L}) \text{ is even} \tag{7.59}$$

$$= \frac{2}{L}\int_{0}^{l} f_{even}(x)\cos(\frac{n\pi x}{L})dx \tag{7.60}$$

which are the Fourier Cosine coefficients of $f(x)$ on $(0,L)$

$$b_n = \frac{1}{L}\int_{-L}^{L} f_{even}(x)\sin(\frac{n\pi x}{L})dx = 0 \tag{7.61}$$

since $f_{even}(x)\sin(\frac{n\pi x}{L})$ is odd. Thus the Fourier Cosine Series of $f(x)$ on $(0,L)$ can be considered as the Fourier expansion of $f_{even}(x)$ on $(-L,L)$, and therefore also as expansion of the periodic extension of $f_{even}(x)$. It will converge as in the Fourier Convergence Theorem to this periodic extension.

This also means that if we want to compute the Fourier Series of an even function, we can just compute the Fourier Cosine Series of its restriction to $(0,L)$. It is very important that this only be attempted if the function we are starting with is even.

■ **Example 7.25** Write down the even extension of $f(x) = L - x$ on $(0,L)$ and compute its Fourier Series.

Solution: The even extension will be

$$f_{even}(x) = \begin{cases} L-x & 0 < x < L \\ L+x & -L < x < 0 \end{cases} . \tag{7.62}$$

Its Fourier Series is the same as the Fourier Cosine Series of $f(x)$, by the previous discussion. So we can just compute the coefficients. Thus we have

$$f_{even}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}), \tag{7.63}$$

where

$$a_0 = \frac{1}{L}\int_0^L f(x)dx = \frac{1}{L}\int_0^L (L-x)dx = \frac{L}{2} \tag{7.64}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L})dx \tag{7.65}$$

$$= \frac{2}{L}\int_0^L (L-x)\cos(\frac{n\pi x}{L})dx \tag{7.66}$$

$$= \frac{2}{L}[\frac{L(L-x)}{n\pi}\sin(\frac{n\pi x}{L}) - \frac{L^2}{n^2\pi^2}\cos(\frac{n\pi x}{L})]_0^L \tag{7.67}$$

$$= \frac{2}{L}\left(\frac{L^2}{n^2\pi^2}(-\cos(n\pi)+\cos(0))\right) \tag{7.68}$$

$$= \frac{2L}{n^2\pi^2}((-1)^{n+1}+1). \tag{7.69}$$

So we have

$$f_{even}(x) = \frac{L}{2} + \sum_{n=1}^{\infty} \frac{2L}{n^2\pi^2}((-1)^{n+1}+1). \tag{7.70}$$

■

■ **Example 7.26** Write down the even extension of

$$f(x) = \begin{cases} \frac{3}{2} & 0 \le x < \frac{3}{2} \\ x - \frac{3}{2} & \frac{3}{2} \le x \le 3 \end{cases} \tag{7.71}$$

and compute its Fourier Series.

Solution: Using Equation (7.57) we see that the even extension is

$$f_{even}(x) = \begin{cases} x - \frac{3}{2} & \frac{3}{2} < x < 3 \\ \frac{3}{2} & 0 \le x < \frac{3}{2} \\ \frac{3}{2} & -\frac{3}{2} < x < 0 \\ -x - \frac{3}{2} & -3 \le x \le -\frac{3}{2} \end{cases}. \tag{7.72}$$

We just need to compute the Fourier Cosine coefficients of the original $f(x)$ on $(0,3)$.

$$
\begin{aligned}
a_0 &= \frac{1}{3}\int_0^3 f(x)dx && (7.73)\\
&= \frac{1}{3}\left(\int_0^{3/2}\frac{3}{2}dx+\int_{3/2}^3 x-\frac{3}{2}dx\right) && (7.74)\\
&= \frac{1}{3}(\frac{9}{4}+\frac{9}{8})=\frac{9}{8} && (7.75)\\
a_n &= \frac{2}{3}\int_0^3 f(x)\cos(\frac{n\pi x}{3})dx && (7.76)\\
&= \frac{2}{3}\left(\int_0^{3/2}\frac{3}{2}\cos(\frac{n\pi x}{3})dx+\int_{3/2}^3(x-\frac{3}{2})\cos(\frac{n\pi x}{3})dx\right) && (7.77)\\
&= \frac{2}{3}\left(\frac{9}{2n\pi}\sin(\frac{n\pi x}{3})|_0^{3/2}+\frac{3(x-\frac{3}{2})}{n\pi}\sin(\frac{n\pi x}{3})|_{3/2}^3+\frac{9}{n^2\pi^2}\cos(\frac{n\pi x}{3})|_{3/2}^3\right) && (7.78)\\
&= \frac{2}{3}\left(\frac{9}{2n\pi}\sin(\frac{n\pi}{2})+\frac{9}{n^2\pi^2}\left(cos(n\pi)-\cos(\frac{n\pi}{2})\right)\right) && (7.79)\\
&= \frac{6}{n\pi}\left(\frac{1}{2}\sin(\frac{n\pi}{2})+\frac{1}{n\pi}\left((-1)^n-\cos(\frac{n\pi}{2})\right)\right) && (7.80)\\
&= \frac{6}{n\pi}\left(\frac{1}{n\pi}((-1)^n-\cos(\frac{n\pi}{2}))+\frac{1}{2}\sin(\frac{n\pi}{2})\right). && (7.81)
\end{aligned}
$$

So the Fourier Series is

$$
f_{even}=\frac{9}{8}+\frac{6}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{n\pi}((-1)^n-\cos(\frac{n\pi}{2}))+\frac{1}{2}\sin(\frac{n\pi}{2})\right)\cos(\frac{n\pi x}{3}). \quad (7.82)
$$

■

8. Partial Differential Equations

8.1 Introduction to Basic Classes of PDEs

8.2 Introduction to PDEs

Many physical and geometric problems rely on models formed by partial differential equations (PDEs). Here the known functions depend on more than one variable, for example space and time. In previous chapters we studied ODEs equations which have limited use in modeling physical systems. Usually they are restricted to the simplest situations such as spring-mass systems or population dynamics. Now we wish to consider problems form a wider range of fields such as elasticity, thermodynamics, electrostatics, quantum mechanics, and population dynamics.

Throughout the remainder of the Chapter we focus on *initial value problems* (IVP) and *boundary value problems* (BVP) for common physical systems such as a vibrating string, temperature distribution in a material, or an elastic membrane.

In this section, we briefly outline the basic concepts involved in studying PDEs and the six most basic PDEs common to math and physics and describe the physical situations where each comes about.

8.2.1 Basics of Partial Differential Equations

Definition 8.2.1 A **partial differentiation equation** (PDE) is an equation involving one or more partial derivatives of an unknown function that depends on more than one variable.

Definition 8.2.2 The **order** of a PDE is the order of the highest derivative.

Definition 8.2.3 A PDE is **linear** if it is of the first degree in the unknown function and its partial derivatives, otherwise we call it **nonlinear**.

R The remainder of the course focuses on second order linear PDEs, which have a surprisingly wide range of applications.

Example 8.1 Determine if the following PDEs are linear and what their order is:

- $u_x x + 2uu_x = 3$
- $u_{xxx} + \sin(u) = 0$
- $u_x + 3u = 5\sin(x)$
- $(u_x)^3 + u_x x = x^3$

■

Definition 8.2.4 We call a *linear* PDE **homogeneous** if each of its terms contains either u or one of its partial derivatives, otherwise we call the equation **nonhomogeneous**.

Example 8.2 In the previous example determine which of the equations is homogeneous. ■

8.2.2 Laplace's Equation - Type: Elliptical

$$\Delta u = 0 \tag{8.1}$$

Applications include gravitational potential u in a region with no mass, electrostatic potential in a charge free region, the steady state temperature distribution in a region without sources or sinks, or the velocity of an incompressible fluid with no vortices or sinks.

8.2.3 Poisson's Equation

$$\Delta u = f(x,y,z) \tag{8.2}$$

Applications include everything for Laplace, but f represents a source term either charge, electrical, force, or heat source.

8.2.4 Diffusion/Heat Equation - Type: Parabolic

$$\Delta u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t} \tag{8.3}$$

The quantity u can represent a non-steady state temperature distribution in a region without heat sources, concentration of a diffusing substance. Here α^2 is known as the *thermal diffusivity*.

8.2.5 Wave Equation - Type: Hyperbolic

$$\Delta u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \tag{8.4}$$

The quantity u represents displacement from equilibrium of a vibrating string or membrane, in electrostatics it can be the current or potential along a transmission line, or u can be the component of the electric or magnetic field in an electromagnetic wave.

8.2.6 Helmholtz Equation

$$\Delta F + k^2 F = 0 \tag{8.5}$$

The time-independent form of the wave equation.

8.2.7 Schrödinger Equation

$$= \frac{\hbar}{2m}\Delta\Psi + V\Psi = i\hbar\frac{\partial}{\partial t}\Psi \tag{8.6}$$

The wave equation of quantum mechanics where m is the particle mass, $\hbar$ is Planck's constant, $i = \sqrt{-1}$, and V is the potential energy of the particle. The wave function Ψ is complex and its absolute value squared is proportional to the position probability of the particle.

8.2.8 Solutions to PDEs

Definition 8.2.5 A **solution** of a PDE in some region R of the space of the independent variables is a function that has all the partial derivatives appearing in the PDE in some domain D containing R, and satisfies the PDE everywhere in R.

(R) Solutions to the same equation can look very different. Consider the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

The following five functions are all solutions (verify): 1. $u = x^2 - y^2$, 2. $e^x \cos(y)$, 3. $u = \sin(x)\cosh(y)$, 4. $u = 5$, 5. $u = \ln(x^2 + y^2)$. A unique solution is obtained by combining the PDE with initial and/or boundary conditions.

Definition 8.2.6 If there is a condition prescribing the values of the unknown function u on the boundary of domain R, we call these conditions **boundary conditions**. When t is one of the variables we can describe the unknown function u or its derivatives at time $t = 0$, we call these conditions **initial conditions**.

Theorem 8.2.1 *(Principle of Superposition)* If u_1 and u_2 are solutions of a **homogeneous linear PDE** in some region R, then

$$u = c_1 u_1 + c_2 u_2 \tag{8.7}$$

with any constants c_1, c_2 is also a solution of that PDE in the region R.

■ **Example 8.3** *(Similar to ODE)* Find solutions u of the PDE $u_{xx} - u = 0$ where $u = u(x,y)$.

Solution: Since there are no y-derivatives, then we can solve this PDE like we would $u'' - u = 0$. Using the characteristic equation we find $r^2 - 1 = 0 \Rightarrow r = \pm 1$. Thus, this ODE has solution $u(x) = c_1 e^x + c_2 e^{-x}$ for constants c_1, c_2. To solve the PDE we must remember these constant could also be functions of y, so the solution of the PDE is

$$u(x,y) = c_1(y)e^x + c_2(y)e^{-x}$$

for arbitrary functions of y, $c_1(y), c_2(y)$. ■

■ **Example 8.4** *(Similar to ODE)* Find solutions $u = u(x,y)$ of the PDE $u_{xy} = u_x$.

Solution: Let $u_x = p$, then $p_y = u_{xy} = -u_x = -p$. Solving the equation for p gives $p = c(x)e^{-y}$, then integrate with respect to x to get u:

$$u(x,y) = f(x)e^{-y} + g(x), \qquad f(x) = \int c(x)dx.$$

■

8.3 Laplace's Equations and Steady State Temperature Problems

We will consider the two-dimensional and three-dimensional Laplace Equations

$$(2D): \quad u_{xx} + u_{yy} = 0, \tag{8.8}$$
$$(3D): \quad u_{xx} + u_{yy} + u_{zz} = 0. \tag{8.9}$$

8.3.1 Dirichlet Problem for a Rectangle

We want to find the function u satisfying Laplace's Equation

$$u_{xx} + u_{yy} = 0 \tag{8.10}$$

in the rectangle $0 < x < a, 0 < y < b$, and satisfying the boundary conditions

$$\begin{aligned} u(x,0) &= 0, \quad u(x,b) = 0, \quad 0 < x < a, && (8.11)\\ u(0,y) &= 0, \quad u(a,y) = f(y), \quad 0 \leq y \leq b. && (8.12) \end{aligned}$$

We need four boundary conditions for the four spatial derivatives.

Start by using Separation of Variables and assume $u(x,y) = X(x)Y(y)$. Substitute u into Equation (8.54). This yields

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda,$$

where λ is a constant. We obtain the following system of ODEs

$$\begin{aligned} X'' - \lambda X &= 0 && (8.13)\\ Y'' + \lambda Y &= 0. && (8.14) \end{aligned}$$

From the boundary conditions we find

$$\begin{aligned} X(0) &= 0 && (8.15)\\ Y(0) &= 0, Y(b) = 0. && (8.16) \end{aligned}$$

We first solve the ODE for Y, which we have seen numerous times before. Using the BCs we find there are nontrivial solutions if and only if λ is an eigenvalue

$$\lambda = \left(\frac{n\pi}{b}\right)^2, \quad n = 1, 2, 3, \dots$$

and $Y_n(y) = \sin(\frac{n\pi y}{b})$, the corresponding eigenfunction. Now substituting in for λ we want to solve the ODE for X. This is another problem we have seen regularly and the solution is

$$X_n(x) = c_1 \cosh\left(\frac{n\pi x}{b}\right) + c_2 \sinh\left(\frac{n\pi x}{b}\right)$$

The BC implies that $c_1 = 0$. So the fundamental solution to the problem is

$$u_n(x,y) = \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right).$$

By linear superposition the general solution is

$$u(x,y) = \sum_{n=1}^{\infty} c_n u_n(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right).$$

Using the last boundary condition $u(a,y) = f(y)$ solve for the coefficients c_n.

$$u(a,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi a}{b}\right) \sin\left(\frac{n\pi y}{b}\right) = f(y)$$

Using the Fourier Since Series coefficients we find

$$c_n = \frac{2}{b \sinh(\frac{n\pi a}{b})} \int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy.$$

8.3.2 Dirichlet Problem For A Circle

Consider solving Laplace's Equation in a circular region $r < a$ subject to the boundary condition

$$u(a,\theta) = f(\theta)$$

where f is a given function on $0 \le \theta \le 2\pi$. In polar coordinates Laplace's Equation becomes

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0. \tag{8.17}$$

Try Separation of Variables in Polar Coordinates

$$u(r,\theta) = R(r)\Theta(\theta),$$

plug into the differential equation, Equation (8.114). This yields

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

or

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

where λ is a constant. We obtain the following system of ODEs

$$r^2R'' + rR' - \lambda R = 0, \tag{8.18}$$
$$\Theta'' + \lambda\theta = 0. \tag{8.19}$$

Since we have no homogeneous boundary conditions we must use instead the fact that the solutions must be bounded and also periodic in Θ with period 2π. It can be shown that we need λ to be real. Consider the three cases when $\lambda < 0, \lambda = 0, \lambda > 0$.

If $\lambda < 0$, let $\lambda = -\mu^2$, where $\mu > 0$. So we find the equation for Θ becomes $\Theta'' - \mu^2\Theta = 0$. So

$$\Theta(\theta) = c_1e^{\mu\theta} + c_2e^{-\mu\theta}$$

Θ can only be periodic if $c_1 = c_2 = 0$, so λ cannot be negative (Since we do not get any nontrivial solutions.

If $\lambda = 0$, then the equation for Θ becomes $\Theta'' = 0$ and thus

$$\Theta(\theta) = c_1 + c_2\theta$$

For Θ to be periodic $c_2 = 0$. Then the equation for R becomes

$$r^2R'' + rR' = 0.$$

This equation is an Euler equation and has solution

$$R(r) = k_1 + k_2\ln(r)$$

Since we also need the solution bounded as $r \to \infty$, then $k_2 = 0$. So $u(r,\theta)$ is a constant, and thus proportional to the solution $u_0(r,\theta) = 1$.

If $\lambda > 0$, we let $\lambda = \mu^2$, where $\mu > 0$. Then the system of equations becomes

$$r^2R'' + rR' - \mu^2R = 0 \tag{8.20}$$
$$\Theta'' + \mu^2\Theta = 0 \tag{8.21}$$

The equation for R is an Euler equation and has the solution

$$R(r) = k_1 r^{\mu} + k_2 r^{-\mu}$$

and the equation for Θ has the solution

$$\Theta(\theta) = c_1 \sin(\mu\theta) + c_2 \cos(\mu\theta).$$

For Θ to be periodic we need μ to be a positive integer n, so $\mu = n$. Thus the solution $r^{-\mu}$is unbounded as $r \to 0$. So $k_2 = 0$. So the solutions to the original problem are

$$u_n(r,\theta) = r^n \cos(n\theta), \quad v_n(r,\theta) = r^n \sin(n\theta), \quad n = 1,2,3,...$$

Together with $u_0(r,\theta) = 1$, by linear superposition we find

$$u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n (c_n \cos(n\theta) + k_n \sin(n\theta)).$$

Using the boundary condition from the beginning

$$u(a,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} a^n (c_n \cos(n\theta) + k_n \sin(n\theta)) = f(\theta)$$

for $0 \leq \theta \leq 2\pi$. We compute to coefficients by using our previous Fourier Series equations

$$c_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \cos(n\theta) d\theta, \quad n = 1,2,3,... \tag{8.22}$$

$$k_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \sin(n\theta) d\theta, \quad n = 1,2,3,... \tag{8.23}$$

Note we need both terms since sine and cosine terms remain throughout the general solution.

■ **Example 8.5** Find the solution $u(x,y)$ of Laplace's Equation in the rectangle $0 < x < a$, $0 < y < b$, that satisfies the boundary conditions

$$u(0,y) = 0, \quad u(a,y) = 0, \quad 0 < y < b \tag{8.24}$$

$$u(x,0) = h(x), \quad u(x,b) = 0, \quad 0 \leq x \leq a \tag{8.25}$$

Answer: Using the method of Separation of Variables, write $u(x,y) = X(x)Y(y)$. We get the following system of ODEs

$$X'' + \lambda X = 0, \quad X(0) = X(a) = 0 \tag{8.26}$$

$$Y'' - \lambda Y = 0, \quad Y(b) = 0 \tag{8.27}$$

It follows that $\lambda_n = (\frac{n\pi}{a})^2$ and $X_n(x) = \sin(\frac{n\pi x}{a})$. The solution of the second ODE gives

$$Y(y) = d_1 \cosh(\lambda(b-y)) + d_2 \sinh(\lambda(b-y)).$$

Using $y(b) = 0$, we find that $d_1 = 0$. Therefore the fundamental solutions are

$$u_n(x,y) = \sin(\frac{n\pi x}{a}) \sinh(\lambda_n(b-y)),$$

and the general solution is

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{a}) \sinh(\frac{n\pi(b-y)}{a}).$$

Using another boundary condition

$$h(x) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{a}) \sinh(\frac{n\pi b}{a}).$$

The coefficients are calculated using the equation from the Fourier Sine Series

$$c_n = \frac{2}{a \sinh(\frac{n\pi b}{a})} \int_0^a h(x) \sin(\frac{n\pi x}{a}) dx.$$

■

■ **Example 8.6** Consider the problem of finding a solution $u(x,y)$ of Laplace's Equation in the rectangle $0 < x < a$, $0 < y < b$, that satisfies the boundary conditions

$$u_x(0,y) = 0, \quad u_x(a,y) = f(y), \quad 0 < y < b, \tag{8.28}$$
$$u_y(x,0) = 0, \quad u_y(x,b) = 0, \quad 0 \le x \le a \tag{8.29}$$

This is an example of a Neumann Problem. We want to find the fundamental set of solutions.

$$X'' - \lambda X = 0, \quad X'(0) = 0 \tag{8.30}$$
$$Y'' + \lambda Y = 0, \quad Y'(0) = Y'(b) = 0. \tag{8.31}$$

The solution to the equation for Y is

$$Y(y) = c_1 \cos(\lambda^{1/2} y) + c_2 \sin(\lambda^{1/2} y),$$

with $Y'(y) = -c_1 \lambda^{1/2} \sin(\lambda^{1/2} y) + c_2 \lambda^{1/2} \cos(\lambda^{1/2} y)$. Using the boundary conditions we find $c_2 = 0$ and the eigenvalues are $\lambda_n = \frac{n^2\pi^2}{b^2}$, for $n = 1,2,3,\ldots$. The corresponding Eigenfunctions are $Y(y) = \cos(\frac{n\pi y}{b})$ for $n = 1,2,3,\ldots$ The solution of the equation for X becomes $X(x) = d_1 \cosh(\frac{n\pi x}{b}) + d_2 \sinh(\frac{n\pi x}{b})$, with

$$X'(x) = d_1 \frac{n\pi}{b} \sinh(\frac{n\pi x}{b}) + d_2 \frac{n\pi}{b} \cosh(\frac{n\pi x}{b}).$$

Using the boundary conditions, $X(x) = d_1 \cosh(\frac{n\pi x}{b})$.So the fundamental set of solutions is

$$u_n(x,y) = \cosh(\frac{n\pi x}{b}) \cos(\frac{n\pi y}{b}), \quad n = 1,2,3,\ldots$$

The general solution is given by

$$u(x,y) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cosh(\frac{n\pi x}{b}) \cos(\frac{n\pi y}{b})$$

■

8.4 Heat Equation and Schrödinger Equation

We will soon see that partial differential equations can be far more complicated than ordinary differential equations. For PDEs, there is no general theory, the methods need to be adapted for smaller groups of equations. This course will only do an introduction, you can find out much more in advanced courses. We will be focusing on a single solution method called **Separation of Variables**, which is pervasive in engineering and mathematics.

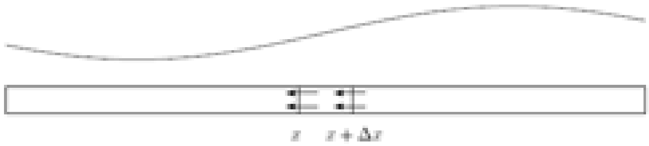


Figure 8.1: Heat Flux across the boundary of a small slab with length Δx. The graph is the graph of temperature at a given time t. In accordance with Fourier's Law, the heat leaves or enters the boundary by flowing from hot to cold; hence at x the flux is opposing the sign of u_x, while at $x + \Delta x$ it is agreeing.

The first partial differential equation to consider is the famous **heat equation** which models the temperature distribution in some object. We will focus on the one-dimensional heat equation, where we want to find the temperature distributions in a one-dimensional bar of length l. In particular we will assume that our bar corresponds to the interval $(0, l)$ on the real line.

The assumption is made purely for simplicity. If we assume we have a real bar, the one-dimensional assumption is equivalent to assuming at every lateral cross-section and every instant of time, the temperature is constant. While this is unrealistic it is not a terrible assumption. Also, if the length is much larger than the width in advanced mathematics one can assume the width is 0 since it is such a small fraction of the length. We are also assuming the bar is perfectly insulated, so the only way heat can enter or leave the bar is through the ends $x = 0$ and $x = l$. So any heat transfer will be one-dimensional.

8.4.1 Derivation of the Heat Equation

Many PDEs come from basic physical laws. Let $u(x,t)$ denote the temperature at a point x at time t. c will be the specific heat of the material the bar is made from (which is the amount of heat needed to raise one unit of mass of this material by one temperature unit) and ρ is the density of the rod. Note that in general, the specific heat and density of the rods do not have to be constants, they may vary with x. We greatly simplify the problem by allowing them to be constant.

Let's consider a small slab of length Δx. We will let $H(t)$ be the amount of heat contained in this slab. The mass of the slab is $\rho \Delta x$ and the heat energy contained in this small region is given by

$$H(t) = cu\rho\Delta x \tag{8.32}$$

On the other hand, within the slab, heat will flow from hot to cold (this is **Fourier's Law**). The only way heat can leave is by leaving through the boundaries, which are at x and $x + \Delta x$ (This is the **Law of Conservation of Energy**). So the change of heat energy of the slab is equal to the heat flux across the boundary. If κ is the conductivity of the bar's material

$$\frac{dH}{dt} = \kappa u_x(x + \Delta x, t) - \kappa u_x(x,t)$$

This is illustrated in Figure 8.4.1. Setting the derivative of $H(t)$ from above equal to the previous equations we find

$$(cu(x,t)\rho\Delta x)_t = \kappa u_x(x + \Delta x, t) - \kappa u_x(x,t)$$

or

$$c\rho u_t(x,t) = \frac{\kappa u_x(x + \Delta x, t) - \kappa u_x(x,t)}{\Delta x}.$$

Figure 8.2: Temperature versus position on a bar. The arrows show time dependence in accordance with the heat equation. The temperature graph is concave up, so the left side of the bar is warming up. While on the right the temperature is concave down and so th right side is cooling down..

If we take the limit as $\Delta x \to 0$, the right hand side is just the x-derivative of $\kappa u_x(x,t)$ or

$$c\rho u_t(x,t) = \kappa u_{xx}(x,t).$$

Setting $k = \frac{\kappa}{c\rho} > 0$, we have the heat equation

$$u_t = k u_{xx}.$$

Notice that the heat equation is a linear PDE, since all of the derivatives of u are only multiplied by constants. What is the constant k? It is called the **Thermal Diffusivity** of the bar and is a measure of how quickly heat spreads through a given material.

How do we interpret the heat equation? Graph the temperature of the bar at a fixed time. Suppose it looks like Figure 2. On the left side the bar is concave up. If the graph is concave up, that means that the second derivative of the temperature (with respect to position x) is positive. The heat equation tells us that the time derivative of the temperature at any of the points on the left side of the bar will be increasing. The left side of the bar will be warming up. Similarly, on the right side of the bar, the graph is concave down. Thus the second x-derivative of the temperature is negative, and so will be the first t-derivative, and we can conclude that the right side of the bar is cooling down.

8.5 Separation of Variables and Heat Equation IVPs

8.5.1 Initial Value Problems

Partial Differential Equations generally have a lot of solutions. To specify a unique one, we will need additional conditions. These conditions are motivated by physics and are initial or boundary conditions. An IVP for a PDE consists for the heat equation, initial conditions, and boundary conditions.

An initial condition specifies the physical state at a given time t_0. For example, and initial condition for the heat equation would be the starting temperature distribution

$$u(x,0) = f(x)$$

This is the only condition required because the heat equation is first order with respect to time. The wave equation, considered in a future section is second order in time and needs two initial conditions.

PDEs are only valid on a given domain. Boundary conditions specify how the solution behaves on the boundaries of the given domain. These need to be specified, because the solution does not exist on one side of the boundary, we might have problems with differentiability there.

Our heat equation was derived for a one-dimensional bar of length l, so the relevant domain in question can be taken to be the interval $0 < x < l$ and the boundary consists of the two points $x = 0$

and $x = l$. We could have derived a two-dimensional heat equation, for example, in which case the domain would be some region in the xy-plane with the boundary being some closed curve.

It will be clear from the physical description of the problem what the appropriate boundary conditions are. We might know at the endpoints $x = 0$ and $x = l$, the temperature $u(0,t)$ and $u(l,t)$ are fixed. Boundary conditions that give the value of the solution are called **Dirichlet Boundary Conditions**. Or we might insulate the ends of the bar, meaning there should be no heat flow out of the boundary. This would yield the boundary conditions $u_x(0,t) = u_x(l,t) = 0$. If the boundary conditions specify the derivative at the boundary, they are called **Neumann Conditions**. If the boundary conditions specify that we have one insulated end and at the other we control the temperature. This is an example of a **Mixed Boundary Condition**.

As we have seen, changing boundary conditions can significantly change the solution. Initially, we will work with homogeneous Dirichlet conditions $u(0,t) = u(l,t) = 0$, giving us the following initial value problem

$$(DE): \quad u_t = ku_{xx} \tag{8.33}$$
$$(BC): \quad u(0,t) = u(l,t) = 0 \tag{8.34}$$
$$(IC): u(x,0) = f(x) \tag{8.35}$$

After we have seen the general method, we will see what happens with homogeneous Neumann conditions. We will discuss nonhomogeneous equations later.

8.5.2 Separation of Variables

Above we have derived the heat equation for the bar of length L. Suppose we have an initial value problem such as Equation (8.33)-(8.35). How should we proceed? We want to try to build a general solution out of smaller solutions which are easier to find.

We start by assuming we have a **separated solution**, where

$$u(x,t) = X(x)T(t).$$

Our solution is the product of a function that depends only on x and a function that depends only on t. We can then try to write down an equation depending only on x and another solution depending only on t before using our knowledge of ODEs to try and solve them.

It should be noted that this is a very special situation and will not occur in general. Even when we can use it sometimes it is hard to move beyond the first step. However, it works for all equations we will be considering in this class, and is a good starting point.

How does this method work? Plug the separated solution into the heat equation.

$$\frac{\partial}{\partial t}[X(x)T(t)] = k\frac{\partial^2}{\partial x^2}[X(x)T(t)] \tag{8.36}$$
$$X(x)T'(t) = kX''(x)T(t) \tag{8.37}$$

Now notice that we can move everything depending on x to one side and everything depending on t to the other.

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}$$

This equation should says that both sides are equal for any x or t we choose. Thus they both must be equal to a constant. Since if what they equal depended on x or t both sides would not be equal for all x and t. So

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

We have written the minus sign for convenience. It will turn out that $\lambda > 0$.

The equation above contains a pair of separate ordinary differential equations

$$X'' + \lambda X = 0 \tag{8.38}$$
$$T' + \lambda kT = 0. \tag{8.39}$$

Notice that our boundary conditions becomes $X(0) = 0$ and $X(l) = 0$. Now the second equation can easily be solved, since we have $T' = -\lambda kT$, so that

$$T(t) = Ae^{-\lambda kt}.$$

The first equation gives a boundary value problem

$$X'' + \lambda X = 0 \quad X(0) = 0 \quad X(l) = 0$$

This should look familiar. The is the basic eigenfunction problem studied in section 10.1. As in that example, it turns out our eigenvalues have to be positive. Let $\lambda = \mu^2$ for $\mu > 0$, our general solution is

$$X(x) = B\cos(\mu x) + C\sin(\mu x).$$

The first boundary condition says $B = 0$. The second condition says that $X(l) = C\sin(\mu l) = 0$. To avoid only having the trivial solution, we must have $\mu l = n\pi$. In other words,

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad \text{and} \quad X_n(x) = \sin\left(\frac{n\pi x}{l}\right)$$

for $n = 1, 2, 3, ...$

So we end up having found infinitely many solutions to our boundary value problem, one for each positive integer n. They are

$$u_n(x,t) = A_n e^{-(\frac{n\pi}{l})^2 kt} \sin\left(\frac{n\pi x}{l}\right).$$

The heat equation is linear and homogeneous. As such, the Principle of Superposition still holds. So a linear combination of solutions is again a solution. So any function of the form

$$u(x,t) = \sum_{n=0}^{N} A_n e^{-(\frac{n\pi}{l})^2 kt} \sin\left(\frac{n\pi x}{l}\right) \tag{8.40}$$

is also a solution to our problem.

Notice we have not used our initial condition yet. We have

$$f(x) = u(x,0) = \sum_{n=0}^{N} A_n \sin\left(\frac{n\pi x}{l}\right).$$

So if our initial condition has this form, the result of superposition Equation (8.40) is in a good form to use the IC. The coefficients A_n just being the associated coefficients from $f(x)$.

■ **Example 8.7** Find the solutions to the following heat equation problem on a rod of length 2.

$$u_t = u_{xx} \tag{8.41}$$
$$u(0,t) = u(2,t) = 0 \tag{8.42}$$
$$u(x,0) = \sin\left(\frac{3\pi x}{2}\right) - 5\sin(3\pi x) \tag{8.43}$$

In this problem, we have $k = 1$. Now we know that our solution will have the form like Equation (8.40), since our initial condition is just the difference of two sine functions. We just need to figure out which terms are represented and what the coefficients A_n are.

Our initial condition is

$$f(x) = \sin(\frac{3\pi x}{2}) - 5\sin(3\pi x)$$

Looking at (8.40) with $l = 2$, we can see that the first term corresponds to $n = 3$ and the second $n = 6$, and there are no other terms. Thus we have $A_3 = 1$ and $A_6 = -5$, and all other $A_n = 0$. Our solution is then

$$u(x,t) = e^{-(\frac{9\pi^2}{4})t}\sin(\frac{3\pi x}{2}) - 5e^{(-9\pi^2)t}\sin(3\pi x).$$

■

There is no reason to suppose that our initial distribution is a finite sum of sine functions. Physically, such situations are special. What do we do if we have a more general initial temperature distribution?

Let's consider what happens if we take an **infinite** sum of our separated solutions. Then our solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-(\frac{n\pi}{l})^2 kt}\sin(\frac{n\pi x}{l}).$$

Now the initial condition gives

$$f(x) = \sum_{n=0}^{\infty} A_n \sin(\frac{n\pi x}{l}).$$

This idea is due to the French Mathematician Joseph Fourier and is called the **Fourier Sine Series** for $f(x)$.

There are several important questions that arise. Why should we believe that our initial condition $f(x)$ ought to be able to be written as an infinite sum of sines? why should we believe that such a sum would converge to anything?

8.5.3 Neumann Boundary Conditions

Now let's consider a heat equation problem with homogeneous Neumann conditions

$$(DE): \quad u_t = u_{xx} \tag{8.44}$$
$$(BC): \quad u_x(0,t) = u_x(l,t) = 0 \tag{8.45}$$
$$(IC): \quad u(x,0) = f(x) \tag{8.46}$$

We will start by again supposing that our solution to Equation (8.44) is separable, so we have $u(x,t) = X(x)T(t)$ and we obtain a pair of ODEs, which are the same as before

$$X'' + \lambda X = 0 \tag{8.47}$$
$$T' + \lambda kT = 0. \tag{8.48}$$

The solution to the first equation is still

$$T(t) = Ae^{-\lambda kt}$$

Now we need to determine the boundary conditions for the second equation. Our boundary conditions are $u_x(0,t)$ and $u_x(l,t)$. Thus they are conditions for $X'(0)$ and $X'(l)$, since the t-derivative is not controlled at all. So we have the boundary value problem

$$X'' + \lambda X = 0 \quad X'(0) = 0 \quad X'(l) = 0.$$

Along the lines of the analogous computation last lecture, this has eigenvalues and eigenfunctions

$$\lambda_n = (\frac{n\pi}{l})^2 \tag{8.49}$$

$$y_n(x) = \cos(\frac{n\pi x}{l}) \tag{8.50}$$

for $n = 0, 1, 2, \ldots$ So the individual solutions to Equation (8.44) have the form

$$u_(x,t) = A_n e^{(\frac{n\pi}{l})^2 kt} \cos(\frac{n\pi x}{l}).$$

Taking finite linear combinations of these work similarly to the Dirichlet case (and is the solution to Equation (8.44) when $f(x)$ is a finite linear combination of constants and cosines, but in general we are interested in knowing when we can take infinite sums, i.e.

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{l})^2 kt} \cos(\frac{n\pi x}{l})$$

Notice how we wrote the $n = 0$ case, as $\frac{1}{2}A_0$. The reason will be clear when talking about Fourier Series. The initial conditions means we need

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{l}).$$

An expression of the form above is called the **Fourier Cosine Series** of $f(x)$.

8.5.4 Other Boundary Conditions

It is also possible for certain boundary conditions to require the "full" Fourier Series of the initial data, this is an expression of the form

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \left(A_n \cos(\frac{n\pi x}{l}) + B_n \sin(\frac{n\pi x}{l}) \right).$$

but in most cases we will work with Dirichlet or Neumann conditions. However, in the process of learning about Fourier sine and cosine series, we will also learn how to compute the full Fourier series of a function.

8.6 Heat Equation Problems

In the previous lecture on the Heat Equation we saw that the product solutions to the heat equation with homogeneous Dirichlet boundary conditions problem

$$u_t = ku_{xx} \tag{8.51}$$

$$u(0,t) = u(l,t) = 0 \tag{8.52}$$

$$u(x,0) = f(x) \tag{8.53}$$

had the form

$$u_n(x,t) = B_n e^{-(\frac{n\pi}{l})kt} \sin(\frac{n\pi x}{l}) \quad n = 1, 2, 3, \ldots$$

Taking linear combinations of these (over each n) gives a general solution to the above problem.

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-(\frac{n\pi}{l})kt} \sin\left(\frac{n\pi x}{l}\right) \tag{8.54}$$

Setting $t = 0$, this implies that we must have

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)$$

In other words, the coefficients in the general solution for the given initial condition are the **Fourier Sine** coefficients of $f(x)$ on $(0,l)$, which are given by

$$B_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

We also, saw that if we instead have a problem with homogeneous Neumann boundary conditions

$$u_t = ku_{xx} \quad 0 < x < l, \quad t > 0 \tag{8.55}$$
$$u_x(0,t) = u_x(l,t) = 0 \tag{8.56}$$
$$u(0,t) = f(x) \tag{8.57}$$

the product solutions had the form

$$u_n(x,t) = A_n e^{-(\frac{n\pi}{l})^2 kt} \cos\left(\frac{n\pi x}{l}\right) \quad n = 1,2,3,...$$

and the general solution has the form

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{l})^2 kt} \cos\left(\frac{n\pi x}{l}\right). \tag{8.58}$$

With $t = 0$ this means that the initial condition must satisfy

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right).$$

and so the coefficients for a particular initial condition are the **Fourier Cosine** coefficients of $f(x)$, given by

$$A_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx.$$

One way to think about this difference is that given the initial data $u(x,0) = f(x)$, the Dirichlet conditions specify the **odd extension** of $f(x)$ as the desired periodic solution, while the Neumann conditions specify the **even extension**. This should make sense since odd functions must have $f(0) = 0$, while even functions must have $f'(0) = 0$.

So to solve a homogeneous heat equation problem, we begin by identifying the type of boundary conditions we have. If we have Dirichlet conditions, we know our solution will have the form of Equation (8.54). All we then have to do is compute the Fourier Sine coefficients of $f(x)$. Similarly, if we have Neumann conditions, we know the solution has the form of Equation (8.114) and we have to compute the Fourier Cosine coefficients of $f(x)$.

R Observe that for any homogeneous Dirichlet problem, the temperature distribution (8.54) will go to 0 as $t \to \infty$. This should make sense because these boundary conditions have a physical interpretation where we keep the ends of our rod at freezing temperature without regulating the heat flow in and out of the endpoints. As a result, if the interior of the rod is initially above freezing, that heat will radiate towards the endpoints and into our reservoirs at the endpoints. On the other hand, if the interior of the rod is below freezing, heat will come from the reservoirs at the endpoints and warm it up until the temperature is uniform.

For the Neumann problem, the temperature distribution (8.114) will converge to $\frac{1}{2}A_0$. Again, this should make sense because these boundary conditions correspond to a situation where we have insulated ends, since we are preventing any heat from escaping the bar. Thus all heat energy will move around inside the rod until the temperature is uniform.

8.6.1 Examples

■ **Example 8.8** Solve the initial value problem

$$u_t = 3u_{xx} \quad 0 < x < 2, \quad t > 0 \tag{8.59}$$
$$u(0,t) = u(2,t) = 0 \tag{8.60}$$
$$u(x,0) = 20. \tag{8.61}$$

This problem has homogeneous Dirichlet conditions, so by (8.54) our general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-3(\frac{n\pi}{2})^2 t} \sin(\frac{n\pi x}{2}).$$

The coefficients for the particular solution are the Fourier Sine coefficients of $u(x,0) = 20$, so we have

$$B_n = \frac{2}{2}\int_0^2 20\sin(\frac{n\pi x}{2})dx \tag{8.62}$$
$$= [-\frac{40}{n\pi}\cos(\frac{n\pi x}{2})]_0^2 \tag{8.63}$$
$$= -\frac{40}{n\pi}(\cos(n\pi) - \cos(0)) \tag{8.64}$$
$$= \frac{40}{n\pi}(1 + (-1)^{n+1}) \tag{8.65}$$

and the solution to the problem is

$$u(x,t) = \frac{40}{\pi}\sum_{n=1}^{\infty} \frac{1+(-1)^{n+1}}{n} e^{-\frac{3n^2\pi^2}{4}t} \sin(\frac{n\pi x}{2}).$$

■

■ **Example 8.9** Solve the initial value problem

$$u_t = 3u_{xx} \quad 0 < x < 2, \quad t > 0 \tag{8.66}$$
$$u_x(0,t) = u_x(2,t) = 0 \tag{8.67}$$
$$u(x,0) = 3x. \tag{8.68}$$

This problem has homogeneous Neumann conditions, so by (8.114) our general solution is

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-3(\frac{n\pi}{2})^2 t} \cos(\frac{n\pi x}{2}).$$

The coefficients for the particular solution are the Fourier Cosine coefficients of $u(x,0) = 3x$, so we have

$$A_0 = \frac{2}{2}\int_0^2 3x dx = 6 \tag{8.69}$$

$$\begin{aligned}
A_n &= \frac{2}{2}\int_0^2 3x\cos(\frac{n\pi x}{2})dx && (8.70)\\
&= [-\frac{6x}{n\pi}\cos(\frac{n\pi x}{2}) + \frac{12}{n^2\pi^2}\sin(\frac{n\pi x}{2})]_0^2 && (8.71)\\
&= -\frac{12}{n\pi}\cos(n\pi) && (8.72)\\
&= \frac{12}{n\pi}(-1)^{n+1} && (8.73)
\end{aligned}$$

and the solution to the problem is

$$u(x,t) = \frac{3}{2} + \frac{12}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-\frac{3n^2\pi^2}{4}t}\cos(\frac{n\pi x}{2}).$$

■

■ **Example 8.10** Solve the initial value problem

$$\begin{aligned}
u_t &= 4u_{xx} \quad 0 < x < 2\pi, \quad t > 0 && (8.74)\\
u(0,t) &= u(2\pi,t) = 0 && (8.75)\\
u(x,0) &= \begin{cases} 1 & 0 < x < \pi \\ x & \pi < x < 2\pi \end{cases} . && (8.76)
\end{aligned}$$

This problem has homogeneous Dirichlet conditions, so our general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2 t}\sin(\frac{nx}{2}).$$

The coefficients for the particular solution are the Fourier Sine coefficients of $u(x,0)$, so we have

$$\begin{aligned}
B_n &= \frac{2}{2\pi}\left(\int_0^{\pi}\sin(\frac{nx}{2})dx + \int_{\pi}^{2\pi}x\sin(\frac{nx}{2})dx\right) && (8.77)\\
&= -\frac{2}{n\pi}\cos(\frac{nx}{2})|_0^{\pi} - \frac{2x}{n\pi}\cos(\frac{nx}{2})|_{\pi}^{2\pi} + \frac{4}{n^2\pi}\sin(\frac{nx}{2})|_{\pi}^{2\pi} && (8.78)\\
&= -\frac{2}{n\pi}(\cos(\frac{nx}{2}) - \cos(0)) - \frac{4}{n}\cos(n\pi) + \frac{2}{n}\cos(\frac{n\pi}{2}) - \frac{4}{n^2\pi}\sin(\frac{n\pi}{2}) && (8.79)\\
&= -\frac{2}{n\pi}(\cos(\frac{n\pi}{2}) - 1) + \frac{4}{n}(-1)^{n+1} + \frac{2}{n}\cos(\frac{n\pi}{2}) - \frac{4}{n^2\pi}\sin(\frac{n\pi}{2}) && (8.80)\\
&= \frac{2}{n}\left(-\frac{1}{\pi}(\cos(\frac{n\pi}{2}) - 1) + 2(-1)^{n+1}\cos(\frac{n\pi}{2}) - \frac{2}{n\pi}\sin(\frac{n\pi}{2})\right) && (8.81)
\end{aligned}$$

and the solution to the problem is

$$u(x,t) = 2\sum_{n=1}^{\infty}\frac{1}{n}\left(-\frac{1}{\pi}(\cos(\frac{n\pi}{2}) - 1) + 2(-1)^{n+1}\cos(\frac{n\pi}{2}) - \frac{2}{n\pi}\sin(\frac{n\pi}{2})\right)e^{-n^2 t}\sin(\frac{nx}{2}).$$

■

8.7 Other Boundary Conditions

So far, we have used the technique of separation of variables to produce solutions to the heat equation

$$u_t = ku_{xx}$$

on $0 < x < l$ with either homogeneous Dirichlet boundary conditions $[u(0,t) = u(l,t) = 0]$ or homogeneous Neumann boundary conditions $[u_x(0,t) = u_x(l,t) = 0]$. What about for some other physically relevant boundary conditions?

8.7.1 Mixed Homogeneous Boundary Conditions

We could have the following boundary conditions

$$u(0,t) = u_x(l,t) = 0$$

Physically, this might correspond to keeping the end of the rod where $x = 0$ in a bowl of ice water, while the other end is insulated.

Use Separation of Variables. Let $u(x,t) = X(x)T(t)$, and we get the pair of ODEs

$$T' = -k\lambda T \tag{8.82}$$

$$X'' = -\lambda X. \tag{8.83}$$

Thus

$$T(t) = Be^{-k\lambda t}.$$

We now have a boundary value problem for X to deal with, where the boundary conditions are $X(0) = X'(l) = 0$. There are only positive eigenvalues, which are given by

$$\lambda_n = \left(\frac{(2n-1)\pi}{2l}\right)^2$$

and their associated eigenfunctions are

$$X_n(x) = \sin\left(\frac{(2n-1)\pi x}{2l}\right).$$

The separated solutions are then given by

$$u_n(x,t) = B_n e^{-(\frac{(2n-1)\pi}{2l})^2 kt} \sin\left(\frac{(2n-1)\pi x}{2l}\right)$$

and the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-(\frac{(2n-1)\pi}{2l})^2 kt} \sin\left(\frac{(2n-1)\pi x}{2l}\right). \tag{8.84}$$

with an initial condition $u(x,0) = f(x)$, we have that

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{(2n-1)\pi x}{2l}\right).$$

This is an example of a specialized sort of Fourier Series, the coefficients are given by

$$B_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{(2n-1)\pi x}{2l}\right)dx.$$

R The convergence for a series like the one above is different than that of our standard Fourier Sine or Cosine series, which converge to the periodic extension of the odd or even extensions of the original function, respectively. Notice that the terms in the sum above are periodic with period $4l$ (as opposed to the $2l$-periodic series we have seen before). In this case, we need to first extend our function $f(x)$, given on $(0,l)$, to a function on $(0,2l)$ symmetric around $x = l$. Then, as our terms are all sines, the convergence on $(-2l,2l)$ will be to the odd extension of this extended function, and the periodic extension of this will be what the series converges to on the entire real line.

■ **Example 8.11** Solve the following heat equation problem

$$\begin{aligned} u_t &= 25u_{xx} && (8.85)\\ u(0,t) &= 0 \quad u_x(10,t) = 0 && (8.86)\\ u(x,0) &= 5. && (8.87) \end{aligned}$$

By (8.84) our general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-25(\frac{(2n-1)\pi}{20})^2 t} \sin\left(\frac{(2n-1)\pi x}{20}\right).$$

The coefficients for the particular solution are given by

$$\begin{aligned} B_n &= \frac{2}{10}\int_0^{10} 5\sin\left(\frac{(2n-1)\pi x}{20}\right)dx && (8.88)\\ &= -\frac{10}{(2n-1)\pi}\cos\left(\frac{(2n-1)\pi x}{20}\right)\Big|_0^{10} && (8.89)\\ &= -\frac{10}{(2n-1)\pi}\left(\cos\left(\frac{(2n-1)\pi}{2}\right) - \cos(0)\right) && (8.90)\\ &= \frac{10}{(2n-1)\pi}. && (8.91) \end{aligned}$$

and the solution to the problem is

$$u(x,t) = \frac{10}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)} e^{-\frac{(2n-1)^2\pi^2}{16}t} \sin\left(\frac{(2n-1)\pi x}{20}\right).$$

■

8.7.2 Nonhomogeneous Dirichlet Conditions

The next type of boundary conditions we will look at are Dirichlet conditions, which fix the value of u at the endpoints $x = 0$ and $x = l$. For the heat equation, this corresponds to fixing the temperature at the ends of the rod. We have already looked at homogeneous conditions where the ends of the rod had fixed temperature 0. Now consider the nonhomogeneous Dirichlet conditions

$$u(0,t) = T_1, \quad u(l,t) = T_2$$

This problem is slightly more difficult than the homogeneous Dirichlet condition problem we have studied. Recall that for separation of variables to work, the differential equations and the boundary conditions must be homogeneous. When we have nonhomogeneous conditions we need to try to split the problem into one involving homogeneous conditions, which we know how to solve, and another dealing with the nonhomogeneity.

R We used a similar approach when we applied the method of Undetermined Coefficients to nonhomogeneous linear ordinary differential equations.

How can we separate the core homogeneous problem from what is causing the inhomogeneity? Consider what happens as $t \to \infty$. We should expect that, since we fix the temperatures at the endpoints and allow free heat flux at the boundary, at some point the temperature will stabilize and we will be at equilibrium. Such a temperature distribution would clearly not depend on time, and we can write

$$\lim_{t\to\infty} u(x,t) = v(x)$$

Notice that $v(x)$ must still satisfy the boundary conditions and the heat equation, but we should not expect it to satisfy the initial conditions (since for large t we are far from where we initially started). A solution such as $v(x)$ which does not depend on t is called a **steady-state** or **equilibrium solution**.

For a steady-state solution the boundary value problem becomes

$$0 = kv'' \quad v(0) = T_1 \quad v(l) = T_2.$$

It is easy to see that solutions to this second order differential equation are

$$v(x) = c_1 x + c_2$$

and applying the boundary conditions, we have

$$v(x) = T_1 + \frac{T_2 - T_1}{l} x.$$

Now, let

$$w(x,t) = u(x,t) - v(x)$$

so that

$$u(x,t) = w(x,t) + v(x).$$

This function $w(x,t)$ represents the **transient** part of $u(x,t)$ (since $v(x)$ is the equilibrium part). Taking derivatives we have

$$u_t = w_t + v_t = w_t \quad \text{and} \quad u_{xx} = w_{xx} + v_{xx} = w_{xx}.$$

Here we use the fact that $v(x)$ is independent of t and must satisfy the differential equation. Also, using the equilibrium equation $v'' = v_{xx} = 0$.

Thus $w(x,t)$ must satisfy the heat equation, as the relevant derivatives of it are identical to those of $u(x,t)$, which is known to satisfy the equation. What are the boundary and initial conditions?

$$w(0,t) = u(0,t) - v(0) = T_1 - T_1 = 0 \tag{8.92}$$
$$w(l,t) = u(l,t) - v(l) = T_2 - T_2 = 0 \tag{8.93}$$
$$w(x,0) = u(x,0) - v(x) = f(x) - v(x) \tag{8.94}$$

where $f(x) = u(x,0)$ is the given initial condition for the nonhomogeneous problem. Now, even though our initial condition is slightly messier, we now have homogeneous boundary conditions, since $w(x,t)$ must solve the problem

$$w_t = kw_{xx} \tag{8.95}$$
$$w(0,t) = w(l,t) = 0 \tag{8.96}$$
$$w(x,0) = f(x) - v(x) \tag{8.97}$$

This is just a homogeneous Dirichlet problem. We know the general solution is

$$w(x,t) = \sum_{n=1}^{\infty} B_n e^{-(\frac{n\pi}{l})^2 kt} \sin(\frac{n\pi x}{l}).$$

where the coefficients are given by

$$B_n = \frac{2}{l} \int_0^l (f(x) - v(x)) \sin(\frac{n\pi x}{l}) dx.$$

Notice that $\lim_{t\to\infty} w(x,t) = 0$, so that $w(x,t)$ is transient.

Thus, the solution to the nonhomogeneous Dirichlet problem

$$u_t = ku_{xx} \tag{8.98}$$
$$u(0,t) = T_1, \quad u(l,t) = T_2 \tag{8.99}$$
$$u(x,0) = f(x) \tag{8.100}$$

is $u(x,t) = w(x,t) + v(x)$, or

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-(\frac{n\pi}{l})^2 lt} \sin(\frac{n\pi x}{l}) + T_1 + \frac{T_2 - T_1}{l} x$$

with coefficients

$$B_n = \frac{2}{l} \int_0^l (f(x) - T_1 - \frac{T_2 - T_1}{l} x) \sin(\frac{n\pi x}{l}) dx.$$

R Do not memorize the formulas but remember what problem $w(x,t)$ has to solve and that the final solution is $u(x,t) = v(x) + w(x,t)$. For $v(x)$, it is not a hard formula, but if one is not sure of it, remember $v_{xx} = 0$ and it has the same boundary conditions as $u(x,t)$. This will recover it.

■ **Example 8.12** Solve the following heat equation problem

$$u_t = 3u_{xx} \tag{8.101}$$
$$u(0,t) = 20, \quad u(40,t) = 100 \tag{8.102}$$
$$u(x,0) = 40 - 3x \tag{8.103}$$

We start by writing

$$u(x,t) = v(x) + w(x,t)$$

where $v(x) = 20 + 2x$. Then $w(x,t)$ must satisfy the problem

$$w_t = 3w_{xx} \tag{8.104}$$
$$w(0,t) = w(40,t) = 0 \tag{8.105}$$
$$w(x,0) = 40 - 3x - (20 + 2x) = 20 - x \tag{8.106}$$

This is a homogeneous Dirichlet problem, so the general solution for $w(x,t)$ will be

$$w(x,t) = \sum_{n=1}^{\infty} e^{-3(\frac{n\pi}{40})^2 t} \sin(\frac{n\pi x}{40}).$$

The coefficients are given by

$$B_n = \frac{2}{40}\int_0^{40}(20-x)\sin\left(\frac{n\pi x}{40}\right)dx \tag{8.107}$$

$$= \frac{1}{20}\left[-\frac{40(20-x)}{n\pi}\cos\left(\frac{n\pi x}{40}\right) - \frac{1600}{n^2\pi^2}\sin\left(\frac{n\pi x}{40}\right)\right]_0^{40} \tag{8.108}$$

$$= \frac{1}{20}\left(\frac{800}{n\pi}\cos(n\pi) + \frac{800}{n\pi}\cos(0)\right) \tag{8.109}$$

$$= \frac{40}{n\pi}((-1)^n + 1). \tag{8.110}$$

So the solution is

$$u(x,t) = 20 + 2x + \frac{40}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n+1}{n}e^{-\frac{3n^2\pi^2}{1600}t}\sin\left(\frac{n\pi x}{40}\right).$$

■

8.7.3 Other Boundary Conditions

There are many other boundary conditions one could use, most of which have a physical interpretation. For example the boundary conditions

$$u(0,t) + u_x(0,t) = 0 \quad u(l,t) + u_x(l,t) = 0$$

say that the heat flux at the end points should be proportional to the temperature. We could also have had nonhomogeneous Neumann conditions

$$u_x(0,t) = F_1 \quad u_x(l,t) = F_2$$

which would specify allowing a certain heat flux at the boundaries. These conditions are not necessarily well suited for the method of separation of variables though and are left for future classes.

8.8 The Schrödinger Equation

Recall the Schrödinger equation

$$-\frac{\hbar}{2m}\Delta\Psi + V\Psi = i\hbar\frac{\partial}{\partial t}\Psi. \tag{8.111}$$

To approach this problem we begin by separating variables by assuming $\Psi = \psi(x,y,z)T(t)$. Substitution into (8.111) gives

$$-\frac{\hbar}{2m}\frac{1}{\psi}\Delta\psi + V = i\hbar\frac{1}{T}\frac{dT}{dt} = E$$

where E is the separation constant (E is the energy of the particle in quantum mechanics). Integrating the time equation in T gives

$$i\hbar\frac{1}{T}\frac{dT}{dt} = E \qquad \Rightarrow \qquad T(t) = e^{-iEt/\hbar}.$$

and the space equation (*Time-independent Schrödinger equation* gives (after multiplication by ψ)

$$-\frac{\hbar}{2m}\Delta\psi + V\psi = E\psi$$

We will only consider the simplest situation here of a 1D problem with $V = 0$

$$-\frac{\hbar}{2m}\frac{d^2\psi}{dx^2} = E\psi \qquad \Leftrightarrow \qquad \frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0.$$

The last equation is the Helmholtz equation with $k^2 = \frac{2mE}{\hbar^2}$. Thus, the solutions are

$$\Psi = \psi(x)T(t) = \sin(kx)e^{-iEt/\hbar}, \cos(kx)e^{-iEt/\hbar}.$$

■ **Example 8.13** The "particle in a box" problem in quantum mechanics requires the solution of the Schrödinger equation with $V = 0$ on $(0,\ell)$ and $\Psi = 0$ at the endpoints $x = 0,\ell$ for all t. The Dirichlet BCs require only the solutions with sine. The basis functions for this problem are the eigenfunctions

$$\Psi_n = \sin\left(\frac{n\pi x}{\ell}\right) r^{-iE_nt/\hbar}$$

and the general solution is a linear combination of these solutions

$$\Psi(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right) r^{-iE_nt/\hbar}$$

■

8.9 Wave Equations and the Vibrating String

8.9.1 Derivation of the Wave Equation

Consider a completely flexible string of length l and constant density ρ. We will assume that the string will only undergo relatively small vertical vibrations, so that points do not move from side to side. An example might be a plucked guitar string. Thus we can let $u(x,t)$ be its displacement from equilibrium at time t. The assumption of complete flexibility means that the tension force is tangent to the string, and the string itself provides no resistance to bending. This means the tension force only depends on the slope of the string.

Take a small piece of string going from x to $x + \Delta x$. Let $\Theta(x,t)$ be the angle from the horizontal of the string. Our goal is to use Newton's Second Law $F = ma$ to describe the motion. What forces are acting on this piece of string?

(a) Tension pulling to the right, which has magnitude $T(x+\Delta x,t)$ and acts at an angle of $\Theta(x+\Delta x,t)$ from the horizontal.

(b) Tension pulling to the left, which has magnitude $T(x,t)$ and acts at an angle of $\Theta(x,t)$ from the horizontal.

(c) Any external forces, which we denote by $F(x,t)$.

Initially, we will assume that $F(x,t) = 0$. The length of the string is essentially $\sqrt{(\Delta x)^2 + (\Delta u)^2}$, so the vertical component of Newton's Law says that

$$\rho\sqrt{(\Delta x)^2 + (\Delta u)^2} u_{tt}(x,t) = T(x+\Delta x,t)\sin(\Theta(x+\Delta x,t)) - T(x,t)\sin(\Theta(x,t)). \tag{8.112}$$

Dividing by Δx and taking the limit as $\Delta x \to 0$, we get

$$\rho\sqrt{1 + (u_x)^2} u_{tt}(x,t) = \frac{\partial}{\partial x}[T(x,t)\sin(\Theta(x,t))]. \tag{8.113}$$

We assumed our vibrations were relatively small. This means that $\Theta(x,t)$ is very close to zero. As a result, $\sin(\Theta(x,t)) \equiv \tan(\Theta(x,t))$. Moreover, $\tan(\Theta(x,t))$ is just the slope of the string $u_x(x,t)$. We conclude, since $\Theta(x,t)$ is small, that $u_x(x,t)$ is also very small. The above equation becomes

$$\rho u_{tt}(x,t) = (T(x,t)u_x(x,t))_x. \tag{8.114}$$

We have not used the horizontal component of Newton's Law yet. Since we assume there are only vertical vibrations, our tiny piece of string can only move vertically. Thus the net horizontal force is zero.

$$T(x+\Delta x,t)\cos(\Theta(x+\Delta x,t)) - T(x,t)\cos(\Theta(x,t)) = 0. \tag{8.115}$$

Dividing by Δx and taking the limit as $\Delta x \to \infty$ yields

$$\frac{\partial}{\partial x}[T(x,t)\cos(\Theta(x,t))] = 0. \tag{8.116}$$

Since $\Theta(x,t)$ is very close to zero, $\cos(\Theta(x,t))$ is close to one. thus we have that $\frac{\partial T}{\partial x}(x,t)$ is close to zero. So $T(x,t)$ is constant along the string, and independent of x. We will also assume that T is independent of t. Then Equation (8.114) becomes the one-dimensional wave equation

$$u_{tt} = c^2 u_{xx} \tag{8.117}$$

where $c^2 = \frac{T}{\rho}$.

8.9.2 The Homogeneous Dirichlet Problem

Now that we have derived the wave equation, we can use Separation of Variables to obtain basic solutions. We will consider homogeneous Dirichlet conditions, but if we had homogeneous Neumann conditions the same techniques would give us a solution. The wave equation is second order in t, unlike the heat equation which was first order in t. We will need to initial conditions in order to obtain a solution, one for the initial displacement and the other for the initial speed.

The relevant wave equation problem we will study is

$$\begin{aligned} u_{tt} &= c^2 u_{xx} && (8.118)\\ u(0,t) &= u(l,t) = 0 && (8.119)\\ u(x,0) &= f(x), \quad u_t(x,0) = g(x) && (8.120) \end{aligned}$$

The physical interpretation of the boundary conditions is that the ends of the string are fixed in place. They might be attached to guitar pegs.

We start by assuming our solution has the form

$$u(x,t) = X(x)T(t). \tag{8.121}$$

Plugging this into the equation gives

$$T''(t)X(x) = c^2 T(t)X''(x). \tag{8.122}$$

Separating variables, we have

$$\frac{X''}{X} = \frac{T''}{c^2 T} = -\lambda \tag{8.123}$$

where λ is a constant. This gives a pair of ODEs

$$\begin{aligned} T'' + c^2\lambda T &= 0 && (8.124)\\ X'' + \lambda X &= 0. && (8.125) \end{aligned}$$

The boundary conditions transform into

$$\begin{aligned} u(0,t) &= X(0)T(t) = 0 \quad \Rightarrow \quad X(0) = 0 && (8.126)\\ u(l,t) &= X(l)T(t) = 0 \quad \Rightarrow \quad X(l) = 0. && (8.127) \end{aligned}$$

This is the same boundary value problem that we saw for the heat equation and thus the eigenvalues and eigenfunctions are

$$\lambda_n = (\frac{n\pi}{l})^2 \tag{8.128}$$

$$X_n(x) = \sin(\frac{n\pi x}{l}) \tag{8.129}$$

for $n = 1, 2, ...$ The first ODE (8.124) is then

$$T'' + (\frac{cn\pi}{l})^2 T = 0, \tag{8.130}$$

and since the coefficient of T is clearly positive this has a general solution

$$T_n(t) = A_n \cos(\frac{n\pi ct}{l}) + B_n \sin(\frac{n\pi ct}{l}). \tag{8.131}$$

There is no reason to think either of these are zero, so we end up with separated solutions

$$u_n(x,t) = \left[A_n \cos(\frac{n\pi ct}{l}) + B_n \sin(\frac{n\pi ct}{l})\right] \sin(\frac{n\pi x}{l}) \tag{8.132}$$

and the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos(\frac{n\pi ct}{l}) + B_n \sin(\frac{n\pi ct}{l})\right] \sin(\frac{n\pi x}{l}). \tag{8.133}$$

We can directly apply our first initial condition, but to apply the second we will need to differentiate with respect to t. This gives us

$$u_t(x,t) = \sum_{n=1}^{\infty} \left[-\frac{n\pi c}{l} A_n \sin(\frac{n\pi ct}{l}) + \frac{n\pi c}{l} B_n \cos(\frac{n\pi ct}{l})\right] \sin(\frac{n\pi x}{l}) \tag{8.134}$$

Plugging in the initial condition then yields the pair of equations

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l}) \tag{8.135}$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \sin(\frac{n\pi x}{l}). \tag{8.136}$$

These are both Fourier Sine series. The first is directly the Fourier Since series for $f(x)$ on $(0,l)$. The second equation is the Fourier Sine series for $g(x)$ on $(0,l)$ with a slightly messy coefficient. The Euler-Fourier formulas then tell us that

$$A_n = \frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi x}{l})dx \tag{8.137}$$

$$\frac{n\pi c}{l} B_n = \frac{2}{l}\int_0^l g(x)\sin(\frac{n\pi x}{l})dx \tag{8.138}$$

$$A_n = \frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi x}{l})dx \tag{8.139}$$

$$B_n = \frac{2}{n\pi c}\int_0^l g(x)\sin(\frac{n\pi x}{l})dx. \tag{8.140}$$

8.9.3 Examples

■ **Example 8.14** Find the solution (displacement $u(x,t)$) for the problem of an elastic string of length L whose ends are held fixed. The string has no initial velocity ($u_t(x,0)=0$) from an initial position

$$u(x,0)=f(x)=\begin{cases}\frac{4x}{L} & 0\le x\le \frac{L}{4}\\ 1 & \frac{L}{4}<x<\frac{3L}{4}\\ \frac{4(L-x)}{L} & \frac{3L}{4}\le x\le L\end{cases} \tag{8.141}$$

By the formulas above we see if we separate variables we have the following equation for T

$$T''+(\frac{cn\pi}{L})^2T=0 \tag{8.142}$$

with the general solution

$$T_n(t)=A_n\cos(\frac{n\pi ct}{L})+B_n\sin(\frac{n\pi ct}{L}). \tag{8.143}$$

since the initial speed is zero, we find $T'(0)=0$ and thus $B_n=0$. Therefore the general solution is

$$u(x,t)=\sum_{n=1}^{\infty}A_n\cos(\frac{n\pi ct}{L})\sin(\frac{n\pi x}{L}). \tag{8.144}$$

where the coefficients are the Fourier Sine coefficients of $f(x)$. So

$$A_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx \tag{8.145}$$

$$= \frac{2}{L}\left[\int_0^{L/4}\frac{4x}{L}\sin(\frac{n\pi x}{L})dx+\int_{L/4}^{3L/4}\sin(\frac{n\pi x}{L})dx+\int_{3L/4}^{L}\frac{4L-4x}{L}\sin(\frac{n\pi x}{L})dx\right] \tag{8.146}$$

$$= 8\frac{\sin(\frac{n\pi}{4})+\sin(\frac{3n\pi}{4})}{n^2\pi^2} \tag{8.147}$$

Thus the displacement of the string will be

$$u(x,t)=\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin(\frac{n\pi}{4})+\sin(\frac{3n\pi}{4})}{\pi^2}\cos(\frac{n\pi ct}{L})\sin(\frac{n\pi x}{L}). \tag{8.148}$$

■

■ **Example 8.15** Find the solution (displacement $u(x,t)$) for the problem of an elastic string of length L whose ends are held fixed. The string has no initial velocity ($u_t(x,0)=0$) from an initial position

$$u(x,0)=f(x)=\frac{8x(L-x)^2}{L^3} \tag{8.149}$$

By the formulas above we see if we separate variables we have the following equation for T

$$T''+(\frac{cn\pi}{L})^2T=0 \tag{8.150}$$

with the general solution

$$T_n(t)=A_n\cos(\frac{n\pi ct}{L})+B_n\sin(\frac{n\pi ct}{L}). \tag{8.151}$$

since the initial speed is zero, we find $T'(0) = 0$ and thus $B_n = 0$. Therefore the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi ct}{L}) \sin(\frac{n\pi x}{L}). \tag{8.152}$$

where the coefficients are the Fourier Sine coefficients of $f(x)$. So

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx && (8.153)\\ &= \frac{2}{L}\int_0^L \frac{8x(L-x)^2}{L^3}\sin(\frac{n\pi x}{L})dx && (8.154)\\ &= 32\frac{2+\cos(n\pi)}{n^3\pi^3} \quad \text{Integrate By Parts} && (8.155)\end{aligned}$$

Thus the displacement of the string will be

$$u(x,t) = \frac{32}{\pi^3}\sum_{n=1}^{\infty} \frac{2+\cos(n\pi)}{n^3} \cos(\frac{n\pi ct}{L}) \sin(\frac{n\pi x}{L}). \tag{8.156}$$

■

8.9.4 D'Alembert's Solution of the Wave Equation, Characteristics

Another approach to solving the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{8.157}$$

can be seen by transforming the equation. Introduce new independent variables:

$$v = x + ct, \qquad w = x - ct \tag{8.158}$$

We can now think of u as a function of v, w instead of x, t. Now compute the appropriate partial derivatives of u using the chain rule

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial x} = u_v + u + w.\\ \frac{\partial^2 u}{\partial x^2} &= (u_v + u_w)_x = (u_v + u_w)_v v_x + (u_v + u_w)_w w_x = u_{vv} + 2u_{vw} + u_{ww}\\ \frac{\partial^2 u}{\partial t^2} &= c^2(u_{vv} - 2u_{vw} + u_{ww}).\end{aligned}$$

Inserting these into the wave equation gives $u_{vw} = 0$. Solve using two successive integrations, first with respect to w and then with respect to v giving

$$\frac{\partial u}{\partial v} = h(v), \quad u(v,w) = \int h(v)dv + \psi(w) = \phi(v) + \psi(w).$$

Thus, replacing v and w by their definitions

$$u(x,t) = \phi(x+ct) + \psi(x-ct) \tag{8.159}$$

This is known as **d'Alembert's solution**. In general given initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$, d'Alembert's solution becomes

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds. \tag{8.160}$$

The idea of d'Alembert's solution is just a special case of the **method of characteristics**. This concerns PDEs of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y). \tag{8.161}$$

PDEs have three general types defined by the coefficients A, B, C and the discriminant $AC - B^2$

Type	Defining Condition	Example in Sec. 12.1
Hyperbolic	$AC - B^2 < 0$	Wave equation (1)
Parabolic	$AC - B^2 = 0$	Heat equation (2)
Elliptic	$AC - B^2 > 0$	Laplace equation (3)

Figure 8.3: Three general types of PDEs, (from Kreysig Adv. Engineering Math).

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