Introduction to Ordinary and Partial Differential Equations

One Semester Course

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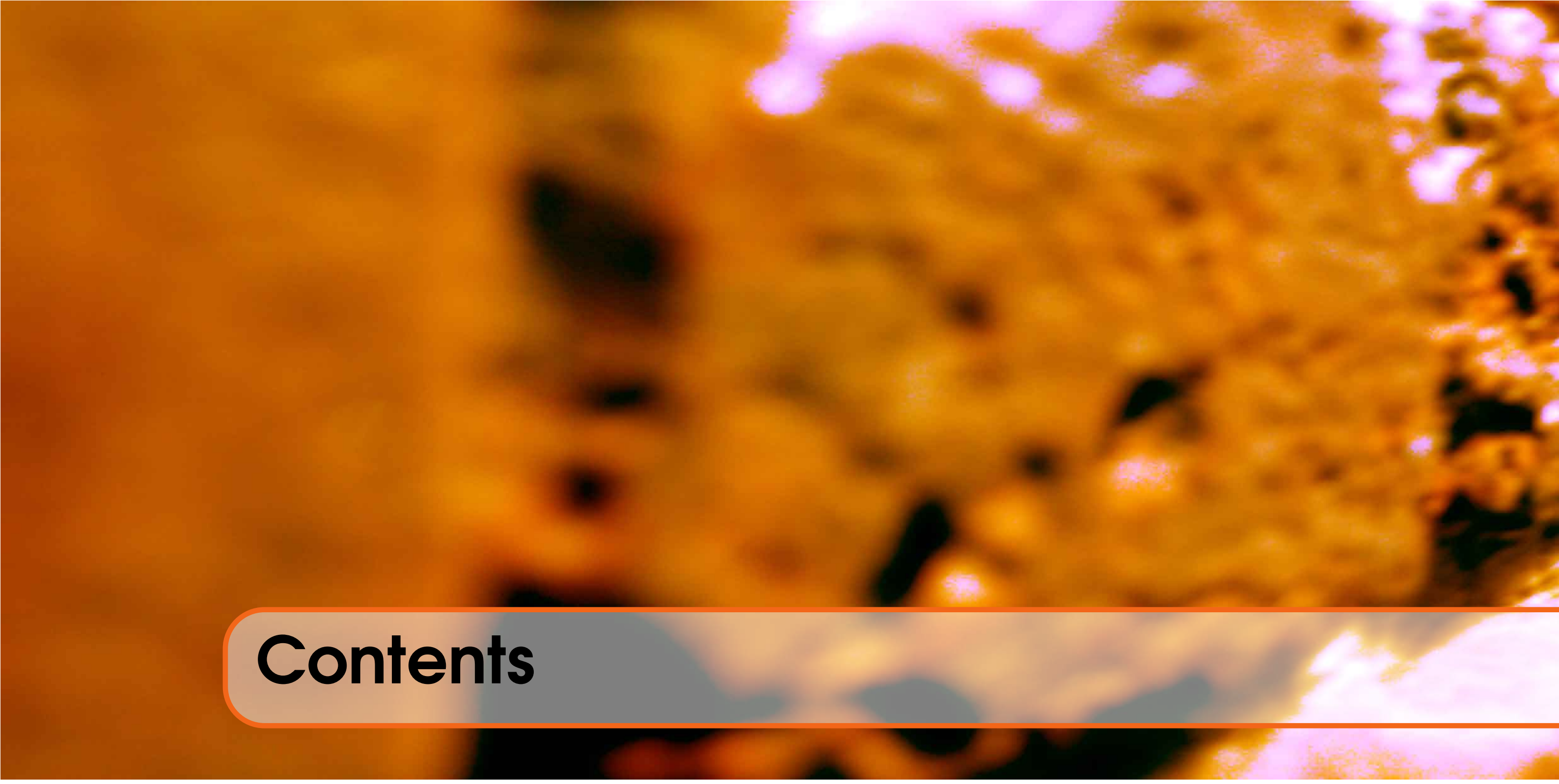

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1. Introduction

1.1 Introduction

This set of lecture notes was built from a one semester course on the Introduction to Ordinary and Differential Equations at Penn State University from 2010-2014. Our main focus is to develop mathematical intuition for solving real world problems while developing our tool box of useful methods. Topics in this course are derived from five principle subjects in Mathematics

(i) **First Order Equations** (Ch. 2)

(ii) **Second Order Linear Equations** (Ch. 3)

(iii) **Higher Order Linear Equations** (Ch. 4)

(iv) **Laplace Transforms** (Ch. 5)

(v) **Systems of Linear Equations** (Ch. 6)

(vi) **Nonlinear Differential Equations and Stability** (Ch. 7)

(vii) **Partial Differential Equations and Fourier Series** (Ch. 8)

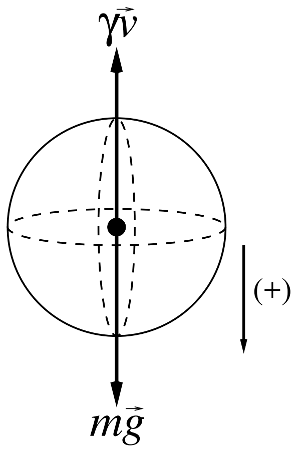
Each class individually goes deeper into the subject, but we will cover the basic tools needed to handle problems arising in physics, materials sciences, and the life sciences. Here we focus on the development of the solution methods for solving those problems.

1.2 Some Basic Mathematical Models; Direction Fields

Definition 1.2.1 A **differential equation** is an equation containing derivatives.

Definition 1.2.2 A differential equation that describes some physical process is often called a **mathematical model**

Example 1.1 *(Falling Object)*



Consider an object falling from the sky. From Newton's Second Law we have

$$F = ma = m\frac{dv}{dt} \tag{1.1}$$

When we consider the forces from the free body diagram we also have

$$F = mg - \gamma v \tag{1.2}$$

where γ is the **drag coefficient**. Combining the two

$$m\frac{dv}{dt} = mg - \gamma v \tag{1.3}$$

Suppose $m = 10kg$ and $\gamma = 2kg/s$. Then we have

$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \tag{1.4}$$

It looks like the direction field tends towards $v = 49m/s$. We plot the direction field by plugging in the values for v and t and letting dv/dt be the slope of a line at that point. ■

Direction Fields are valuable tools in studying the solutions of differential equations of the form

$$\frac{dy}{dt} = f(t,y) \tag{1.5}$$

where f is a given function of the two variables t and y, sometimes referred to as a **rate function**. At each point on the grid, a short line is drawn whose slope is the value of f at the point. This technique provides a good picture of the overall behavior of a solution.

Two Things to keep in mind:

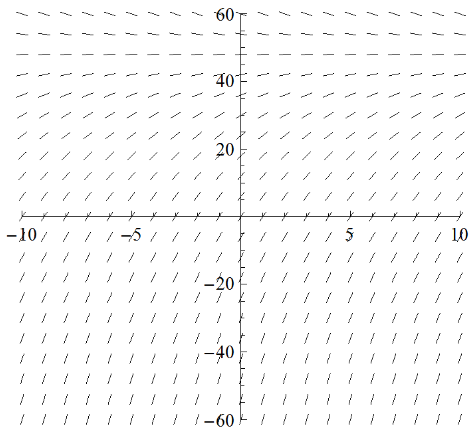


Figure 1.1: Direction field for above example

1. In constructing a direction field we never have to solve the differential equation only evaluate it at points.

2. This method is useful if one has access to a computer because a computer can generate the plots well.

■ **Example 1.2** *(Population Growth)* Consider a population of field mice, assuming there is nothing to eat the field mice, the population will grow at a constant rate. Denote time by t (in months) and the mouse population by $p(t)$, then we can express the model as

$$\frac{dp}{dt} = rp \tag{1.6}$$

where the proportionality factor r is called the **rate constant** or **growth constant**. Now suppose owls are killing mice (15 per day), the model becomes

$$\frac{dp}{dt} = 0.5p - 450 \tag{1.7}$$

note that we subtract 450 rather than 15 because time was measured in months. In general

$$\frac{dp}{dt} = rp - k \tag{1.8}$$

where the growth rate is r and the predation rate k is unspecified. Note the equilibrium solution would be k/r.

Definition 1.2.3 The **equilibrium solution** is the value of $p(t)$ where the system no longer changes, $\frac{dp}{dt} = 0$.

In this example solutions above equilibrium will increase, while solutions below will decrease.

■

Steps to Constructing Mathematical Models:

1. Identify the independent and dependent variables and assign letters to represent them.

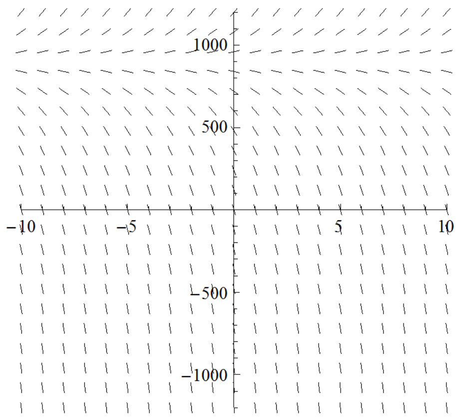


Figure 1.2: Direction field for above example

Often the independent variable is time.
2. Choose the units of measurement for each variable.
3. Articulate the basic principle involved in the problem.
4. Express the principle in the variables chosen above.
5. Make sure each term has the same physical units.
6. We will be dealing with models in this chapter which are single differential equations.

■ **Example 1.3** Draw the direction field for the following, describe the behavior of y as $t \to \infty$. Describe the dependence on the initial value:

$$y' = 2y + 3 \tag{1.9}$$

Ans: For $y > -1.5$ the slopes are positive, and hence the solutions increase. For $y < -1.5$ the slopes are negative, and hence the solutions decrease. All solutions appear to diverge away from the equilibrium solution $y(t) = -1.5$. ■

■ **Example 1.4** Write down a DE of the form $dy/dt = ay + b$ whose solutions have the required behavior as $t \to \infty$. It must approach $\frac{2}{3}$.
Answer: For solutions to approach the equilibrium solution $y(t) = 2/3$, we must have $y' < 0$ for $y > 2/3$, and $y' > 0$ for $y < 2/3$. The required rates are satisfied by the DE $y' = 2 - 3y$. ■

■ **Example 1.5** Find the direction field for $y' = y(y-3)$

■

1.3 Solutions of Some Differential Equations

Last Time: We derived two formulas:

$$m\frac{dv}{dt} = mg - \gamma v \quad \text{(Falling Bodies)} \tag{1.10}$$

$$\frac{dp}{dt} = rp - k \quad \text{(Population Growth)} \tag{1.11}$$

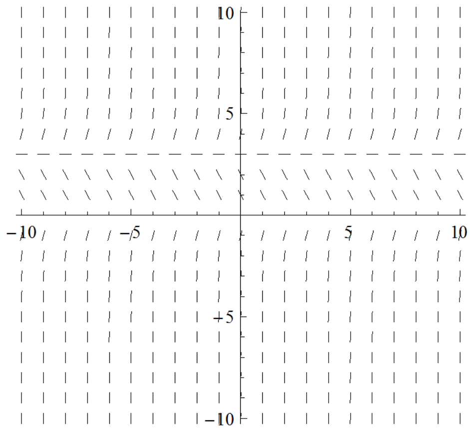


Figure 1.3: Direction field for above example

Both equations have the form:

$$\frac{dy}{dt} = ay - b \tag{1.12}$$

■ **Example 1.6** *(Field Mice / Predator-Prey Model)*
Consider

$$\frac{dp}{dt} = 0.5p - 450 \tag{1.13}$$

we want to now solve this equation. Rewrite equation (1.13) as

$$\frac{dp}{dt} = \frac{p-900}{2}. \tag{1.14}$$

Note $p = 900$ is an equilbrium solution and the system does not change. If $p \neq 900$

$$\frac{dp/dt}{p-900} = \frac{1}{2} \tag{1.15}$$

By Chain Rule we can rewrite as

$$\frac{d}{dt}\left[\ln|p-900|\right] = \frac{1}{2} \tag{1.16}$$

So by integrating both sides we find

$$\ln|p-900| = \frac{t}{2} + C \tag{1.17}$$

Therefore,

$$p = 900 + Ce^{t/2} \tag{1.18}$$

Thus we have infinitely many solutions where a different arbitrary constant C produces a different solution. What if the initial population of mice was 850. How do we account for this? ■

Definition 1.3.1 The additional condition, $p(0) = 850$, that is used to determine C is an example of an **initial condition**.

Definition 1.3.2 The differential equation together with the initial condition form the **initial value problem**

Consider the general problem

$$\frac{dy}{dt} = ay - b \tag{1.19}$$
$$y(0) = y_0 \tag{1.20}$$

The solution has the form

$$y = (b/a) + [y_0 - (b/a)]e^{at} \tag{1.21}$$

when $a \neq 0$ this contains all possible solutions to the general equation and is thus called **the general solution** The geometric representation of the general solution is an infinite family of curves called **integral curves**.

■ **Example 1.7** *(Dropping a ball)* System under consideration:

$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \tag{1.22}$$
$$v(0) = 0 \tag{1.23}$$

From the formula above we have

$$v = \left(\frac{-9.8}{-1/5}\right) + \left[0 - \frac{-9.8}{-1/5}\right]e^{-\frac{t}{5}} \tag{1.24}$$

and the general solution is

$$v = 49 + Ce^{-t/5} \tag{1.25}$$

with the I.C. $C = -49$. ■

1.4 Classifications of Differential Equations

Last Time: We solved some basic differential equations, discussed IVPs, and defined the general solution.

Now we want to classify two main types of differential equations.

Definition 1.4.1 If the unknown function depends on a single independent variable where only ordinary derivatives appear, it is said to be an **ordinary differential equation**. Example

$$y'(x) = xy \tag{1.26}$$

Definition 1.4.2 If the unknown function depends on several variables, and the derivatives are partial derivatives it is said to be a **partial differential equation**.

One can also have a **system of differential equations**

$$dx/dt = ax - \alpha xy \quad (1.27)$$
$$dy/dt = -cy + \gamma xy \quad (1.28)$$

Note: Questions from this section are common on exams.

Definition 1.4.3 The **order** of a differential equation is the order of the highest derivative that appears in the equation.

Ex 1: $y''' + 2e^t y'' + yy' = 0$ has order 3.
Ex 2: $y^{(4)} + (y')^2 + 4y''' = 0$ has order 4. Look at derivatives not powers.

Another way to classify equations is whether they are linear or nonlinear:

Definition 1.4.4 A differential equation is said to be **linear** if $F(t, y, y', y'', ..., y^{(n)}) = 0$ is a linear function in the variables $y, y', y'', ..., y^{(n)}$. i.e. none of the terms are raised to a power or inside a sin or cos.

■ **Example 1.8** a) $y' + y = 2$
b) $y'' = 4y - 6$
c) $y^{(4)} + 3y' + \sin(t)y$ ■

Definition 1.4.5 An equation which is not linear is **nonlinear**.

■ **Example 1.9** a) $y' + t^4 y^2 = 0$
b) $y'' + \sin(y) = 0$
c) $y^{(4)} - \tan(y) + (y''')^3 = 0$ ■

■ **Example 1.10** $\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin(\theta) = 0$.

The above equation can be approximated by a linear equation if we let $\sin(\theta) = \theta$. This process is called **linearization**. ■

Definition 1.4.6 A **solution** of the ODE on the interval $\alpha < t < \beta$ is a function ϕ that satisfies

$$\phi^{(n)}(t) = f[t, \phi(t), ..., \phi^{(n-1)}(t)] \quad (1.29)$$

Common Questions:
1. (Existence) Does a solution exist? Not all Initial Value Problems (IVP) have solutions.

2. (Uniqueness) If a solution exists how many are there? There can be none, one or infinitely many solutions to an IVP.

3. How can we find the solution(s) if they exist? This is the key question in this course.

We will develop many methods for solving differential equations the key will be to identify which method to use in which situation.

II

First Order Differential Equations

2. Solution Methods For First Order Equatio

2.1 Linear Equations; Method of Integrating Factors

Last Time: We classified ODEs and PDEs in terms of Order (the highest derivative taken) and linearity.

Now we start Chapter 2: First Order Differential Equations
All equations in this chapter will have the form

$$\frac{dy}{dt} = f(t,y) \tag{2.1}$$

If f depends linearly on y the equation will be a first order linear equation.

Consider the general equation

$$\frac{dy}{dt} + p(t)y = g(t) \tag{2.2}$$

We said in Chapter 1 if $p(t)$ and $g(t)$ are constants we can solve the equation explicitly. Unfortunately this is not the case when they are not constants. We need the method of **integrating factor** developed by Leibniz, who also invented calculus, where we multiply (2.2) by a certain function $\mu(t)$, chosen so the resulting equation is integrable. $\mu(t)$ is called the **integrating factor**. The challenge of this method is finding it.

Summary of Method:
1. Rewrite the equation as (MUST BE IN THIS FORM)

$$y' + ay = f \tag{2.3}$$

2. Find an integrating factor, which is any function

$$\mu(t) = e^{\int a(t)dt}. \tag{2.4}$$

3. Multiply both sides of (2.3) by the integrating factor.

$$\mu(t)y' + a\mu(t)y = f\mu(t) \tag{2.5}$$

4. Rewrite as a derivative

$$(\mu y)' = \mu f \tag{2.6}$$

5. Integrate both sides to obtain

$$\mu(t)y(t) = \int \mu(t)f(t)dt + C \tag{2.7}$$

and thus

$$y(t) = \frac{1}{\mu(t)}\int \mu(t)f(t)dt + \frac{C}{\mu(t)} \tag{2.8}$$

Now lets see some examples:

■ **Example 2.1** Find the general solution of

$$y' = y + e^{-t} \tag{2.9}$$

Step 1:

$$y' - y = e^{t} \tag{2.10}$$

Step 2:

$$\mu(t) = e^{-\int 1dt} = e^{-t} \tag{2.11}$$

Step 3:

$$e^{-t}(y' - y) = e^{-2t} \tag{2.12}$$

Step 4:

$$(e^{-t}y)' = e^{-2t} \tag{2.13}$$

Step 5:

$$e^{-t}y = \int e^{-2t}dt = -\frac{1}{2}e^{-2t} + C \tag{2.14}$$

Solve for y

$$y(t) = -\frac{1}{2}e^{-t} + Ce^{t} \tag{2.15}$$

■

■ **Example 2.2** Find the general solution of

$$y' = y \sin t + 2te^{-\cos t} \tag{2.16}$$

and $y(0) = 1$.
Step 1:

$$y' - y \sin t = 2te^{-\cos t} \tag{2.17}$$

Step 2:

$$\mu(t) = e^{-\int \sin t dt} = e^{\cos t} \tag{2.18}$$

Step 3:

$$e^{\cos t}(y' - y \sin t) = 2t \tag{2.19}$$

Step 4:

$$(e^{\cos t} y)' = 2t \tag{2.20}$$

Step 5:

$$e^{\cos t} y = t^2 + C \tag{2.21}$$

So the general solution is:

$$y(t) = (t^2 + C)e^{-\cos t} \tag{2.22}$$

With IC

$$y(t) = (t^2 + e)e^{-\cos t} \tag{2.23}$$

■

■ **Example 2.3** Find General Solution to

$$y' = y \tan t + \sin t \tag{2.24}$$

with $y(0) = 2$. Note Integrating factor

$$\mu(t) = e^{-\int \tan dt} = e^{\ln(\cos t)} = \cos t \tag{2.25}$$

Final Answer

$$y(t) = -\frac{\cos t}{2} + \frac{5}{2\cos t} \tag{2.26}$$

■

■ **Example 2.4** Solve

$$2y' + ty = 2 \tag{2.27}$$

with $y(0) = 1$. Integrating Factor

$$\mu(t) = e^{t^2/4} \tag{2.28}$$

Final Answer

$$y(t) = e^{-t^2/4} \int_0^t e^{s^2/4} ds + e^{-t^2/4}. \tag{2.29}$$

■

2.1.1 REVIEW: Integration By Parts

This is the most important integration technique learned in Calculus 2. We will derive the method. Consider the product rule for two functions of t.

$$\frac{d}{dt}\left[uv\right] = u\frac{dv}{dt} + v\frac{du}{dt} \tag{2.30}$$

Integrate both sides from a to b

$$uv\Big|_a^b = \int_a^b u\frac{dv}{dt} + \int_a^b v\frac{du}{dt} \tag{2.31}$$

Rearrange the resulting terms

$$\int_a^b u\frac{dv}{dt} = uv\Big|_a^b - \int_a^b v\frac{du}{dt} \tag{2.32}$$

Practicing this many times will be helpful on the homework. Consider two examples.

■ **Example 2.5** Find the integral $\int_1^9 \ln(t)dt$. First define u, du, dv, and v.

$$u = \ln(t) \qquad dv = dt \tag{2.33}$$

$$du = \frac{1}{t}dt \qquad v = t \tag{2.34}$$

Thus

$$\int_1^9 \ln(t)dt = t\ln(t)\Big|_1^9 - \int_1^9 1dt \tag{2.35}$$

$$= 9\ln(9) - t\Big|_1^9 \tag{2.36}$$

$$= 9\ln(9) - 9 + 1 \tag{2.37}$$

$$= 9\ln(9) - 8 \tag{2.38}$$

■

■ **Example 2.6** Find the integral $\int e^x \cos(x)dx$. First define u, du, dv, and v.

$$u = \cos(x) \qquad dv = e^x dx \tag{2.39}$$

$$du = -\sin(x)dx \qquad v = e^x \tag{2.40}$$

Thus

$$\int e^x \cos(x)dx = e^x\cos(x) - \int e^x \sin(x)dx \tag{2.41}$$

Do Integration By Parts Again

$$u = \sin(x) \qquad dv = e^x dx \tag{2.42}$$

$$du = \cos(x)dx \qquad v = e^x \tag{2.43}$$

So

$$\int e^x \cos(x)dx = e^x \cos(x) - \int e^x \sin(x)dx \tag{2.44}$$

$$= e^x \cos(x) + e^x \sin(x) - \int e^x \cos(x)dx \tag{2.45}$$

$$2\int e^x \cos(x)dx = e^x\big(\cos(x)+\sin(x)\big) \tag{2.46}$$

$$\int e^x \cos(x)dx = \frac{1}{2}e^x\big(\cos(x)+\sin(x)\big)+C \tag{2.47}$$

■

Notice when we do not have limits of integration we need to include the arbitrary constant of integration C.

2.2 Separable Equations

Last Time: We used integration to solve first order equations of the form

$$\frac{dy}{dt} = a(t)y + b(t) \tag{2.48}$$

the method of integrating factor only works on an equation of this form, but we want to handle a more general class of equations

$$\frac{dy}{dt} = f(t,y) \tag{2.49}$$

We want to solve **separable** equations which have the form

$$\frac{dy}{dx} = f(y)g(x) \tag{2.50}$$

The General Solution Method:

$$\text{Step 1: (Separate)} \quad \frac{1}{f(y)}dy = g(x)dx \tag{2.51}$$

$$\text{Step 2: (Integrate)} \quad \int \frac{1}{f(y)}dy = \int g(x)dx \tag{2.52}$$

$$\text{Step 2: (Solve for } y\text{)} \quad F(y) = G(x) + c \tag{2.53}$$

Note only need a constant of integration of one side, could just combine the constants we get on each side. Also, we only solve for y if it is possible, if not leave in **implicit form**.

Definition 2.2.1 An **equilibrium solution** is the value of y which makes $dy/dx = 0$, y remains this constant forever.

■ **Example 2.7** *(Newton's Law of Cooling)* Consider the ODE, where E is a constant:

$$\frac{dB}{dt} = \kappa(E - B) \tag{2.54}$$

with initial condition (IC) $B(0) = B_0$. This is separable

$$\begin{aligned}
\int \frac{dB}{E-B} &= \int \kappa dt && (2.55)\\
-\ln|E-B| &= \kappa t + c && (2.56)\\
E-B &= e^{-\kappa t + c} = Ae^{-\kappa t} && (2.57)\\
B(t) &= E - Ae^{-\kappa t} && (2.58)\\
B(0) &= E - A && (2.59)\\
A &= E - B_0 && (2.60)\\
B(t) &= E - \frac{E-B_0}{e^{\kappa t}} && (2.61)
\end{aligned}$$

■

■ Example 2.8

$$\frac{dy}{dt} = 6y^2 x, \quad y(1) = \frac{1}{3}. \tag{2.62}$$

Separate and Solve:

$$\begin{aligned}
\int \frac{dy}{y^2} &= \int 6x dx && (2.63)\\
-\frac{1}{y} &= 3x^2 + c && (2.64)\\
y(1) &= 1/3 && (2.65)\\
-3 &= 3(1) + c \Rightarrow c = -6 && (2.66)\\
-\frac{1}{y} &= 3x^2 - 6 && (2.67)\\
y(x) &= \frac{1}{6-3x^2} && (2.68)
\end{aligned}$$

What is the **interval of validity** for this solution? Problem when $6 - 3x^2 = 0$ or when $x = \pm\sqrt{2}$. So possible intervals of validity: $(-\infty, -\sqrt{2}), (-\sqrt{2}, \sqrt{2}), (\sqrt{2}, \infty)$. We want to choose the one containing the initial value for x, which is x = 1, so the interval of validity is $(-\sqrt{2}, \sqrt{2})$. ■

■ Example 2.9

$$y' = \frac{3x^2 + 2x - 4}{2y - 2}, \quad y(1) = 3 \tag{2.69}$$

There are no equilibrium solutions.

$$\begin{aligned}
\int 2y - 2dy &= \int 3x^2 + 2x - 4dx && (2.70)\\
y^2 - 2y &= x^3 + x^2 - 4x + c && (2.71)\\
y(1) &= 3 \Rightarrow c = 5 && (2.72)\\
y^2 - 2y + 1 &= x^3 + x^2 - 4x + 6 \quad \text{(Complete the Square)} && (2.73)\\
(y-1)^2 &= x^3 + x^2 - 4x + 6 && (2.74)\\
y(x) &= 1 \pm \sqrt{x^3 + x^2 - 4x + 6} && (2.75)
\end{aligned}$$

There are two solutions we must choose the appropriate one. Use the IC to determine only the positive solution is correct.

$$y(x) = 1 + \sqrt{x^3 + x^2 - 4x + 6} \tag{2.76}$$

We need the terms under the square root to be positive, so the interval of validity is values of x where $x^3 + x^2 - 4x + 6 \geq 0$. Note $x = 1$ is in here so IC is in interval of validity. ■

■ Example 2.10

$$\frac{dy}{dx} = \frac{xy^3}{1+x^2}, \quad y(0) = 1 \tag{2.77}$$

One equilibrium solution, $y(x) = 0$, which is not our case (since it does not meet the IC). So separate:

$$\int \frac{dy}{y^3} = \int \frac{x}{1+x^2} dx \tag{2.78}$$

$$-\frac{1}{2y^2} = \frac{1}{2}\ln(1+x^2) + c \tag{2.79}$$

$$y(0) = 1 \Rightarrow c = -\frac{1}{2} \tag{2.80}$$

$$y^2 = \frac{1}{1 - \ln(1+x^2)} \tag{2.81}$$

$$y(x) = \frac{1}{\sqrt{1 - \ln(1+x^2)}} \tag{2.82}$$

Determine the interval of validity. Need

$$\ln(1+x^2) < 1 \Rightarrow x^2 < e - 1 \tag{2.83}$$

So the interval of validity is $-\sqrt{e-1} < x < \sqrt{e-1}$. ■

■ Example 2.11

$$\frac{dy}{dx} = \frac{y-1}{x^2+1} \tag{2.84}$$

The equilibrium solution is $y(x) = 1$ and our IC is $y(0) = 1$, so in this case the solution is the constant function $y(s) = 1$. ■

■ Example 2.12

$$\textit{(Review IBP)} \frac{dy}{dt} = e^{y-t}\sec(y)(1+t^2), \quad y(0) = 0 \tag{2.85}$$

Separate by rewriting, and using Integration By Parts (IBP)

$$\frac{dy}{dt} = \frac{e^y e^{-t}}{\cos(y)}(1+t^2) \tag{2.86}$$

$$\int e^{-y}\cos(y)dy = \int e^{-t}(1+t^2)dt \tag{2.87}$$

$$\frac{e^{-y}}{2}(\sin(y) - \cos(y)) = -e^{-t}(t^2 + 2t + 3) + \frac{5}{2} \tag{2.88}$$

Won't be able to find an explicit solution so leave in implicit form. In the implicit form it is difficult to find the interval of validity so we will stop here. ■

2.3 Modeling With First Order Equations

Last Time: We solved separable ODEs and now we want to look at some applications to real world situations

There are two key questions to keep in mind throughout this section:
1. How do we write a differential equation to model a given situation?
2. What can the solution tell us about that situation?

■ **Example 2.13** *(Radioactive Decay)*

$$\frac{dN}{dt} = -\lambda N(t), \tag{2.89}$$

where $N(t)$ is the number of atoms of a radioactive isotope and $\lambda > 0$ is the decay constant. The equation is separable, and if the initial data is $N(0) = N_0$, the solution is

$$N(t) = N_0 e^{-\lambda t}. \tag{2.90}$$

so we can see that radioactive decay is exponential. ■

■ **Example 2.14** *(Newton's Law of Cooling)* If we immerse a body in an environment with a constant temperature E, then if $B(t)$ is the temperature of the body we have

$$\frac{dB}{dt} = \kappa(E - B), \tag{2.91}$$

where $\kappa > 0$ is a constant related to the material of the body and how it conducts heat. This equation is separable. We solved it before with the initial condition $B(0) = B_0$ to get

$$B(t) = E - \frac{E - B_0}{e^{\kappa t}}. \tag{2.92}$$

■

Approaches to writing down a model describing a situation:
1. Remember the derivative is the **rate of change**. It's possible that the description of the problem tells us directly what the rate of change is. Newton's Law of Cooling tells us the rate of change of the body's temperature was proportional to the difference in temperature between the body and the environment. All we had to do was set the relevant terms equal.

2. There are also cases where we are not explicitly given the formula for the rate of change. But we may be able to use the physical description to define the rate of change and then set the derivative equal to that. Note: The derivative = increase - decrease. This type of thinking is only applicable to first order equations since higher order equations are not formulated as rate of change equals something.

3. We may just be adapting a known differential equation to a particular situation, i.e. Newton's Second Law $F = ma$. It is either a first or second order equation depending on if you define it for position for velocity. Combine all forces and plug in value for F

to yield the differential equation. Used for falling bodies, harmonic motion, and pendulums.

4. The last possibility is to determine two different expressions for the same quantity and set the equal to derive a differential equation. Useful when discussing PDEs later in the course.

The first thing one must do when approaching a modeling problem is determining which of the four situations we are in. It is crucial to practice this identification now it will be useful on exams and later sections. Secondly, your differential equation should not depend on the initial condition. The IC only tells the starting position and should not effect how a system evolves.

Type I: (Interest)

Suppose there is a bank account that gives $r\%$ interest per year. If I withdraw a constant w dollars per month, what is the differential equation modeling this?

Ans: Let t be time in years, and denote the balance after t years as $B(t)$. $B'(t)$ is the rate of change of my account balance from year to year, so it will be the difference between the amount added and the amount withdrawn. The amount added is interest and the amount withdrawn is $12w$. Thus

$$B'(t) = \frac{r}{100}B(t) - 12w \tag{2.93}$$

This is a linear equation, so we can solve by integrating factor. Note: UNITS ARE IMPORTANT, w is withdrawn each month, but $12w$ is withdrawn per year.

■ **Example 2.15** Bill wants to take out a 25 year loan to buy a house. He knows that he can afford maximum monthly payments of \$400. If the going interest rate on housing loans is 4%, what is the largest loan Bill can take out so that he will be able to pay it off in time?

Ans: Measure time t in years. The amount Bill owes will be $B(t)$. We want $B(25) = 0$. The 4% interest rate will take the form of $.04B$ added. He can make payments of $12 \times 400 = 4800$ each year. So the IVP will be

$$B'(t) = .04B(t) - 4800, \quad B(25) = 0 \tag{2.94}$$

This is a linear equation in standard form, use integrating factor

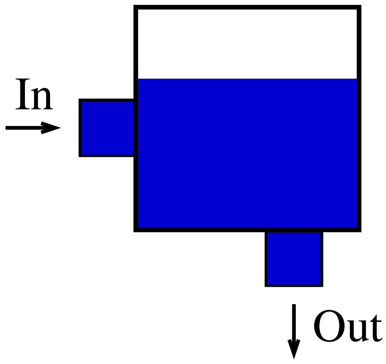
$$\begin{aligned}
B'(t) - .04B(t) &= -4800 && (2.95)\\
\mu(t) &= e^{\int -.04dt} = e^{-.04t} && (2.96)\\
(e^{-\frac{4}{100}t}B(t))' &= -4800e^{-\frac{4}{100}t} && (2.97)\\
e^{-\frac{4}{100}t}B(t) &= -4800\int e^{-\frac{4}{100}t}dt = 120000e^{-\frac{4}{100}t} + c && (2.98)\\
B(t) &= 120000 + ce^{\frac{4}{100}t} && (2.99)\\
B(25) &= 0 = 120000 + ce \Rightarrow c = -120000e^{-1} && (2.100)\\
B(t) &= 120000 - 120000e^{\frac{4}{100}(t-25)} && (2.101)
\end{aligned}$$

We want the size of the loan, which is the amount Bill begins with $B(0)$:

$$B(0) = 120000 - 120000e^{-1} = 120000(1 - e^{-1}) \tag{2.102}$$

■

Type II: (Mixing Problems)



We have a mixing tank containing some liquid inside. Contaminant is being added to the tank at some constant rate and the mixed solution is drained out at a (possibly different) rate. We will want to find the amount of contaminant in the tank at a given time.
How do we write the DE to model this process? Let $P(t)$ be the amount of pollutant (Note: Amount of pollutant, not the concentration) in the tank at time t. We know the amount of pollutant that is entering and leaving the tank each unit of time. So we can use the second approach

$$\text{Rate of Change of} P(t) = \text{Rate of entry of contaminant} - \text{Rate of exit of contaminant} \tag{2.103}$$

The **rate of entry** can be defined in different ways. 1. Directly adding contaminant i.e. pipe adding food coloring to water. 2. We might be adding solution with a known concentration of contaminant to the tank (amount = concentration x volume).
What is the **rate of exit**? Suppose that we are draining the tank at a rate of r_{out}. The amount of contaminant leaving the tank will be the amount contained in the drained solution, that is given by rate x concentration. We know the rate, and we need the concentration. This will just be the concentration of the solution in the tank, which is in turn given by the amount of contaminant in the tank divided by the volume.

$$\text{Rate of exit of contaminant} = \text{Rate of drained solution} \times \frac{\text{Amount of Contaminant}}{\text{Volume of Tank}} \tag{2.104}$$

or

$$\text{Rate of exit of contaminant} = r_{out}\frac{P(t)}{V(t)}. \tag{2.105}$$

What is $V(t)$? The Volume is decreasing by r_{out} at each t. Is there anything being added to the volume? That depends if we are adding some solution to the tank at a certain rate r_{in}, that will add to the in-tank volume. If we directly add contaminant not in solution, nothing is added. So determine which situation by reading the problem. In the first case if the initial volume is V_0, we'll get $V(t) = V_0 + t(r_{in} - r_{out})$, and in the second, $V(t) = V_0 - tr_{out}$.

■ **Example 2.16** Suppose a 120 gallon well-mixed tank initially contains 90 lbs. of salt mixed with 90 gal. of water. Salt water (with a concentration of 2 lb/gal) comes into the tank at a rate of 4 gal/min. The solution flows out of the tank at a rate of 3 gal/min. How much salt is in the tank when it is full?

Ans: We can immediately write down the expression for volume $V(t)$. How much liquid is entering each minute? 4 gallons. How much is leaving the tank in the same minute? 3 gallons. So each minute the Volume increases by 1 gallon, and we have $V(t) = 90 + (4-3)t = 90 + t$. This tells us the tank will be full at $t = 30$.
We let $P(t)$ be the amount of salt (in pounds) in the tank at time t. Ultimately, we want to determine $P(30)$, since this is when the tank will be full. We need to determine the rates at which salt is entering and leaving the tank. How much salt is entering? 4 gallons of salt water enter the tank each minute, and each of those gallons has 2lb. of salt dissolved in it. Hence we are adding 8 lbs. of salt to the tank each minute. How much is exiting the tank? 3 gallons leave each minute, and the concentration in each of those gallons is $P(t)/V(t)$. Recall

$$\text{Rate of Change of} P(t) = \text{Rate of entry of contaminant} - \text{Rate of exit of contaminant} \tag{2.106}$$

$$\text{Rate of exit of contaminant} = \text{Rate of drained solution} \times \frac{\text{Amount of Contaminant}}{\text{Volume of Tank}} \tag{2.107}$$

$$\frac{dP}{dt} = (4gal/min)(2lb/gal) - (3gal/min)(\frac{P(t)lb}{V(t)gal}) = 8 - \frac{3P(t)}{90+t} \tag{2.108}$$

This is the ODE for the salt in the tank, what is the IC? P(0) = 90 as given by the problem. Now we have an IVP so solve (since linear) using integrating factor

$$\begin{aligned}
\frac{dP}{dt} + \frac{3}{90+t}P(t) &= 8 && (2.109)\\
\mu(t) &= e^{\int \frac{3}{90+t}dt} = e^{3\ln(90+t)} = (90+t)^3 && (2.110)\\
((90+t)^3 P(t))' &= 8(90+t)^3 && (2.111)\\
(90+t)^3 P(t) &= \int 8(90+t)^3 dt = 2(90+t)^4 + c && (2.112)\\
P(t) &= 2(90+t) + \frac{c}{(90+t)^3} && (2.113)\\
P(0) &= 90 = 2(90) + \frac{c}{90^3} \Rightarrow c = -(90)^4 && (2.114)\\
P(t) &= 2(90+t) - \frac{90^4}{(90+t)^3} && (2.115)
\end{aligned}$$

Remember we wanted $P(30)$ which is the amount of salt when the tank is full. So

$$P(30) = 240 - \frac{90^4}{120^3} = 240 - 90(\frac{3}{4})^3 = 240 - 90(\frac{27}{64}). \tag{2.116}$$

We could ask for amount of salt at anytime before overflow and all would be the same besides last step where we replace 30 with the time wanted. ■

Exercise: What is the concentration of the tank when the tank is full?

■ **Example 2.17** A full 20 liter tank has 30 grams of yellow food coloring dissolved in it. If a yellow food coloring solution (with concentration of 2 grams/liter) is piped into the tank at a rate of 3 liters/minute while the well mixed solution is drained out of the tank at a rate of 3 liters/minute, what is the limiting concentration of yellow food coloring solution in the tank?

Ans: The ODE would be

$$\frac{dP}{dt} = (3L/min)(2g/L) - (3L/min)\frac{P(t)g}{V(t)L} = 6 - \frac{3P}{20} \tag{2.117}$$

Note that volume is constant since we are adding and removing the same amount at each time step. Use the method of integrating factor.

$$\begin{aligned}
\mu(t) &= e^{\int \frac{3}{20}dt} = e^{\frac{3}{20}t} && (2.118)\\
(e^{\frac{3}{20}t}P(t))' &= 6e^{\frac{3}{20}t} && (2.119)\\
e^{\frac{3}{20}t}P(t) &= \int 6e^{\frac{3}{20}t}dt = 40e^{\frac{3}{20}t} + c && (2.120)\\
P(t) &= 40 + \frac{c}{e^{\frac{3}{20}t}} && (2.121)\\
P(0) &= 20 = 40 + c \Rightarrow c = -20 && (2.122)\\
P(t) &= 40 - \frac{20}{e^{\frac{3}{20}t}}. && (2.123)
\end{aligned}$$

Now what will happen to the concentration in the limit, or as $t \to \infty$. We know the volume will always be 20 liters.

$$\lim_{t\to\infty} \frac{P(t)}{V(t)} = \lim_{t\to\infty} \frac{40 - 20e^{-\frac{3}{20}t}}{20} = 2 \tag{2.124}$$

So the limiting concentration is $2g/L$. Why does this make physical sense? After a period of time the concentration of the mixture will be exactly the same as the concentration of the incoming solution. It turns out that the same process will work if the concentration of the incoming solution is variable. ■

■ **Example 2.18** A 150 gallon tank has 60 gallons of water with 5 pounds of salt dissolved in it. Water with a concentration of $2 + \cos(t)$ lbs/gal comes into the tank at a rate of 9 gal/hr. If the well mixed solution leaves the tank at a rate of 6 gal/hour, how much salt is in the tank when it overflows?

Ans: The only difference is the incoming concentration is variable. Given the Volume starts at 600 gal and increases at a rate of 3 gal/min

$$\frac{dP}{dt} = 9(2+\cos(t)) - \frac{6P}{60+3t} \tag{2.125}$$

Our IC is $P(0) = 5$ and use the method of integrating factor

$$\begin{aligned}
\mu(t) &= e^{\int \frac{6}{60+3t}dt} = e^{2\ln(20+t)} = (20+t)^2. && (2.126)\\
((20+t)^2P(t))' &= 9(2+\cos(t))(20+t)^2 && (2.127)\\
(20+t)^2P(t) &= \int 9(2+\cos(t))(20+t)^2dt && (2.128)\\
&= 9(\frac{2}{3}(20+t)^3 + (20+t)^2\sin(t) + 2(20+t)\cos(t) - 2\sin(t)) + c && (2.129)\\
P(t) &= 9(\frac{2}{3}(20+t) + \sin(t) + \frac{2\cos(t)}{20+t} - \frac{2\sin(t)}{(20+t)^2}) + \frac{c}{(20+t)^2} && (2.130)\\
P(0) &= 5 = 9(\frac{2}{3}(20) + \frac{2}{20}) + \frac{c}{400} = 120 + \frac{9}{10} + \frac{c}{400} && (2.131)\\
c &= -46360 && (2.132)
\end{aligned}$$

We want to know how much salt is in the tank when it overflows. This happens when the volume hits 150, or at $t = 30$.

$$P(30) = 300 + 9\sin(30) + \frac{18\cos(30)}{50} - \frac{18\sin(30)}{2500} - \frac{46360}{2500} \tag{2.133}$$

So $P(t) \approx 272.63$ pounds. ■

We could make the problem more complicated by assuming that there will be a change in the situation if the solution ever reached a critical concentration. The process would still be the same, we would just need to solve two different but limited IVPs.

Type III: (Falling Bodies)
Lets consider an object falling to the ground. This body will obey Newton's Second Law of Motion,

$$m\frac{dv}{dt} = F(t,v) \tag{2.134}$$

where m is the object's mass and F is the net force acting on the body. We will look at the situation where the only forces are air resistance and gravity. It is crucial to be careful with the signs. Throughout this course **downward displacements and forces are positive**. Hence the force due to gravity is given by $F_G = mg$, where $g \approx 10m/s^2$ is the gravitational constant.
Air Resistance acts against velocity. If the object is moving up air resistance works downward, always in opposite direction. We will assume air resistance is linearly dependant on velocity (ie $F_A = \alpha v$, where F_A is the force due to air resistance). This is not realistic, but it simplifies the problem. So $F(t,v) = F_G + F_A = 10 - \alpha v$, and our ODE is

$$m\frac{dv}{dt} = 10m - \alpha v \tag{2.135}$$

■ **Example 2.19** A 50 kg object is shot from a cannon straight up with an initial velocity of 10 m/s off the very tip of a bridge. If the air resistance is given by $5v$, determine the velocity of the mass at any time t and compute the rock's terminal velocity.

Ans: Two parts: 1. When the object is moving upwards and 2. When the object is moving downwards. If we look at the forces it turns out we get the same DE

$$50v' = 500 - 5v \tag{2.136}$$

The IC is $v(0) = -10$, since we shot the object upwards. Our DE is linear and we can use integrating factor

$$\begin{aligned}
v' + \frac{1}{10}v &= 10 && (2.137)\\
\mu(t) &= e^{\frac{t}{10}} && (2.138)\\
(e^{\frac{t}{10}}v(t))' &= 10e^{\frac{t}{10}} && (2.139)\\
e^{\frac{t}{10}}v(t) &= \int 10e^{\frac{t}{10}}dt = 100e^{\frac{t}{10}} + c && (2.140)\\
v(t) &= 100 + \frac{c}{e^{\frac{t}{10}}} && (2.141)\\
v(0) &= -10 = 100 + c \Rightarrow c = -110 && (2.142)\\
v(t) &= 100 - \frac{110}{e^{\frac{t}{10}}}. && (2.143)
\end{aligned}$$

What is the terminal velocity of the rock? The terminal velocity is given by the limit of the velocity as $t \to \infty$, which is 100. We could also have computed the velocity of the rock when it hit the ground if we knew the height of the bridge (integrate to get position). ■

■ **Example 2.20** A 60kg skydiver jumps out of a plane with no initial velocity. Assuming the magnitude of air resistance is given by $0.8|v|$, what is the appropriate initial value problem modeling his velocity?

Ans: Air Resistance is an upward force, while gravity is acting downward. So our force should be

$$F(t,v) = mg - .8v \tag{2.144}$$

thus our IVP is

$$60v' = 60g - .8v, \quad v(0) = 0 \tag{2.145}$$

■

2.4 Existence and Uniqueness

Last Time: We developed 1st Order ODE models for physical systems and solved them using the methods of Integrating Factor and Separable Equations.

In Section 1.3 we noted three common questions we would be concerned with this semester.

1. (Existence) Given an IVP, does a solution exist?
2. (Uniqueness) If a solution exists, is it unique?
3. If a solution exists, how do we find it?

We have spent a lot of time on developing methods, now we will spend time on the first two questions. **Without Solving** an IVP, what information can we derive about the existence and uniqueness of solutions? Also we will note strong differences between linear and nonlinear equations.

2.4.1 Linear Equations

While we will focus on first order linear equations, the same basic ideas work for higher order linear equations.

Theorem 2.4.1 *(Fundamental Theorem of Existence and Uniqueness for Linear Equations)* Consider the IVP

$$y' + p(t)y = q(t), \quad y(t_0) = y_0. \tag{2.146}$$

If $p(t)$ and $q(t)$ are continuous functions on an open interval $\alpha < t_0 < \beta$, then there exists a unique solution to the IVP defined on the interval (α, β).

REMARK: The same result holds for general IVPs. If we have the IVP

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \ldots + a_1(t)y' + a_0(t)y = g(t), \quad y(t_0) = y_0, \ldots, y^{(n-1)}(t_0) = y_0^{(n-1)} \tag{2.147}$$

then if $a_i(t)$ (for $i = 0, \ldots, n-1$) and $g(t)$ are continuous on an open interval $\alpha < t_0 < \beta$, there exists a unique solution to the IVP defined on the interval (α, β).

What does Theorem 1 tell us?
(1) If the given linear differential equation is nice, not only do we know **EXACTLY ONE** solution exists. In most applications knowing a solution is unique is more important than knowing a solution exists.
(2) If the interval (α, β) is the largest interval on which $p(t)$ and $q(t)$ are continuous, then (α, β) is the interval of validity to the unique solution guaranteed by the theorem. Thus given a "nice" IVP there is no need to solve the equation to find the interval of validity. The interval only depends on t_0 since the interval must contain it, but does not depend on y_0.

■ **Example 2.21** Without solving, determine the interval of validity for the solution to the following IVP

$$(t^2 - 9)y' + 2y = \ln|20 - 4t|, \quad y(4) = -3 \tag{2.148}$$

Ans: If we look at Theorem 1, we need to write our equation in the form given in Theorem 1 (i.e. coefficient of y' is 1). So rewrite as

$$y' + \frac{2}{t^2 - 9} = \frac{\ln|20 - 4t|}{t^2 - 9} \tag{2.149}$$

Next we identify where either of the two other coefficients are discontinuous. By removing those points we find all intervals of validity. Then the last step is to identify which interval of validity contains t_0.

Using the notation in Theorem 1, $p(t)$ is discontinuous when $t = \pm 3$, since at those points we are dividing by zero. $q(t)$ is discontinuous at $t = 5$, since the natural log of 0 does not exists (only defined on $(0,\infty)$). This yields four intervals of validity where both $p(t)$ and $q(t)$ are continuous

$$(-\infty,-3), \quad (-3,3), \quad (3,5), \quad (5,\infty) \tag{2.150}$$

Notice the endpoints are where $p(t)$ and $q(t)$ are discontinuous, guaranteeing within each interval **both** are continuous. Now all that is left is to identify which interval contains $t_0 = 4$. Thus our interval of validity is $(3,5)$. ■

REMARK: The other intervals of validity we found are intervals of validity for the same differential equation, but for different initial conditions. For example, if our IC was $y(2) = 5$ then the interval of validity must contain 2, so the answer would be $(-3,3)$.

What happens if our IC is at one of the bad points where $p(t)$ and $q(t)$ are discontinuous? Unfortunately we are unable to conclude anything, since the theorem does not apply. On the other hand we cannot say that a solution does not exist just because the hypothesis are not met, so the bottom line is that we cannot conclude anything.

■ **Example 2.22** Without solving, find the interval of validity for the following IVP

$$\cos(x)y' = \sin(x)y - \sqrt{x-1}, \quad y(\frac{3}{2}) = 0 \tag{2.151}$$

First we need to put the equation in the form of Theorem 1

$$y' - \tan(x)y = -\frac{\sqrt{x-1}}{\cos(x)} \tag{2.152}$$

Using the notation in Theorem 1, $p(t)$ is discontinuous at $x = \frac{n\pi}{2}$ for odd integers n and $q(t)$ is discontinuous there and for any $x < 1$. Thus we can list the possible intervals of validity

$$(1,\frac{\pi}{2}),(\frac{\pi}{2},\frac{3\pi}{2}),\ldots,(\frac{(2n+1)}{2},\frac{(2n+3)\pi}{2}) \tag{2.153}$$

for all positive integers n. Since the IC is $y(\frac{3}{2}) = 0$, then the I.O.V. must contain $\frac{3}{2}$. Therefore the answer is $(1,\frac{\pi}{2})$. ■

2.4.2 Nonlinear Equations

We saw in the linear case every "nice enough" equation has a unique solution except for if the initial conditions are ill-posed. But even this seemingly simple nonlinear equation

$$(\frac{dt}{dx})^2 + x^2 + 1 = 0 \tag{2.154}$$

has no real solutions.

So we have the following revision of Theorem 1 that applies to nonlinear equations as well. Since this is applied to a broader class the conclusions are expected to be weaker.

Theorem 2.4.2 Consider the IVP

$$y' = f(t,y), \quad y(t_0) = y_0. \tag{2.155}$$

If f and $\frac{\partial f}{\partial y}$ are continuous functions on some rectangle $\alpha < t_0 < \beta, \gamma < y_0 < \delta$ containing the point (t_0, y_0), then there is a unique solution to the IVP defined on some interval (a,b) satisfying $\alpha < a < t_0 < b \le \beta$.

OBSERVATION:
(1) Unlike Theorem 1, Theorem 2 does not tell us the interval of a unique solution guaranteed by it. Instead, it tells us the largest possible interval that the solution will exist in, we would need to actually solve the IVP to get the interval of validity.
(2) For nonlinear differential equations, the value of y_0 may affect the interval of validity, as we will see in a later example. We want our IC to NOT lie on the boundary of a region where f or its partial derivative are discontinuous. Then we find the largest t-interval on the line $y = y_0$ containing t_0 where everything is continuous.

REMARK: Theorem 2 refers to **partial derivative** $\frac{\partial f}{\partial y}$ of the function of two variables $f(t,y)$. We will talk extensively about this later, but for now we treat t as a constant and take a normal derivative with respect to y. For example

$$f(t,y) = t^2 - 2y^3 t, \quad \text{then} \quad \frac{\partial f}{\partial y} = -6y^2 t. \tag{2.156}$$

■ **Example 2.23** Determine the largest possible interval of validity for the IVP

$$y' = x\ln(y), \quad y(2) = e \tag{2.157}$$

We have $f(x,y) = x\ln(y)$, so $\frac{\partial f}{\partial y} = \frac{x}{y}$. f is discontinuous when $y \le 0$, and f_y (partial derivative with respect to y) is discontinuous when $y = 0$. Since our IC $y(2) = e > 0$ there is no problem since y_0 is never in the discontinuous region. Since there are no discontinuities involving x, then the rectangle is $-\infty < x_0 < \infty, 0 < y_0 < \infty$. Thus the theorem concludes that the unique solution exists somewhere inside $(-\infty, \infty)$. ■

REMARK: Note that this basically told us nothing, and nonlinear problems are quite harder to deal with than linear.

■ **Example 2.24** Determine the largest possible interval of validity for the IVP

$$y' = \sqrt{y - t^2}, \quad y(0) = 1. \tag{2.158}$$

$f(t,y) = \sqrt{y - t^2}$ and $f_y = \frac{1}{2\sqrt{y - t^2}}$. The region of discontinuities is given by $y \le t^2$. Our IC is $y(0) = 1$ does not lie in this region, so we can continue. The line $y = 1$ is continuous for $-1 < t < 1$, so our conclusion is that the interval of validity of the guaranteed unique solution is contained somewhere within $(-1, 1)$. ■

What can happen if the conditions of Theorem 2 are NOT met?

■ **Example 2.25** Determine all possible solutions to the IVP

$$y' = y^{\frac{1}{3}}, \quad y(0) = 0. \tag{2.159}$$

First note this does not satisfy the conditions of the theorem, since $f_y = \frac{1}{3y^{2/3}}$ is not continuous at $y_0 = y = 0$. Now solve the equation it is separable. Notice the equilibrium solution is $y = 0$. This satisfies the IC, but let's solve the equation.

$$\int y^{-1/3} dy = \int dt \tag{2.160}$$
$$\frac{3}{2} y^{2/3} = t + c \tag{2.161}$$
$$y(0) = 0 \tag{2.162}$$
$$y(t) = \pm(\frac{2}{3}t)^{\frac{3}{2}} \tag{2.163}$$
$$\tag{2.164}$$

The IC does not rule out either of these possibilities, so we end up with three possible solutions (these two and the equilibrium solution $y(t) \equiv 0$). ■

In our class we will be mostly dealing with nice equations and unique solutions, but be aware this is not always the case. Consider the next example which illustrates the dependence of the interval of validity on y_0.

■ **Example 2.26** Determine the interval of validity for the IVP

$$y' = y^2, \quad y(0) = y_0 \tag{2.165}$$

First notice its nonlinear so Theorem 1 does not apply. y^2 is continuous everywhere, so for every y_0 there will be a unique solution. It is defined somewhere in $(-\infty, \infty)$. So let's solve. Notice first the equilibrium solution if $y_0 = 0$ we have $y \equiv 0$. So assume $y_0 \neq 0$.

$$\int \frac{1}{y^2} dy = \int dt \tag{2.166}$$
$$-\frac{1}{y} = t + c \tag{2.167}$$
$$c = -\frac{1}{y_0} \tag{2.168}$$
$$-\frac{1}{y} = t - \frac{1}{y_0} \tag{2.169}$$
$$y(t) = \frac{y_0}{1 - y_0 t} \tag{2.170}$$

What is the interval of validity? The only point of discontinuity is $t = \frac{1}{y_0}$. So the two possible intervals of validity are

$$(-\infty, \frac{1}{y_0}), \quad (\frac{1}{y_0}, \infty) \tag{2.171}$$

The correct choice will be the interval containing $t_0 = 0$. But this will depend on y_0. If $y_0 > 0$, 0 will be contained in the interval $(-\infty, \frac{1}{y_0})$ and so this is the interval of validity. On the other hand, if $y_0 < 0$, 0 is contained inside $(\frac{1}{y_0}, \infty)$ and so this is the interval of validity. Thus we have the following possible intervals of validity, depending on y_0.
(1) If $y_0 > 0$, $(-\infty, \frac{1}{y_0})$ is the interval of validity
(2) If $y_0 = 0$, $(-\infty, \infty)$ is the interval of validity
(3) If $y_0 < 0$, $(-\frac{1}{y_0}, \infty)$ is the interval of validity ■

2.4.3 Summary

We established conditions for existence and uniqueness of solutions to first order ODEs. Intervals of validity for linear equations do not depend on the initial choice of y_0, while nonlinear equations may. Secondly, we can find intervals of validity for solutions for linear equations without having to solve the equation. For a nonlinear equation, we would need to solve the equation to get the actual interval of validity. But we can still find all places where the interval of validity definitely will not be defined.

2.5 Autonomous Equations with Population Dynamics

Last Time: We focused on the differences between linear and nonlinear equations as well as identifying intervals of validity without solving any initial value problems (IVP).

2.5.1 Autonomous Equations

First order differential equations relate the slope of a function to the values of the function and the independent variable. We can visualize this using direction fields. This in principle can be very complicated and it might be hard to determine which initial values correspond to which outcomes. However, there is a special class of equations, called **autonomous equations**, where this process is simplified. The first thing to note is autonomous equations do not depend on t

$$y' = f(y) \tag{2.172}$$

REMARK: Notice that all autonomous equations are separable.

What we need to know to study the equation qualitatively is which values of y make y' zero, positive, or negative. The values of y making $y' = 0$ are the **equilibrium solutions**. They are constant solutions and are indicated on the ty-plane by horizontal lines.

After we establish the equilibrium solutions we can study the positivity of $f(y)$ on the intermediate intervals, which will tell us whether the equilibrium solutions attract nearby initial conditions (in which case they are called **asymptotically stable**), repel them (**unstable**), or some combination of them (**semi-stable**).

■ **Example 2.27** Consider

$$y' = y^2 - y - 2 \tag{2.173}$$

Start by finding the equilibrium solutions, values of y such that $y' = 0$. In this case we need to solve $y^2 - y - 2 = (y-2)(y+1) = 0$. So the equilibrium solutions are $y = -1$ and $y = 2$. There are constant solutions and indicated by horizontal lines. We want to understand their stability. If we plot $y^2 - y - 2$ versus y, we can see that on the interval $(-\infty, -1), f(y) > 0$. On the interval $(-1, 2), f(y) < 0$ and on $(2, \infty), f(y) > 0$. Now consider the initial condition.

(1) If the IC $y(t_0) = y_0 < -1, y' = f(y) > 0$ and $y(t)$ will increase towards -1.
(2) If the IC $-1 < y_0 < 2, y' = f(y) < 0$, so the solution will decrease towards -1. Since the solutions below -1 go to -1 and the solutions above -1 go to -1, we conclude $y(t) = -1$

is an asymptotically stable equilibrium.
(3) If $y_0 > 2, y' = f(y) > 0$, so the solution increases away from 2. So at $y(t) = 2$ above and below solutions move away so this is an unstable equilibrium. ■

■ **Example 2.28** Consider

$$y' = (y-4)(y+1)^2 \tag{2.174}$$

The equilibrium solutions are $y = -1$ and $y = 4$. To classify them, we graph $f(y) = (y-4)(y+1)^2$.
(1) If $y < -1$, we can see that $f(y) < 0$, so solutions starting below -1 will tend towards $-\infty$.
(2) If $-1 < y_0 < 4, f(y) < 0$, so solutions starting here tend downwards to -1. So $y(t) = 1$ is semistable.
(3) If $y > 4, f(y) > 0$, solutions starting above 4 will asymptotically increase to ∞, so $y(t) = 4$ is unstable since no nearby solutions converge to it. ■

2.5.2 Populations

The best examples of autonomous equations come from population dynamics. The most naive model is the "Population Bomb" since it grows without any deaths

$$P'(t) = rP(t) \tag{2.175}$$

with $r > 0$. The solution to this differential equation is $P(t) = P_0e^{rt}$, which indicates that the population would increase exponentially to ∞. This is not realistic at all.

A better and more accurate model is the "Logistic Model"

$$P'(t) = rP(1-\frac{P}{N}) = rP - \frac{r}{N}P^2 \tag{2.176}$$

where $N > 0$ is some constant. With this model we have a birth rate of rP and a mortality rate of $\frac{r}{N}P^2$. The equation is separable so let's solve it.

$$\begin{aligned}
\frac{dP}{P(1-\frac{P}{N})} &= rdt && (2.177)\\
\int (\frac{1}{P} + \frac{1/N}{1-P/N})dP &= \int rdt && (2.178)\\
\ln|P| - \ln|1-\frac{P}{N}| &= rt + c && (2.179)\\
\frac{P}{1-\frac{P}{N}} &= Ae^{rt} && (2.180)\\
P &= Ae^{rt} = \frac{1}{N}Ae^{rt}P && (2.181)\\
P(t) &= \frac{Ae^{rt}}{1+\frac{A}{N}e^{rt}} = \frac{AN}{Ne^{-rt}+A} && (2.182)
\end{aligned}$$

if $P(0) = P_0$, then $A = \frac{P_0N}{N-P_0}$ to yield

$$P(t) = \frac{P_0N}{(N-P_0)e^{-rt}+P_0} \tag{2.183}$$

In its present form its hard to analyze what is going on so let's apply the methods from the first section to analyze the stability.

Looking at the logistic equation, we can see that our equilibrium solutions are $P = 0$ and $P = N$. Graphing $f(P) = rP(1-\frac{N}{P})$, we see that
(1) If $P < 0, f(P) < 0$
(2) If $0 < P < N, f(P) > 0$
(3) If $P > N, f(P) < 0$
Thus 0 is unstable while while N is asymptotically stable, so we can conclude for initial population $P_0 > 0$

$$\lim_{t\to\infty} P(t) = N \tag{2.184}$$

So what is N? It is the carrying capacity for the environment. If the population exists, it will grow towards N, but the closer it gets to N the slower the population will grow. If the population starts off greater then the carrying capacity for the environment $P_0 > N$, then the population will die off until it reaches that stable equilibrium position. And if the population starts off at N, the births and deaths will balance out perfectly and the population will remain exactly at $P_0 = N$.

Note: It is possible to construct similar models that have unstable equilibria above 0.

EXERCISE: Show that the equilibrium population $P(t) = N$ is unstable for the autonomous equation

$$P'(t) = rP(\frac{P}{N} - 1). \tag{2.185}$$

2.6 Exact Equations

Last Time: We solved problems involving population dynamics, plotted phase portraits, and determined the stability of equilibrium solutions.

The final category of first order differential equations we will consider are **Exact Equations**. These nonlinear equations have the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{2.186}$$

where $y = y(x)$ is a function of x and find the

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{2.187}$$

where these two derivatives are **partial derivatives**.

2.6.1 Multivariable Differentiation

If we want a partial derivative of $f(x,y)$ with respect to x we treat y as a constant and differentiate normally with respect to x. On the other hand, if we want a partial derivative of $f(x,y)$ with respect to y we treat x as a constant and differentiate normally with respect to y.

Example 2.29 Let $f(x,y) = x^2y = y^2$. Then

$$\frac{\partial f}{\partial x} = 2xy \tag{2.188}$$
$$\frac{\partial f}{\partial y} = x^2 + 2y. \tag{2.189}$$

■

Example 2.30 Let $f(x,y) = y\sin(x)$

$$\frac{\partial f}{\partial x} = y\cos(y) \tag{2.190}$$
$$\frac{\partial f}{\partial y} = \sin(x) \tag{2.191}$$

■

We also need the crucial tool of the multivariable chain rule. If we have a function $\Phi(x,y(x))$ depending on some variable x and a function y depending on x, then

$$\frac{d\Phi}{dx} = \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{dy}{dx} = \Phi_x + \Phi_y y' \tag{2.192}$$

2.6.2 Exact Equations

Start with an example to illustrate the method.

Example 2.31 Consider

$$2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0 \tag{2.193}$$

The first step in solving an exact equation is to find a certain function $\Phi(x,y)$. Finding $\Phi(x,y)$ is most of the work. For this example it turns out

$$\Phi(x,y) = y^2 + (x^2+1)y - 3x^3 \tag{2.194}$$

Notice if we compute the partial derivatives of Φ, we obtain

$$\Phi_x(x,y) = 2xy - 9x^2 \tag{2.195}$$
$$\Phi_y(x,y) = 2y + x^2 + 1. \tag{2.196}$$

Looking back at the differential equation, we can rewrite it as

$$\Phi_x + \Phi_y \frac{dy}{dx} = 0. \tag{2.197}$$

Thinking back to the chain rule we can express as

$$\frac{d\Phi}{dx} = 0 \tag{2.198}$$

Thus if we integrate, $\Phi = c$, where c is a constant. So the general solution is

$$y^2 + (x^2+1)y - 3x^3 = c \tag{2.199}$$

for some constant c. If we had an initial condition, we could use it to find the particular solution to the initial value problem. ■

Let's investigate the last example further. An exact equation has the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{2.200}$$

with $M_y(x,y)=N_x(x,y)$. The key is to construct $\Phi(x,y)$ such that the DE turns into

$$\frac{d\Phi}{dx}=0 \tag{2.201}$$

by using the multivariable chain rule. Thus we require $\Phi(x,y)$ satisfy

$$\Phi_x(x,y) = M(x,y) \tag{2.202}$$
$$\Phi_y(x,y) = N(x,y) \tag{2.203}$$

REMARK: A standard fact from multivariable calculus is that mixed partial derivatives commute. That is why we want $M_y=N_x$, so $M_y=\Phi_{xy}$ and $N_x=\Phi_{yx}$, and so these should be equal for Φ to exist. Make sure you check the function is exact before wasting time on the wrong solution process.

Once we have found Φ, then $\frac{d\Phi}{dx}=0$, and so

$$\Phi(x,y)=c \tag{2.204}$$

yielding an implicit general solution to the differential equation.

So the majority of the work is computing $\Phi(x,y)$. How can we find this desired function, let's retry Example 3, filling in the details.

Example 2.32 Solve the initial value problem

$$2xy-9x^2+(2y+x^2+1)\frac{dy}{dx}=0, \quad y(0)=2 \tag{2.205}$$

Let's begin by checking the equation is in fact exact.

$$M(x,y) = 2xy-9x^2 \tag{2.206}$$
$$N(x,y) = 2y+x^2+1 \tag{2.207}$$
$$\tag{2.208}$$

Then $M_y=2x=N_x$, so the equation is exact.

Now how do we find $\Phi(x,y)$? We have $\Phi_x=M$ and $\Phi_y=N$. Thus we could compute Φ in one of two ways

$$\Phi(x,y)=\int Mdx \quad \text{or} \quad \Phi(x,y)=\int Ndy. \tag{2.209}$$

In general it does not usually matter which you choose, one may be easier to integrate than the other. In this case

$$\Phi(x,y)=\int 2xy-9x^2dx=x^2y-3x^3+h(y). \tag{2.210}$$

Notice since we only integrate with respect to x we can have an arbitrary function only depending on y. If we differentiate $h(y)$ with respect to x we still get 0 like an arbitrary constant c. So in order to have the highest accuracy we take on an arbitrary function of y. Note if we integrated N with respect to y we would get an arbitrary function of x. DO NOT FORGET THIS!

Now all we need is to find $h(y)$. We know if we differentiate Φ with respect to x, then $h(y)$ will vanish which is unhelpful. So instead differentiate with respect to y, since $\Phi_y = N$ in order to be exact. so any terms in N that aren't in Φ_y must be $h'(y)$.

So $\Phi_y = x^2 + h'(y)$ and $N = x^2 + 2y + 1$. Since these are equal we have $h'(y) = 2y + 1$, an so

$$h(y) = \int h'(y)dy = y^2 + y \tag{2.211}$$

REMARK: We will drop the constant of integration we get from integrating h since it will combine with the constant c that we get in the solution process.

Thus, we have

$$\Phi(x,y) = x^2y - 3x^3 + y^2 + y = y^2 + (x^2+1)y - 3x^3, \tag{2.212}$$

which is precisely the Φ that we used in Example 3. Observe

$$\frac{d\Phi}{dx} = 0 \tag{2.213}$$

and thus $\Phi(x,y) = y^2 + (x^2+1)y - 3x^3 = c$ for some constant c. To compute c, we'll use our initial condition $y(0) = 2$

$$2^2 + 2 = c \Rightarrow c = 6 \tag{2.214}$$

and so we have a particular solution of

$$y^2 + (x^2+1)y - 3x^3 = 6 \tag{2.215}$$

This is a quadratic equation in y, so we can complete the square or use quadratic formula to get an explicit solution, which is the goal when possible.

$$y^2 + (x^2+1)y - 3x^3 = 6 \tag{2.216}$$

$$y^2 + (x^2+1)y + \frac{(x^2+1)^2}{4} = 6 + 3x^3 + \frac{(x^2+1)^2}{4} \tag{2.217}$$

$$(y + \frac{x^2+1}{2})^2 = \frac{x^4 + 12x^3 + 2x^2 + 25}{4} \tag{2.218}$$

$$y(x) = \frac{-(x^2+1) \pm \sqrt{x^4 + 12x^3 + 2x^2 + 25}}{2} \tag{2.219}$$

Now we use the initial condition to figure out whether we want the $+$ or $-$ solution. Since $y(0) = 2$ we have

$$2 = y(0) = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2} = 2, -3 \tag{2.220}$$

Thus we see we want the $+$ so our particular solution is

$$y(x) = \frac{-(x^2+1) + \sqrt{x^4 + 12x^3 + 2x^2 + 25}}{2} \tag{2.221}$$

■

■ **Example 2.33** Solve the initial value problem

$$2xy^2+2=2(3-x^2y)y', \quad y(-1)=1. \tag{2.222}$$

First we need to put it in the standard form for exact equations

$$2xy^2+2-2(3-x^2y)y'=0. \tag{2.223}$$

Now, $M(x,y)=2xy^2+2$ and $N(x,y)=-2(3-x^2y)$. So $M_y=4xy=N_x$ and the equation is exact.

The next step is to compute $\Phi(x,y)$. We choose to integrate N this time

$$\Phi(x,y)=\int N dy=\int 2x^2y-6dy=x^2y^2-6y+h(x). \tag{2.224}$$

To find $h(x)$, we compute $\Phi_x=2xy^2+h'(x)$ and notice that for this to be equal to M, $h'(x)=2$. Hence $h(x)=2x$ and we have an implicit solution of

$$x^2y^2-6y+2x=c. \tag{2.225}$$

Now, we use the IC $y(-1)=1$:

$$1-6-2=c \Rightarrow c=-7 \tag{2.226}$$

So our implicit solution is

$$x^2y^2-6y+2x+7=0. \tag{2.227}$$

Again complete the square or use quadratic formula

$$\begin{aligned} y(x) &= \frac{6\pm\sqrt{36-4x^2(2x+7)}}{2x^2} && (2.228)\\ &= \frac{3\pm\sqrt{9-2x^3-7x^2}}{x^2} && (2.229)\end{aligned}$$

and using the IC, we see that we want $-$ solution, so the explicit particular solution is

$$y(x)=\frac{3-\sqrt{9-2x^3-7x^2}}{x^2} \tag{2.230}$$

■

■ **Example 2.34** Solve the IVP

$$\frac{2ty}{t^2+1}-2t-(4-\ln(t^2+1))y'=0, \quad y(2)=0 \tag{2.231}$$

and find the solution's interval of validity.

This is already in the right form. Check if it is exact, $M(t,y)=\frac{2ty}{t^2+1}-2t$ and $N(t,y)=\ln(t^2+1)-4$, so $M_y=\frac{2t}{t^2+1}=N_t$. Thus the equation is exact. Now compute $\Phi(x,y)$. Integrate M

$$\begin{aligned} \Phi=\int M dt &= \int \frac{2ty}{t^2+1}dt=y\ln(t^2+1)-t^2+h(y). && (2.232)\\ \Phi_y &= \ln(t^2+1)+h'(y)=\ln(t^2+1)-4=N && (2.233)\end{aligned}$$

so we conclude $h'(y) = -4$ and thus $h(y) = -4y$. So our implicit solution is then

$$y\ln(t^2+1) - t^2 - 4y = c \tag{2.234}$$

and using the IC we find $c = -4$. Thus the particular solution is

$$y\ln(t^2+1) - t^2 - 4y = -4 \tag{2.235}$$

Solve explicitly to obtain

$$y(x) = \frac{t^2-4}{\ln(t^2+1)-4}. \tag{2.236}$$

Now let's find the interval of validity. We do not have to worry about the natural log since $t^2+1 > 0$ for all t. Thus we want to avoid division by 0.

$$\begin{aligned} \ln(t^2+1)-4 &= 0 && (2.237)\\ \ln(t^2+1) &= 4 && (2.238)\\ t^2 &= e^4-1 && (2.239)\\ t &= \pm\sqrt{e^4-1} && (2.240) \end{aligned}$$

So there are three possible intervals of validity, we want the one containing $t = 2$, so $(-\sqrt{e^4-1}, \sqrt{e^4-1})$. ■

■ **Example 2.35** Solve

$$3y^3e^{3xy} - 1 + (2ye^{3xy} + 3xy^2e^{3xy})y' = 0, \quad y(1) = 2 \tag{2.241}$$

We have

$$M_y = 9y^2e^{3xy} + 9xy^3e^{3xy} = N_x \tag{2.242}$$

Thus the equation is exact. Integrate M

$$\Phi = \int M dx = \int 3y^3e^{3xy} - 1 = y^2e^{3xy} - x + h(y) \tag{2.243}$$

and

$$\Phi_y = 2ye^{3xy} + 3xy^2e^{3xy} + h'(y) \tag{2.244}$$

Comparing Φ_y to N, we see that they are already identical, so $h'(y) = 0$ and $h(y) = 0$. So

$$y^2e^{3xy} - x = c \tag{2.245}$$

and using the IC gives $c = 4e^6 - 1$. Thus our implicit particular solution is

$$y^2e^{3xy} - x = 4e^6 - 1, \tag{2.246}$$

and we are done because we will not be able to solve this explicitly. ■

III

Linear Higher Order Equations

3. Solutions to Second Order Linear Equati

3.1 Second Order Linear Differential Equations

Last Time: We studied exact equations, which were our last type of first order differential equations and the method for solving them. Now we start Chapter 3: Second Order Linear Equations.

3.1.1 Basic Concepts

The example of a second order equation which we have seen many times before is Newton's Second Law when expressed in terms of position $s(t)$ is

$$m\frac{d^2s}{dt^2} = F(t,s',s) \tag{3.1}$$

One of the most basic 2nd order equations is $y'' = -y$. By inspection, we might notice that this has two obvious nonzero solutions: $y_1(t) = \cos(t)$ and $y_2(t) = \sin(t)$. But consider $9\cos(t) - 2\sin(t)$? This is also a solution. Anything of the form $y(t) = c_1\cos(t) + c_2\sin(t)$, where c_1 and c_2 are arbitrary constants. Every solution if no conditions are present has this form.

■ **Example 3.1** Find all of the solutions to $y'' = 9y$

We need a function whose second derivative is 9 times the original function. What function comes back to itself without a sign change after two derivatives? Always think of the exponential function when situations like this arise. Two possible solutions are $y_1(t) = e^{3t}$ and $y_2(t) = e^{-3t}$. In fact so are any combination of the two. This is the principal of **linear superposition**. So $y(t) = c_1e^{3t} + c_2e^{-3t}$ are infinitely many solutions.

EXERCISE: Check that $y_1(t) = e^{3t}$ and $y_2(t) = e^{-3t}$ are solutions to $y'' = 9y$. ■

The general form of a second order linear differential equation is

$$p(t)y'' + q(t)y' + r(t)y = g(t). \tag{3.2}$$

We call the equation **homogeneous** if $g(t) = 0$ and **nonhomogeneous** if $g(t) \neq 0$.

Theorem 3.1.1 *(Principle of Superposition)* If $y_1(t)$ and $y_2(t)$ are solutions to a second order linear homogeneous differential equation, then so is any linear combination

$$y(t) = c_1 y_1(t) + c_2 y_2(t). \tag{3.3}$$

This follows from the homogeneity and the fact that a derivative is a linear operator. So given any two solutions to a homogeneous equation we can find infinitely more by combining them. The main goal is to be able to write down a general solution to a differential equation, so that with some initial conditions we could uniquely solve an IVP. We want to find $y_1(t)$ and $y_2(t)$ so that the general solution to the differential equation is $y(t) = c_1 y_1(t) + c_2 y_2(t)$. By different we mean solutions which are not constant multiples of each other.

Now reconsider $y'' = -y$. We found two different solutions $y_1(t) = \cos(t)$ and $y_2(t) = \sin(t)$ and any solution to this equation can be written as a linear combination of these two solutions, $y(t) = c_1 \cos(t) + c_2 \sin(t)$. Since we have two constants and a 2nd order equation we need two initial conditions to find a particular solution. We are generally given these conditions in the form of y and y' defined at a particular t_0. So a typical problem might look like

$$p(t)y'' + q(t)y' + r(t)y = 0, \quad y'(t_0) = y'_0, \quad y(t_0) = y_0 \tag{3.4}$$

■ **Example 3.2** Find a particular solution to the initial value problem

$$y'' + y = 0, \quad y(0) = 2, \quad y'(0) = -1 \tag{3.5}$$

We have established the general solution to this equation is

$$y(t) = c_1 \cos(t) + c_2 \sin(t) \tag{3.6}$$

To apply the initial conditions, we'll need to know the derivative

$$y'(t) = -c_1 \sin(t) + c_2 \cos(t) \tag{3.7}$$

Plugging in the initial conditions yields

$$2 = c_1 \tag{3.8}$$
$$-1 = c_2 \tag{3.9}$$

so the particular solution is

$$y(t) = 2\cos(t) - \sin(t). \tag{3.10}$$

■

Sometimes when applying initial conditions we will have to solve a system of equations, other times it is as easy as the previous example.

3.1.2 Homogeneous Equations With Constant Coefficients

We will start with the easiest class of second order linear homogeneous equations, where the coefficients $p(t)$, $q(t)$, and $r(t)$ are constants. The equation will have the form

$$ay'' + by' + c = 0. \tag{3.11}$$

How do we find solutions to this equation? From calculus we can find a function that is linked to its derivatives by a multiplicative constant, $y(t) = e^{rt}$. Now that we have a candidate plug it into the differential equation. First calculate the derivatives $y'(t) = re^{rt}$ and $y''(t) = r^2e^{rt}$.

$$a(r^2e^{rt}) + b(re^{rt}) + ce^{rt} = 0 \tag{3.12}$$

$$e^{rt}(ar^2 + br + c) = 0 \tag{3.13}$$

What can we conclude? If $y(t) = e^{rt}$ is a solution to the differential equation, then $e^{rt}(ar^2 + br + c) = 0$. Since $e^{rt} \neq 0$, then $y(t) = e^{rt}$ will solve the differential equation as long as r is a solution to

$$ar^2 + br + c = 0. \tag{3.14}$$

This equation is called the **characteristic equation** for $ay'' + by' + c = 0$.

Thus, to find a solution to a linear second order homogeneous constant coefficient equation, we begin by writing down the characteristic equation. Then we find the roots r_1 and r_2 (not necessarily distinct or real). So we have the solutions

$$y_1(t) = e^{r_1t}, \quad y_2(t) = e^{r_2t}. \tag{3.15}$$

Of course, it is also possible these are the same, since we might have a repeated root. We will see in a future section how to handle these. In fact, we have three cases.

■ **Example 3.3** Find two solutions to the differential equation $y'' - 9y = 0$ (Example 1). The characteristic equation is $r^2 - 9 = 0$, and this has roots $r = \pm 3$. So we have two solutions $y_1(t) = e^{3t}$ and $y_2(t) = e^{-3t}$, which agree with what we found earlier. ■

The three cases are the same as the three possibilities for types of roots of quadratic equations:
(1) Real, distinct roots $r_1 \neq r_2$.
(2) Complex roots $r_1, r_2 = \alpha \pm \beta i$.
(3) A repeated real root $r_1 = r_2 = r$.
We'll look at each case more closely in the lectures to come.

3.2 Solutions of Linear Homogeneous Equations and the Wronskian

Last Time: We studied linear homogeneous equations, the principle of linear superposition, and the characteristic equation.

3.2.1 Existence and Uniqueness

Given an initial value problem involving a linear second order equation, when does a solution exist? We had a theroem in the previous chapter for the first order case so the following theorem will cover second order equations.

Theorem 3.2.1 Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0'. \tag{3.16}$$

If $p(t)$, $q(t)$, and $g(t)$ are all continuous on some interval (a,b) such that $a < t_0 < b$, then the initial value problem has a unique solution defined on (a,b).

3.2.2 Wronskian

Let's suppose we are given the initial value problem

$$p(t)y'' + q(t)y' + r(t)y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0' \tag{3.17}$$

and that we know two solutions $y_1(t)$ and $y_2(t)$. Since the differential equation is linear and homogeneous, the Principle of Superposition says that any linear combination

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \tag{3.18}$$

is also a solution. When is this the general solution? For this to be the case it must satisfy its initial conditions. As long as t_0 does not make any of the coefficient discontinuous, Theorem 1 says $y(t)$ meeting the initial conditions is the general solution. Start by differentiating our candidate $y(t)$ and using the initial conditions

$$y_0 = y(t_0) = c_1 y_1(t_0) + c_2 y_2(t_0) \tag{3.19}$$
$$y_0' = y'(t_0) = c_1 y_1'(t_0) + c_2 y_2'(t_0) \tag{3.20}$$

Solve this system of equations to get

$$c_1 = \frac{y_0 - c_2 y_2(t_0)}{y_1(t_0)} \tag{3.21}$$

Thus

$$y_0' = \frac{y_0 y_1'(t_0) - c_2 y_2(t_0) y_1'(t_0)}{y_1(t_0)} + c_2 y_2'(t_0) \tag{3.22}$$
$$= \frac{y_0 y_1'(t_0) - c_2 y_2(t_0) y_1'(t_0) + c_2 y_2'(t_0) y_1(t_0)}{y_1(t_0)} \tag{3.23}$$

and we compute

$$c_2 = \frac{y_0' y_1(t_0) - y_0 y_1'(t_0)}{y_1(t_0) y_2'(t_0) - y_2(t_0) y_1'(t_0)} \tag{3.24}$$
$$c_1 = \frac{y_0' y_2(t_0) - y_0 y_2'(t_0)}{y_1(t_0) y_2'(t_0) - y_2(t_0) y_1'(t_0)} \tag{3.25}$$
$$\tag{3.26}$$

Notice that c_1 and c_2 have the same quantity in their denominators, so the only time we can solve for c_1 and c_2 is when this quantity is NOT zero.

Definition 3.2.1 The quantity

$$W(y_1,y_2)(t_0) = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0) \tag{3.27}$$

is called the **Wronskian** of y_1 and y_2 at t_0.

REMARK:
(1) When it's clear what the two functions are, we will often denote the Wronskian by W.
(2) We can think of the Wronskian, $W(y_1,y_2)(t)$, as a function of t and can be evaluated at any t where y_1 and y_2 are defined. For any two solutions satisfying the initial conditions we need the Wronskian $W(y_1,y_2)$ to be nonzero at any value t_0 where Theorem 1 applies.
(3) We could have solved the system of equations for $y(t_0)$ and $y'(t_0)$ by Cramer's Rule from Linear Algebra and we have the following formula for the Wronskian

$$W(y_1,y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}. \tag{3.28}$$

We will generally represent the Wronskian as a determinant.

Two solutions will form the general solution if they satisfy the general initial conditions. The above computation showed that this will be the case so long as

$$W(y_1,y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0) \neq 0 \tag{3.29}$$

If $y_1(t)$ and $y_2(t)$ are solutions to our second order equation and $W(y_1,y_2) \neq 0$, then the two solutions are said to be a **fundamental set of solutions** and the general solution is

$$y(t) = c_1y_1(t) + c_2y_2(t). \tag{3.30}$$

In other words, two solutions are "different" enough to form a general solution if they are a fundamental set of solutions.

■ **Example 3.4** If r_1 and r_2 are distinct real roots of the characteristic equation for $ay'' + by' + cy = 0$, check that

$$y_1(t) = e^{r_1t} \quad \text{and} \quad y_2(t) = e^{r_2t} \tag{3.31}$$

form a fundamental set of solutions.

To show this, we compute the Wronskian

$$W = \begin{vmatrix} e^{r_1t} & e^{r_2t} \\ r_1e^{r_1t} & r_2e^{r_2t} \end{vmatrix} = r_2e^{(r_1+r_2)t} - r_1e^{(r_2+r_1)} = (r_2-r_1)e^{(r_2+r_1)t} \tag{3.32}$$

Since the exponentials are never zero and $r_2 \neq r_1$, we conclude that $W \neq 0$ and so as claimed y_1 and y_2 form a fundamental set of solutions for the differential equation and the general solution is

$$y(t) = c_1y_1(t) + c_2y_2(t). \tag{3.33}$$

■

■ **Example 3.5** Consider

$$2t^2y'' + ty' - 3y = 0. \tag{3.34}$$

given that $y_1(t) = t^{-1}$ is a solution. Show $y_2(t) = t^{3/2}$ form a fundamental set of solutions. To do this, we compute the Wronskian

$$W = \begin{vmatrix} t^{-1} & t^{3/2} \\ -t^{-2} & \frac{3}{2}t^{1/2} \end{vmatrix} = \frac{3}{2}t^{-\frac{1}{2}} + t^{-\frac{1}{2}} = \frac{5}{2\sqrt{t}} \tag{3.35}$$

Thus $W \neq 0$, so they are a fundamental set of solutions. Notice we cannot plug in $t = 0$, but this is OK since we cannot plug $t = 0$ into the solution anyway since it would make the coefficients in standard for discontinuous. So the general solution is

$$y(t) = c_1t^{-1} + c_2t^{\frac{3}{2}} \tag{3.36}$$

■

■ **Example 3.6** Consider

$$t^2y'' + 2ty' - 2y = 0. \tag{3.37}$$

We are given that $y_1(t) = t$ is a solution and want to test $y_2(t) = t^{-2}$ as our other solution. Check the Wronskian

$$W = \begin{vmatrix} t & t^{-2} \\ 1 & -2t^{-3} \end{vmatrix} = -2t^{-2} - t^{-2} = -3t^{-2} \neq 0. \tag{3.38}$$

So the solutions are a fundamental set of solutions, and the general solution is

$$y(t) = c_1t + c_2t^{-2} \tag{3.39}$$

■

The last question is how we know if a fundamental set of solutions will exist for a given differential equation. The following theorem has the answer.

Theorem 3.2.2 Consider the differential equation

$$y'' + p(t)y' + q(t) = 0 \tag{3.40}$$

where $p(t)$ and $q(t)$ are continuous on some interval (a,b). Suppose $a < t_0 < b$. If $y_1(t)$ is a solution satisfying the initial conditions

$$y(t_0) = 1, \quad y'(t_0) = 0 \tag{3.41}$$

and $y_2(t)$ is a solution satisfying

$$y(t_0) = 0, \quad y'(t_0) = 1, \tag{3.42}$$

then $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions.

We cannot use this to compute our fundamental set of solutions, but the importance is it assures us that as long as $p(t)$ and $q(t)$ are continuous, then a fundamental set of solutions will exist.

3.2.3 Linear Independence

Consider two functions $f(t)$ and $g(t)$ and the equation

$$c_1 f(t) + c_2 g(t) = 0. \tag{3.43}$$

Notice that $c_1 = 0$ and $c_2 = 0$ always solve this equation, regardless of what f and g are.

Definition 3.2.2 If there are nonzero constants c_1 and c_2 such that the above equation is satisfied for all t, then f and g are said to be **linearly dependent**. On the other hand, if the only constants for which the equation holds are $c_1 = c_2 = 0$, then f and g are said to be **linearly independent**.

REMARK: Two functions are linearly dependent when they are constant multiples of each other. So there are nonzero c_1 and c_2 such that

$$f(t) = -\frac{c_2}{c_1} g(t) \tag{3.44}$$

■ **Example 3.7** Determine if the following pairs of functions are linearly dependent or independent.
(1) $f(x) = 9\cos(2x), g(x) = 2\cos^2(x) - 2\sin^2(x)$
(2) $f(t) = 2t^2, g(t) = t^4$

(1) Consider

$$9c_1 \cos(2x) + 2c_2(\cos^2(x) - \sin^2(x)) = 0. \tag{3.45}$$

We want to determine if there are nonzero constants c_1 and c_2 so this equation is true. Note the trig identity $\cos(2x) = \cos^2(x) - \sin^2(x)$. So our equation becomes

$$9c_1 \cos(2x) + 2c_2 \cos(2x) = 0 \tag{3.46}$$

$$(9c_1 + 2c_2)\cos(2x) = 0 \tag{3.47}$$

This equation is true for $c_1 = 2$ and $c_2 = -9$, thus f and g are linearly dependent.
(2) Consider

$$2c_1 t^2 + c_2 t^4 = 0 \tag{3.48}$$

If this is true differentiate both sides and it will still be true

$$4c_1 t + 4c_2 t^3 = 0 \tag{3.49}$$

Solve for c_1 and c_2. The second equation tells us $c_1 = -c_2 t^2$. Plug into the first equation to get $-c_2 t^4 = 0$, which is only true when $c_2 = 0$. If $c_2 = 0$, then $c_1 = 0$, so f and g are linearly independent. ■

This can be involved and sometimes it is unclear how to proceed. The Wronskian helps identify when two functions are linearly independent.

Theorem 3.2.3 Given two functions $f(t)$ and $g(t)$ which are differentiable on some interval (a,b),
(1) If $W(f,g)(t_0) \neq 0$ for some $a < t_0 < b$, then $f(t)$ and $g(t)$ are linearly independent on (a,b) and
(2) If $f(t)$ and $g(t)$ are linearly dependent on (a,b), then $W(f,g)(t) = 0$ for all $a < t < b$.

REMARK: BE CAREFUL, this theorem DOES NOT say that if $W(f,g)(x) = 0$ then f and g are linearly dependent. It's possible for two linearly independent functions to have a zero Wronskian.

Let's use the theorem to check an earlier example.

■ **Example 3.8** (1) $f(t) = 9\cos(2x), g(x) = 2\cos^2(x) - 2\sin^2(x)$.

$$W = \begin{vmatrix} 9\cos(2x) & 2\cos^2(x) - 2\sin^2(x) \\ -18\sin(2x) & -4\cos(x)\sin(x) - 4\cos(x)\sin(x) \end{vmatrix} \tag{3.50}$$

$$= \begin{vmatrix} 9\cos(2x) & 2\cos(2x) \\ -18\sin(2x) & -4\sin(2x) \end{vmatrix} \tag{3.51}$$

$$= -36\cos(2x)\sin(2x) + 36\cos(2x)\sin(2x) = 0 \tag{3.52}$$

We get zero which we expected since the two functions are linearly dependent.
(2) Now let's take $f(t) = 2t^2$ and $g(t) = t^4$.

$$W = \begin{vmatrix} 2t^2 & t^4 \\ 4t & 4t^3 \end{vmatrix} \tag{3.53}$$

$$= 8t^5 - 4t^5 = 4t^5 \tag{3.54}$$

The Wronskian will be nonzero so long as $t = 0$, which is OK, we just do not want it to be zero for all t. ■

3.2.4 More On The Wronskian

We have established when the Wronskian is nonzero the two functions are linearly independent. We also have seen when y_1 and y_2 are solutions to the linear homogeneous equation

$$p(t)y'' + q(t)y' + r(t)y = 0 \tag{3.55}$$

$W(y_1, y_2)(t) \neq 0$ is precisely the condition for the general solution of the differential equation to be

$$y(t) = c_1y_1(t) + c_2y_2(t) \tag{3.56}$$

where y_1 and y_2 form a fundamental set of solutions.

3.2.5 Abel's Theorem

Through the discussion of the Wronskian we have yet to use the differential equation. If y_1 and y_2 are solutions to a linear homogeneous equation we can say more about the Wronskian.

Theorem 3.2.4 Suppose $y_1(t)$ and $y_2(t)$ solve the linear homogeneous equation

$$y''(t) + p(t)y' + q(t)y = 0, \tag{3.57}$$

where $p(t)$ and $q(t)$ are continuous on some interval (a,b). Then, for $a < t < b$, their Wronskian is given by

$$W(y_1,y_2)(t) = W(y_1,y_2)(t_0)e^{-\int_{t_0}^{t} p(x)dx}, \tag{3.58}$$

where t_0 is in (a,b).

If $W(y_1,y_2)(t_0) \neq 0$ at some point t_0 in the interval (a,b), then Abel's Theorem tell us that the Wronskian can't be zero for any t in (a,b), since exponentials are never zero. We can thus change our initial data without worry that our general solution will change.

Another advantage to Abel's Theorem is that it lets us compute the general form of the Wronskian of any two solutions to the differential equation without knowing them explicitly. The formulation in the theorem is not computationally useful, but we might not have a precise t_0 in mind. But applying the Fundamental Theorem of Calculus

$$W(y_1,y_2)(t) = W(y_1,y_2)(t_0)e^{-\int_{t_0}^{t} p(x)dx} = ce^{-\int p(t)dt}. \tag{3.59}$$

What is c? We only care that $c \neq 0$.

■ **Example 3.9** Compute, up to a constant, the Wronskian of two solutions y_1 and y_2 of the differential equation

$$t^4y'' - 2t^3y' - t^3y = 0. \tag{3.60}$$

First we put the equation in the form of Abel's Theorem.

$$y'' - \frac{2}{t}y' - t^4y = 0 \tag{3.61}$$

So, Abel's Theorem tells us

$$W = ce^{-\int -\frac{2}{t}dt} = ce^{2\ln(t)} = ct^2 \tag{3.62}$$

The main reason this is important is it is an alternative way to compute the Wronskian. We know by Abel's Theorem

$$W(y_1,y_2)(t) = ce^{-\int p(t)dt}. \tag{3.63}$$

On the other hand, by definition

$$W(y_1,y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t) \tag{3.64}$$

Setting these equal, if we know one solution $y_1(t)$, we're left with a first order differential equation for y_2 that we can then solve. ■

Example 3.10 Suppose we want to find a general solution to $2t^2y''+ty'-3y=0$ and we're given that $y_1(t)=t^{-1}$ is a solution. We need to find a second solution that will form a fundamental set of solutions with y_1. Let's compute the Wronskian both ways.

$$ce^{-\int\frac{1}{2t}dt} = W(t^{-1},y_2)(t) = y_2't^{-1}+y_2t^{-2} \tag{3.65}$$
$$y_2't^{-1}+y_2t^{-2} = ce^{-\frac{1}{2}\ln(t)} = ct^{-\frac{1}{2}} \tag{3.66}$$

This is a first order linear equation with integrating factor $\mu(t)=e^{\int t^{-1}dt}=e^{\ln(t)}=t$. Thus

$$(ty_2)' = ct^{\frac{3}{2}} \tag{3.67}$$
$$ty_2 = \frac{2}{5}ct^{\frac{5}{2}}+k \tag{3.68}$$
$$y_2(t) = \frac{2}{5}ct^{\frac{3}{2}}+kt^{-1} \tag{3.69}$$

Now we can choose constants c and k. Notice that k is the coefficient of t^{-1}, which is just $y_1(t)$. So we do not have to worry about that term, and we can take $k=0$. We can similarly take $c=\frac{5}{2}$, and so we'll get $y_2(t)=t^{\frac{3}{2}}$. ■

3.3 Complex Roots of the Characteristic Equation

Last Time: We considered the Wronskian and used it to determine when we have solutions to a second order linear equation or if given one solution we can find another which is linearly independent.

3.3.1 Review Real, Distinct Roots

Recall that a second order linear homogeneous differential equation with constant coefficients

$$ay''+by'+cy=0 \tag{3.70}$$

is solved by $y(t)=e^{rt}$, where r solves the **characteristic equation**

$$ar^2+br+c=0 \tag{3.71}$$

So when there are two distinct roots $r_1\neq r_2$, we get two solutions $y_1(t)=e^{r_1t}$ and $y_2(t)=e^{r_2t}$. Since they are distinct we can immediately conclude the general solution is

$$y(t)=c_1e^{r_1t}+c_2e^{r_2t} \tag{3.72}$$

Then given initial conditions we can solve c_1 and c_2.

Exercises:

(1) $y''+3y'-18y=0, \quad y(0)=0, \quad y'(0)=-1.$
ANS: $y(t)=\frac{1}{9}e^{-6t}-\frac{1}{9}e^{3t}$.
(2) $y''-7y'+10y=0, \quad y(0)=3, \quad y(0)=2$
ANS: $y(t)=-\frac{4}{3}e^{5t}+\frac{13}{3}e^{2t}$
(3) $2y''-5y'+2y=0, \quad y(0)=-3, \quad y'(0)=3$
ANS: $y(t)=-6e^{\frac{1}{2}t}+3e^{2t}$.

(4) $y'' + 5y' = 0, \quad y(0) = 2, \quad y'(0) = -5$
ANS: $y(t) = 1 + e^{-5t}$
(5) $y'' - 2y' - 8 = 0, \quad y(2) = 1, \quad y'(2) = 0$
ANS: $y(t) = \frac{1}{3e^8}e^{4t} + \frac{2e^4}{3}e^{-2t}$
(6) $y'' + y' - 3y = 0$
ANS: $y(t) = c_1 e^{\frac{-1+\sqrt{13}}{2}t} + c_2 e^{\frac{-1-\sqrt{13}}{2}t}$.

3.3.2 Complex Roots

Now suppose the characteristic equation has complex roots of the form $r_{1,2} = \alpha \pm i\beta$. This means we have two solutions to our differential equation

$$y_1(t) = e^{(\alpha+i\beta)t}, \quad y_2(t) = e^{(\alpha-i\beta)t} \tag{3.73}$$

This is a problem since $y_1(t)$ and $y_2(t)$ are complex-valued. Since our original equation was both simple and had real coefficients, it would be ideal to find two real-valued "different" enough solutions so that we can form a real-valued general solution. There is a way to do this.

Theorem 3.3.1 *(Euler's Formula)*

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{3.74}$$

In other words, we can write an imaginary exponential as a sum of sin and cos. How do we establish this fact? There are two ways:

(1) **Differential Equations**: First we want to write $e^{i\theta} = f(\theta) + ig(\theta)$. We also have

$$f' + ig' = \frac{d}{d\theta}[e^{i\theta}] = ie^{i\theta} = if - g. \tag{3.75}$$

Thus $f' = -g$ and $g' = f$, so $f'' = -f$ and $g'' = -g$. Since $e^0 = 1$, we know that $f(0) = 1$ and $g(0) = 0$. We conclude that $f(\theta) = \cos(\theta)$ and $g(\theta) = \sin(\theta)$, so

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{3.76}$$

(2) **Taylor Series**: Recall that the Taylor series for e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \tag{3.77}$$

while the Taylor series for $\sin(x)$ and $\cos(x)$ are

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \tag{3.78}$$

$$\cos(x) = \sum_{n-0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \tag{3.79}$$

$$\tag{3.80}$$

If we set $x = i\theta$ in the first series, we get

$$\begin{aligned}
e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} && (3.81)\\
&= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - ... && (3.82)\\
&= (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ...) + i(\theta - \frac{\theta^3}{3!} + \frac{i\theta^5}{5!} - ...) && (3.83)\\
&= \sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n}}{(2n!)} + i\sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n+1}}{(2n+1)!} && (3.84)\\
&= \cos(\theta) + i\sin(\theta) && (3.85)
\end{aligned}$$

So we can write our two complex exponentials as

$$\begin{aligned}
e^{(\alpha+i\beta)t} &= e^{\alpha t}e^{i\beta t} = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t)) && (3.86)\\
e^{(\alpha-i\beta)t} &= e^{\alpha t}e^{-i\beta t} = e^{\alpha t}(\cos(\beta t) - i\sin(\beta t)) && (3.87)
\end{aligned}$$

where the minus sign pops out of the sign in the second equation since sin is odd and cos is even. Notice our new expression is still complex-valued. However, by the Principle of Superposition, we can obtain the following solutions

$$\begin{aligned}
y_1(t) &= \frac{1}{2}(e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))) + \frac{1}{2}(e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))) = e^{\alpha t}\cos(\beta t) && (3.88)\\
y_2(t) &= \frac{1}{2i}(e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))) - \frac{1}{2i}(e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))) = e^{\alpha t}\sin(\beta t) && (3.89)
\end{aligned}$$

EXERCISE: Check that $y_1(t) = e^{\alpha t}\cos(\beta t)$ and $y_2(t) = e^{\alpha t}\sin(\beta t)$ are in fact solutions to the beginning differential equation when the roots are $\alpha \pm i\beta$.

So now we have two real-valued solutions $y_1(t)$ and $y_2(t)$. It turns out they are linearly independent, so if the roots of the characteristic equation are $r_{1,2} = \alpha \pm i\beta$, we have the general solution

$$y(t) = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t) \qquad (3.90)$$

Let's consider some examples:

■ **Example 3.11** Solve the IVP

$$y'' - 4y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = -2 \qquad (3.91)$$

The characteristic equation is

$$r^2 - 4r + 9 = 0 \qquad (3.92)$$

which has roots $r_{1,2} = 2 \pm i\sqrt{5}$. Thus the general solution and its derivatives are

$$\begin{aligned}
y(t) &= c_1 e^{2t}\cos(\sqrt{5}t) + c_2 e^{2t}\sin(\sqrt{5}t) && (3.93)\\
y'(t) &= 2c_1 e^{2t}\cos(\sqrt{5}t) - \sqrt{5}c_1 e^{2t}\sin(\sqrt{5}t) + 2c_2 e^{2t}\sin(\sqrt{5}t) + \sqrt{5}c_2 e^{2t}\cos(\sqrt{5}t) && (3.94)
\end{aligned}$$

If we apply the initial conditions, we get

$$\begin{aligned} 0 &= c_1 && (3.95) \\ -2 &= 2c_1 + \sqrt{5}c_2 && (3.96) \end{aligned}$$

which is solved by $c_1 = 0$ and $c_2 = -\frac{2}{\sqrt{5}}$. So the particular solution is

$$y(t) = -\frac{2}{\sqrt{5}}e^{2t}\sin(\sqrt{5}t). \tag{3.97}$$

■

■ **Example 3.12** Solve the IVP

$$y'' - 8y' + 17y = 0, \quad y(0) = 2, \quad y'(0) = 5. \tag{3.98}$$

The characteristic equation is

$$r^2 - 8r + 17 = 0 \tag{3.99}$$

which has roots $r_{1,2} = 4 \pm i$. Hence the general solution and its derivatives are

$$\begin{aligned} y(t) &= c_1 e^{4t}\cos(t) = c_2 e^{4t}\sin(t) && (3.100) \\ y'(t) &= 4c_1 e^{4t}\cos(t) - c_1 e^{4t}\sin(t) + 4c_2 e^{4t}\sin(t) + c_2 e^{4t}\cos(t) && (3.101) \end{aligned}$$

and plugging in initial conditions yields the system

$$\begin{aligned} 2 &= c_1 && (3.102) \\ 5 &= 4c_1 + c_2 && (3.103) \end{aligned}$$

so we conclude $c_1 = 2$ and $c_2 = -3$ and the particular solution is

$$y(t) = 2e^{4t}\cos(t) - 3e^{4t}\sin(t) \tag{3.104}$$

■

■ **Example 3.13** Solve the IVP

$$4y'' + 12y' + 10y = 0, \quad y(0) = -1, \quad y'(0) = 3 \tag{3.105}$$

The characteristic equation is

$$4r^2 + 12r + 10 = 0 \tag{3.106}$$

which has roots $r_{1,2} = -\frac{3}{2} \pm \frac{1}{2}i$. So the general solution and its derivative are

$$\begin{aligned} y(t) &= c_1 e^{\frac{3}{2}t}\cos(\frac{t}{2}) + c_2 e^{\frac{3}{2}t}\sin(\frac{t}{2}) && (3.107) \\ y'(t) &= \frac{3}{2}c_1 e^{\frac{3}{2}t}\cos(\frac{t}{2}) - \frac{1}{2}c_1 e^{\frac{3}{2}t}\sin(\frac{t}{2}) + \frac{3}{2}c_2 e^{\frac{3}{2}t}\sin(\frac{t}{2}) + \frac{1}{2}c_2 e^{\frac{3}{2}t}\cos(\frac{t}{2}) && (3.108) \end{aligned}$$

Plugging in the initial condition yields

$$\begin{aligned} -1 &= c_1 && (3.109) \\ 3 &= \frac{3}{2}c_1 + \frac{1}{2}c_2 && (3.110) \end{aligned}$$

which has solution $c_1 = -1$ and $c_2 = 9$. The particular solution is

$$y(t) = -e^{\frac{3}{2}t}\cos(\frac{t}{2}) + 9e^{\frac{3}{2}t}\sin(\frac{t}{2}) \tag{3.111}$$

■

Example 3.14 Solve the IVP

$$y'' + 4y = 0, \quad y(\frac{\pi}{4}) = -10, \quad y'(\frac{\pi}{4}) = 4. \tag{3.112}$$

The characteristic equation is

$$r^2 + 4 = 0 \tag{3.113}$$

which has roots $r_{1,2} = \pm 2i$. The general solution and its derivatives are

$$\begin{aligned} y(t) &= c_1 \cos(2t) + c_2 \sin(2t) & (3.114)\\ y'(t) &= -2c_1 \sin(2t) + 2c_2 \cos(2t). & (3.115) \end{aligned}$$

The initial conditions give the system

$$\begin{aligned} -10 &= c_2 & (3.116)\\ 4 &= -2c_1 & (3.117) \end{aligned}$$

so we conclude that $c_1 = -2$ and $c_2 = -10$ and the particular solution is

$$y(t) = -2\cos(2t) - 10\sin(2t). \tag{3.118}$$

■

3.4 Repeated Roots of the Characteristic Equation and Reduction of Order

Last Time: We considered cases of homogeneous second order equations where the roots of the characteristic equation were complex.

3.4.1 Repeated Roots

The last case of the characteristic equation to consider is when the characteristic equation has repeated roots $r_1 = r_2 = r$. This is a problem since our usual solution method produces the same solution twice

$$y_1(t) = e^{r_1 t} = e^{r_2 t} = y_2(t) \tag{3.119}$$

But these are the same and are not linearly independent. So we will need to find a second solution which is "different" from $y_1(t) = e^{rt}$. What should we do?

Start by recalling that if the quadratic equation $ar^2 + br + c = 0$ has a repeated root r, it must be $r = -\frac{b}{2a}$. Thus our solution is $y_1(t) = e^{-\frac{b}{2a}}$. We know any constant multiple of $y_1(t)$ is also a solution. These will still be linearly dependent to $y_1(t)$. Can we find a solution of the form

$$y_2(t) = v(t)y_1(t) = v(t)e^{-\frac{b}{2a}t} \tag{3.120}$$

i.e. y_2 is the product of a function of t and y_1.

Differentiate $y_2(t)$:

$$
\begin{aligned}
y_2'(t) &= v'(t)e^{-\frac{b}{2a}t} - \frac{b}{2a}v(t)e^{-\frac{b}{2a}t} && (3.121)\\
y_2''(t) &= v''(t)e^{-\frac{b}{2a}} - \frac{b}{2a}v'(t)e^{-\frac{b}{2a}t} - \frac{b}{2a}v'(t)e^{-\frac{b}{2a}t} + \frac{b^2}{4a^2}v(t)e^{-\frac{b}{2a}t} && (3.122)\\
&= v''(t)e^{-\frac{b}{2a}t} - \frac{b}{a}v'(t)e^{-\frac{b}{2a}t} + \frac{b^2}{4a^2}v(t)e^{-\frac{b}{2a}t}. && (3.123)
\end{aligned}
$$

Plug in differential equation:

$$
\begin{aligned}
a(v''e^{-\frac{b}{2a}t} - \frac{b}{a}v'e^{-\frac{b}{2a}t} + \frac{b^2}{4a^2}ve^{-\frac{b}{2a}t}) + b(v'e^{-\frac{b}{2a}t} - \frac{b}{2a}ve^{-\frac{b}{2a}t}) + c(ve^{-\frac{b}{2a}t}) &= 0 && (3.124)\\
e^{-\frac{b}{2a}t}\left(av'' + (-b+b)v' + (\frac{b^2}{4a} - \frac{b^2}{2a} + c)v\right) &= 0 && (3.125)\\
e^{-\frac{b}{2a}t}\left(av'' - \frac{1}{4a}(b^2-4ac)v\right) &= 0 && (3.126)
\end{aligned}
$$

Since we are in the repeated root case, we know the discriminant $b^2-4ac=0$. Since exponentials are never zero, we have

$$av''=0 \Rightarrow v''=0 \tag{3.127}$$

We can drop the a since it cannot be zero, if a were zero it would be a first order equation! So what does v look like

$$v(t)=c_1t+c_2 \tag{3.128}$$

for constants c_1 and c_2. Thus for any such $v(t)$, $y_2(t)=v(t)e^{-\frac{b}{2a}t}$ will be a solution. The most general possible $v(t)$ that will work for us is c_1t+c_2. Take $c_1=1$ and $c_2=0$ to get a specific $v(t)$ and our second solution is

$$y_2(t)=te^{-\frac{b}{2a}t} \tag{3.129}$$

and the general solution is

$$y(t)=c_1e^{-\frac{b}{2a}t}+c_2te^{-\frac{b}{2a}t} \tag{3.130}$$

REMARK: Here's another way of looking at the choice of constants. Suppose we do not make a choice. Then we have the general solution

$$
\begin{aligned}
y(t) &= c_1e^{-\frac{b}{2a}t} + c_2(ct+k)e^{-\frac{b}{2a}t} && (3.131)\\
&= c_1e^{-\frac{b}{2a}t} + c_2cte^{-\frac{b}{2a}t} + c_2ke^{-\frac{b}{2a}t} && (3.132)\\
&= (c_1+c_2k)e^{-\frac{b}{2a}t} + c_2cte^{-\frac{b}{2a}t} && (3.133)
\end{aligned}
$$

since they are all constants we just get

$$y(t)=c_1e^{-\frac{b}{2a}t}+c_2te^{-\frac{b}{2a}t} \tag{3.134}$$

To summarize: if the characteristic equation has repeated roots $r_1=r_2=r$, the general solution is

$$y(t)=c_1e^{rt}+c_2te^{rt} \tag{3.135}$$

Now for examples:

Example 3.15 Solve the IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = -1, \quad y'(0) = 6 \tag{3.136}$$

The characteristic equation is

$$r^2 - 4r + 4 = 0 \tag{3.137}$$
$$(r-2)^2 = 0 \tag{3.138}$$

so we see that we have a repeated root $r = 2$. The general solution and its derivative are

$$y(t) = c_1 e^{2t} + c_2 t e^{2t} \tag{3.139}$$
$$y'(t) = 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t e^{2t} \tag{3.140}$$

and plugging in initial conditions yields

$$-1 = c_1 \tag{3.141}$$
$$6 = 2c_1 + c_2 \tag{3.142}$$

so we have $c_1 = -1$ and $c_2 = 8$. The particular solution is

$$y(t) = -e^{2t} + 6te^{2t} \tag{3.143}$$

■

Example 3.16 Solve the IVP

$$16y'' + 40y' + 25y = 0, \quad y(0) = -1, \quad y'(0) = 2. \tag{3.144}$$

The characteristic equation is

$$16r^2 + 40r + 25 = 0 \tag{3.145}$$
$$(4r+5)^2 = 0 \tag{3.146}$$

and so we conclude that we have a repeated root $r = -\frac{5}{4}$ and the general solution and its derivative are

$$y(t) = c_1 e^{-\frac{5}{4}t} + c_2 t e^{-\frac{5}{4}t} \tag{3.147}$$
$$y'(t) = -\frac{5}{4} c_1 e^{-\frac{5}{4}t} + c_2 e^{-\frac{5}{4}t} - \frac{5}{4} c_2 t e^{-\frac{5}{4}t} \tag{3.148}$$

Plugging in the initial conditions yields

$$-1 = c_1 \tag{3.149}$$
$$2 = -\frac{5}{4} c_1 + c_2 \tag{3.150}$$

so $c_1 = -1$ and $c_2 = \frac{5}{4}$. The particular solution is

$$y(t) = -e^{-\frac{5}{4}t} + \frac{3}{4} t e^{-\frac{5}{4}t} \tag{3.151}$$

■

3.4.2 Reduction of Order

We have spent the last few lectures analyzing second order linear homogeneous equations with constant coefficients, i.e. equations of the form

$$ay'' + by' + cy = 0 \tag{3.152}$$

Let's now consider the case when the coefficients are not constants

$$p(t)y'' + q(t)y' + r(t)y = 0 \tag{3.153}$$

In general this is not easy, but if we can guess a solution, we can use the techniques developed in the repeated roots section to find another solution. This method will be called **Reduction Of Order**. Consider a few examples

■ **Example 3.17** Find the general solution to

$$2t^2y'' + ty' - 3y = 0 \tag{3.154}$$

given that $y_1(t) = t^{-1}$ is a solution.

ANS: Think back to repeated roots. We know we had a solution $y_1(t)$ and needed to find a distinct solution. What did we do? We asked which nonconstant function $v(t)$ make $y_2(t) = v(t)y_1(t)$ is also a solution. The y_2 derivatives are

$$\begin{aligned} y_2 &= vt^{-1} && (3.155)\\ y_2' &= v't^{-1} - vt^{-2} && (3.156)\\ y_2'' &= v''t^{-1} - v't^{-2} + 2vt^{-3} = v''t^{-1} - 2v't^{-2} + 2vt^{-3} && (3.157) \end{aligned}$$

The next step is to plug into the original equation so we can solve for v:

$$\begin{aligned} 2t^2(v''t^{-1} - 2v't^{-2} + 2vt^{-3}) + t(v't^{-1} - vt^{-2}) - 3vt^{-1} &= 0 && (3.158)\\ 2v''t - 4v' + 4vt^{-1} + v' - vt^{-1} - 3vt^{-1} &= 0 && (3.159)\\ 2tv'' - 3v' &= 0 && (3.160) \end{aligned}$$

Notice that the only terms left involve v'' and v', not v. This also happened in the repeated root case. The v term should always disappear at this point, so we have a check on our work. If there is a v term left we have done something wrong.

Now we know that if y_2 is a solution, the function v must satisfy

$$2tv'' - 3v' = 0 \tag{3.161}$$

But this is a second order linear homogeneous equation with nonconstant coefficients. Let $w(t) = v'(t)$. By changing variables our equation becomes

$$w' - \frac{3}{2t}w = 0. \tag{3.162}$$

So by Integrating Factor

$$\begin{aligned} \mu(t) &= e^{\int -\frac{3}{2t}dt} = e^{-\frac{3}{2}\ln(t)} = t^{-\frac{3}{2}} && (3.163)\\ (t^{-\frac{3}{2}}w)' &= 0 && (3.164)\\ t^{-\frac{3}{2}}w &= c && (3.165)\\ w(t) &= ct^{\frac{3}{2}} && (3.166) \end{aligned}$$

So we know what $w(t)$ must solve the equation. But to solve our original differential equation, we do not need $w(t)$, we need $v(t)$. Since $v'(t) = w(t)$, integrating w will give our v

$$\begin{aligned} v(t) &= \int w(t)dt && (3.167)\\ &= \int ct^{\frac{3}{2}t}dt && (3.168)\\ &= \frac{2}{5}ct^{\frac{5}{2}} + k && (3.169) \end{aligned}$$

Now this is the general form of $v(t)$. Pick $c = 5/2$ and $k = 0$. Then $v(t) = t^{\frac{5}{2}}$, so $y_2(t) = v(t)y_1(t) = t^{\frac{3}{2}}$, and the general solution is

$$y(t) = c_1 t^{-1} + c_2 t^{\frac{3}{2}} \tag{3.170}$$

■

Reduction of Order is a powerful method for finding a second solution to a differential equation when we do not have any other method, but we need to have a solution to begin with. Sometimes even finding the first solution is difficult.

We have to be careful with these problems sometimes the algebra is tedious and one can make sloppy mistakes. Make sure the v terms disappears when we plug in the derivatives for y_2 and check the solution we obtain in the end in case there was an algebra mistake made in the solution process.

■ **Example 3.18** Find the general solution to

$$t^2y'' + 2ty' - 2y = 0 \tag{3.171}$$

given that

$$y_1(t) = t \tag{3.172}$$

is a solution.

Start by setting $y_2(t) = v(t)y_1(t)$. So we have

$$\begin{aligned} y_2 &= tv && (3.173)\\ y_2' &= tv' + v && (3.174)\\ y_2'' &= tv'' + v' + v' = tv'' + 2v'. && (3.175) \end{aligned}$$

Next, we plug in and arrange terms

$$\begin{aligned} t^2(tv'' + 2v') + 2t(tv' + v) - 2tv &= 0 && (3.176)\\ t^3v'' + 2t^2v' + 2t^2v' + 2tv - 2tv &= 0 && (3.177)\\ t^3v'' + 4t^2v' &= 0. && (3.178) \end{aligned}$$

Notice the v drops out as desired. We make the change of variables $w(t) = v'(t)$ to obtain

$$t^3w' + 4t^2w = 0 \tag{3.179}$$

which has integrating factor $\mu(t) = t^4$.

$$\begin{aligned} (t^4 w)' &= 0 && (3.180) \\ t^4 w &= c && (3.181) \\ w(t) &= ct^{-4} && (3.182) \end{aligned}$$

So we have

$$\begin{aligned} v(t) &= \int w(t)dt && (3.183) \\ &= \int ct^{-4}dt && (3.184) \\ &= -\frac{c}{3}t^{-3} + k. && (3.185) \end{aligned}$$

A nice choice for the constants is $c = -3$ and $k = 0$, so $v(t) = t^{-3}$, which gives a second solution of $y_2(t) = v(t)y_1(t) = t^{-2}$. So our general solution is

$$y(t) = c_1 t + c_2 t^{-2} \tag{3.186}$$

■

3.5 Nonhomogeneous Equations with Constant Coefficients

Last Time: We considered cases of homogeneous second order equations where the roots of the characteristic equation were repeated real roots. Then we looked at the method of reduction of order to produce a second solution to an equation given the first solution.

3.5.1 Nonhomogeneous Equations

A second order nonhomogeneous equation has the form

$$p(t)y'' + q(t)y' + r(t)y = g(t) \tag{3.187}$$

where $g(t) \neq 0$. How do we get the general solution to these?

Suppose we have two solutions $Y_1(t)$ and $Y_2(t)$. The Principle of Superposition no longer holds for nonhomogeneous equations. We cannot just take a linear combination of the two to get another solution. Consider the equation

$$p(t)y'' + q(t)y' + r(t)y = 0 \tag{3.188}$$

which we will call the **associated homogeneous equation**.

Theorem 3.5.1 Suppose that $Y_1(t)$ and $Y_2(t)$ are two solutions to equation (3.187) and that $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions to (3.188). Then $Y_1(t) - Y_2(t)$ is a solution to Equation (3.188) and has the form

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t) \tag{3.189}$$

Notice the notation used, it will be standard. Uppercase letters are solutions to the nonhomogeneous equation and lower case letters to denote solutions to the homogeneous equation.

Let's verify the theorem by plugging in $Y_1 - Y_2$ to (3.188)

$$\begin{aligned}
p(t)(Y_1 - Y_2)'' + q(t)(Y_1 - Y_2)' + r(t)(Y_1 - Y_2) &= 0 &(3.190)\\
\big(p(t)Y_1'' + q(t)Y_1' + r(t)Y_1\big) - \big(p(t)Y_2'' + q(t)Y_2' + r(t)Y_2\big) &= 0 &(3.191)\\
g(t) - g(t) &= 0 &(3.192)\\
0 &= 0 &(3.193)
\end{aligned}$$

So we have that $Y_1(t) - Y_2(t)$ solves equation (3.188). We know that $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions to equation (3.188) and so any solution can be written as a linear combination of them. Thus for constants c_1 and c_2

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t) \tag{3.194}$$

So the difference of any two solutions of (3.187) is a solution to (3.188). Suppose we have a solution to (3.187), which we denote by $Y_p(t)$. Let $Y(t)$ denote the general solution. We have seen

$$Y(t) - Y_p(t) = c_1 y_1(t) + c_2 y_2(t) \tag{3.195}$$

or

$$Y(t) = c_1 y_1(t) + c_2 y_2(t) + Y_p(t) \tag{3.196}$$

where y_1 and y_2 are a fundamental set of solutions to $Y(t)$. We will call

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t) \tag{3.197}$$

the **complimentary solution** and $Y_p(t)$ a **particular solution**. So, the general solution can be expressed as

$$Y(t) = y_c(t) + Y_p(t). \tag{3.198}$$

Thus, to find the general solution of (3.187), we'll need to find the general solution to (3.188) and then find some solution to (3.187). Adding these two pieces together give the general solution to (3.187).

If we vary a solution to (3.187) by just adding in some solution to Equation (3.188), it will still solve Equation (3.187). Now the goal of this section is to find some particular solution $Y_p(t)$ to Equation (3.187). We have two methods. The first is the method of **Undetermined Coefficients**, which reduces the problem to an algebraic problem, but only works in a few situations. The other called Variation of Parameters is a much more general method that always works but requires integration which may or may not be tedious.

3.5.2 Undetermined Coefficients

The major disadvantage of this solution method is that it is only useful for constant coefficient differential equations, so we will focus on

$$ay'' + by' + cy = g(t) \tag{3.199}$$

for $g(t) \neq 0$. The other disadvantage is it only works for a small class of $g(t)$'s.

Recall that we are trying to find some particular solution $Y_p(t)$ to Equation (3.199). The idea behind the method is that for certain classes of nonhomogeneous terms, we're able to make a good educated guess as to how $Y_p(t)$ should look, up to some unknown coefficients. Then we plug our guess into the differential equation and try to solve for the coefficients. If we can, our guess was correct and we have determined $Y_p(t)$. If we cannot solve for the coefficients, then we guessed incorrectly and we will need to try again.

3.5.3 The Basic Functions

There are three types of basic types of nonhomogeneous terms $g(t)$ that can be used for this method: exponentials, trig functions (sin and cos), and polynomials. One we know how they work individually and combination will be similar.

Exponentials

Let's walk through an example where $g(t)$ is an exponential and see how to proceed.

■ **Example 3.19** Determine a particular solution to

$$y'' - 4y' - 12y = 2e^{4t}. \tag{3.200}$$

How can we guess the form of $Y_p(t)$? When we plug $Y_p(t)$ into the equation, we should get $g(t) = 2e^{4t}$. We know that exponentials never appear or disappear during differentiation, so try

$$Y_p(t) = Ae^{4t} \tag{3.201}$$

for some coefficient A. Differentiate, plug in, and see if we can determine A. Plugging in we get

$$\begin{aligned} 16Ae^{4t} - 4(4Ae^{4t}) - 12Ae^{4t} &= 2e^{4t} && (3.202)\\ -12Ae^{4t} &= 2e^{4t} && (3.203) \end{aligned}$$

For these to be equal we need A to satisfy

$$-12A = 2 \Rightarrow A = -\frac{1}{6}. \tag{3.204}$$

So with this choice of A, our guess works, and the particular solution is

$$Y_p(t) = -\frac{1}{6}e^{4t}. \tag{3.205}$$

■

Consider the following full problem:

■ **Example 3.20** Solve the IVP

$$y'' - 4y' - 12y = 2e^{4t}, \quad y(0) = -\frac{13}{6}, \quad y'(0) = \frac{7}{3}. \tag{3.206}$$

We know the general solution has the form

$$y(t) = y_c(t) + Y_p(t) \tag{3.207}$$

where the complimentary solution $y_c(t)$ is the general solution to the associated homogeneous equation

$$y'' - 4y' - 12y = 0 \tag{3.208}$$

and $Y_p(t)$ is the particular solution to the original differential equation. From the previous example we know

$$Y_p(t) = -\frac{1}{6}e^{4t}. \tag{3.209}$$

What is the complimentary solution? Our associated homogeneous equation has constant coefficients, so we need to find roots of the characteristic equation.

$$r^2 - 4r - 12 = 0 \tag{3.210}$$
$$(r-6)(r+2) = 0 \tag{3.211}$$

So we conclude that $r_1 = 6$ and $r_2 = -2$. These are distinct roots, so the complimentary solution will be

$$y_c(t) = c_1e^{6t} + c_2e^{-2t} \tag{3.212}$$

We must be careful to remember the initial conditions are for the non homogeneous equation, not the associated homogeneous equation. Do not apply them at this stage to y_c, since that is not a solution to the original equation.

So our general solution is the sum of $y_c(t)$ and $Y_p(t)$. We'll need it and its derivative to apply the initial conditions

$$y(t) = c_1e^{6t} + c_2e^{-2t} - \frac{1}{6}e^{4t} \tag{3.213}$$
$$y'(t) = 6c_1e^{6t} - 2c_2e^{2t} - \frac{2}{3}e^{4t} \tag{3.214}$$

Now apply the initial conditions

$$-\frac{13}{6} = y(0) = c_1 + c_2 - \frac{1}{6} \tag{3.215}$$
$$\frac{7}{3} = y'(0) = 6c_1 - 2c_2 - \frac{2}{3} \tag{3.216}$$

This system is solved by $c_1 = -\frac{1}{8}$ and $c_2 = -\frac{15}{8}$, so our solution is

$$y(t) = -\frac{1}{8}e^{6t} - \frac{15}{8}e^{-2t} - \frac{1}{6}e^{4t}. \tag{3.217}$$

■

Trig Functions

The second class of nonhomogeneous terms for which we can use this method are trig functions, specifically sin and cos.

■ **Example 3.21** Find a particular solution for the following IVP

$$y'' - 4y' - 12y = 6\cos(4t). \tag{3.218}$$

In the first example the nonhomogeneous term was exponential, and we know when we differentiate exponentials they persist. In this case, we've got a cosine function. When we differentiate a cosine, we get sine. So we expect an initial guess to require a sine term in addition to cosine. Try

$$Y_p(t) = A\cos(4t) + B\sin(4t). \tag{3.219}$$

Now differentiate and plug in

$$-16A\cos(4t) - 16B\sin(4t) - 4\big(-4A\sin(4t) + 4B\cos(4t)) - 12(A\cos(4t) + B\sin(4t)\big) = 13\cos(4t) \tag{3.220}$$

$$(-16A - 16B - 12A)\cos(4t) + (-16B + 16A - 12B)\sin(4t) = 13\cos(4t) \tag{3.221}$$

$$(-28A - 16B)\cos(4t) + (16A - 28B)\sin(4t) = 13\cos(4t) \tag{3.222}$$

To solve for A and B set the coefficients equal. Note that the coefficient for $\sin(4t)$ on the right hand side is 0. So we get the system of equations

$$\cos(4t) \;:\; -28A - 16B = 13 \tag{3.223}$$

$$\sin(4t) \;:\; 16A - 28B = 0. \tag{3.224}$$

This system is solved by $A = -\frac{7}{20}$ and $B = -\frac{1}{5}$. So a particular solution is

$$Y_p(t) = -\frac{7}{20}\cos(4t) - \frac{1}{5}\sin(4t) \tag{3.225}$$

■

Note that the guess would have been the same if $g(t)$ had been sine instead of cosine.

Polynomials

The third and final class of nonhomogeneous term we can use with this method are polynomials.

■ **Example 3.22** Find a particular solution to

$$y'' - 4y' - 12y = 3t^3 - 5t + 2. \tag{3.226}$$

In this case, $g(t)$ is a cubic polynomial. When differentiating polynomials the order decreases. So if our initial guess is a cubic, we should capture all terms that will arise. Our guess

$$Y_p(t) = At^3 + Bt^2 + Ct + D. \tag{3.227}$$

Note that we have a t^2 term in our equation even though one does not appear in $g(t)$! Now differentiate and plug in

$$6At + 2B - 4(3At^2 + 2Bt + C) - 12(At^3 + Bt^2 + Ct + D) = 3t^2 - 5t + 2 \tag{3.228}$$

$$-12At^3 + (12A - 12B)t^2 + (6A - 8B - 12C)t + (2B - 4C - 12D) = 3t^2 - 5t + 2 \tag{3.229}$$

We obtain a system of equations by setting coefficients equal

$$t^3 \quad : \quad -12A = 3 \Rightarrow A = -\frac{1}{4} \tag{3.230}$$
$$t^2 \quad : \quad -12A - 12B = 0 \Rightarrow B = \frac{1}{4} \tag{3.231}$$
$$t \quad : \quad 6A - 8B - 12C = -5 \Rightarrow C = \frac{1}{8} \tag{3.232}$$
$$1 \quad : \quad 2B - 4C - 12D = 2 \Rightarrow D = -\frac{1}{6} \tag{3.233}$$

So a particular solution is

$$Y_p(t) = -\frac{1}{4}t^3 + \frac{1}{4}t^2 + \frac{1}{8}t - \frac{1}{6} \tag{3.234}$$

■

Summary

Given each of the basic types, we make the following guess

$$ae^{\alpha t} \quad \Rightarrow \quad Ae^{\alpha t} \tag{3.235}$$
$$a\cos(\alpha t) \quad \Rightarrow \quad A\cos(\alpha t) + B\sin(\alpha t) \tag{3.236}$$
$$a\sin(\alpha t) \quad \Rightarrow \quad A\cos(\alpha t) + B\sin(\alpha t) \tag{3.237}$$
$$a_n t^n + a_{n-1}t^{n-1} + \ldots + a_1 t + a_0 \quad \Rightarrow \quad A_n t^n + A_{n-1}t^{n-1} + \ldots + A_1 t + A_0 \tag{3.238}$$

3.5.4 Products

The idea for products is to take products of our forms above.

■ **Example 3.23** Find a particular solution to

$$y'' - 4y' - 12y = te^{4t} \tag{3.239}$$

Start by writing the guess for the individual pieces. $g(t)$ is the product of a polynomial and an exponential. Thus guess for the polynomial is $At + B$ while the guess for the exponential is Ce^{4t}. So the guess for the product should be

$$Ce^{4t}(At + B) \tag{3.240}$$

We want to minimize the number of constants, so

$$Ce^{4t}(At + B) = e^{4t}(ACt + BC). \tag{3.241}$$

Rewrite with two constants

$$Y_p(t) = e^{4t}(At + B) \tag{3.242}$$

Notice this is the guess as if it was just t with the exponential multiplied to it. Differentiate and plug in

$$16e^{4t}(At + B) + 8Ae^{4t} - 4(4e^{4t}(At + B) + Ae^{4t}) - 12e^{4t}(At + B) = te^{4t} \tag{3.243}$$
$$(16A - 16A - 12A)t^{4t} + (16B + 8A - 16B - 4A - 12B)e^{4t} = te^{4t} \tag{3.244}$$
$$-12Ate^{4t} + (4A - 12B)e^{4t} = te^{4t} \tag{3.245}$$

Then we set the coefficients equal

$$te^{4t}: \quad -12A = 1 \quad \Rightarrow \quad A = -\frac{1}{12} \tag{3.246}$$

$$e^{4t}: \quad (4A - 12B) = 0 \quad \Rightarrow \quad B = -\frac{1}{36} \tag{3.247}$$

So, a particular solution for this differential equation is

$$Y_p(t) = e^{4t}(-\frac{1}{12}t - \frac{1}{36}) = -\frac{e^{4t}}{36}(3t+1). \tag{3.248}$$

■

Basic Rule: If we have a product with an exponential write down the guess for the other piece and multiply by an exponential without any leading coefficient.

■ **Example 3.24** Find a particular solution to

$$y'' - 4y' - 12y = 29e^{5t}\sin(3t). \tag{3.249}$$

We try the following guess

$$Y_p(t) = e^{5t}(A\cos(3t) + B\sin(3t)). \tag{3.250}$$

So differentiate and plug in

$$\begin{aligned} &25e^{5t}(A\cos(3t)+B\sin(3t)) + 30e^{5t}(-A\sin(3t)+B\cos(3t)+ \\ &9e^{5t}(-A\cos(3t)-B\sin(3t)) - 4(5e^{5t}(A\cos(3t)+B\sin(3t))+ \\ &3e^{5t}(-A\sin(3t)+B\cos(3t))) - 12e^{5t}(A\cos(3t)+B\sin(3t)) = 29e^{5t}\sin(3t) \end{aligned} \tag{3.251}$$

Gather like terms

$$(-16A+18B)e^{5t}\cos(3t) + (-18A-16B)e^{5t}\sin(3t) = 29e^{5t}\sin(3t) \tag{3.252}$$

Set the coefficients equal

$$e^{5t}\cos(3t): \quad -16A + 18B = 0 \tag{3.253}$$

$$e^{5t}\sin(3t): \quad -18A - 16B = 29 \tag{3.254}$$

This is solved by $A = -\frac{9}{10}$ and $B = -\frac{4}{5}$. So a particular solution to this differential equation is

$$Y_p(t) = e^{5t}(-\frac{9}{10}t - \frac{4}{5}) = -\frac{e^{5t}}{10}(9t+8). \tag{3.255}$$

■

■ **Example 3.25** Write down the form of the particular solution to

$$y'' - 4y' - 12y = g(t) \tag{3.256}$$

for the following $g(t)$:
(1) $g(t) = (9t^2 - 103t)\cos(t)$

Here we have a product of a quadratic and a cosine. The guess for the quadratic is

$$At^2 + Bt + C \tag{3.257}$$

and the guess for the cosine is

$$D\cos(t) + E\sin(t). \tag{3.258}$$

Multiplying the two guesses gives

$$(At^2 + Bt + C)(D\cos(t)) \quad + \quad (At^2 + Bt + C)(E\sin(t)) \tag{3.259}$$
$$(ADt^2 + BDt + CD)\cos(t) \quad + \quad (AEt^2 + BEt + CE)\sin(t). \tag{3.260}$$

Each of the coefficients is a product of two constants, which is another constant. Simply to get our final guess

$$Y_p(t) = (At^2 + Bt + C)\cos(t) + (Dt^2 + Et + F)\sin(t) \tag{3.261}$$

This is indicative of the general rule for a product of a polynomial and a trig function. Write down the guess for the polynomial, multiply by cosine, then add to that the guess for the polynomial multiplied by a sine.

(2) $g(t) = e^{-2t}(3 - 5t)\cos(9t)$

This homogeneous term has all three types of special functions. So combining the two general rules above, we get

$$Y_p(t) = e^{-2t}(At + B)\cos(9t) + e^{-2t}(Ct + D)\sin(9t). \tag{3.262}$$

■

3.5.5 Sums

We have the following important fact. If Y_1 satisfies

$$p(t)y'' + q(t)y' + r(t)y = g_1(t) \tag{3.263}$$

and Y_2 satisfies

$$p(t)y'' + q(t)y' + r(t)y = g_2(t) \tag{3.264}$$

then $Y_1 + Y_2$ satisfies

$$p(t)y'' + q(t)y' + r(t)y = g_1(t) + g_2(t) \tag{3.265}$$

This means that if our nonhomogeneous term $g(t)$ is a sum of terms we can write down the guesses for each of those terms and add them together for our guess.

■ **Example 3.26** Find a particular solution to

$$y'' - 4y' - 12y = e^{7t} + 12. \tag{3.266}$$

Our nonhomogeneous term $g(t) = e^{7t} + 12$ is the sum of an exponential $g_1(t) = e^{7t}$ and a 0 degree polynomial $g_2(t) = 12$. The guess is

$$Y_p(t) = Ae^{7t} + B \tag{3.267}$$

This cannot be simplified, so this is our guess. Differentiate and plug in

$$49Ae^{7t} - 28Ae^{7t} - 12Ae^{7t} - 12B = e^{7t} + 12 \tag{3.268}$$
$$9Ae^{7t} - 12B = e^{7t} + 12. \tag{3.269}$$

Setting the coefficients equal gives $A = \frac{1}{9}$ and $B = -1$, so our particular solution is

$$Y_p(t) = \frac{1}{9}e^{7t} - 1. \tag{3.270}$$

■

■ **Example 3.27** Write down the form of a particular solution to

$$y'' - 4y' - 12y = g(t) \tag{3.271}$$

for each of the following $g(t)$:
(1) $g(t) = 2\cos(t) - 9\sin(3t)$

Our guess for the cosine is

$$A\cos(3t) + B\sin(3t) \tag{3.272}$$

Additionally, our guess for the sine is

$$C\cos(3t) + D\sin(3t) \tag{3.273}$$

So if we add the two of them together, we obtain

$$A\cos(3t) + B\sin(3t) + C\cos(3t) + D\sin(3t) = (A+C)\cos(3t) + (B+D)\sin(3t) \tag{3.274}$$

But $A+C$ and $B+D$ are just some constants, so we can replace them with the guess

$$Y_p(t) = A\cos(3t) + B\sin(3t). \tag{3.275}$$

(2) $g(t) = \sin(t) - 2\sin(14t) - 5\cos(14t)$

Start with a guess for the $\sin(t)$

$$A\cos(t) + B\sin(t). \tag{3.276}$$

Since they have the same argument, the previous example showed we can combine the guesses for $\cos(14t)$ and $\sin(14t)$ into

$$C\cos(14t) + D\sin(14t) \tag{3.277}$$

So the final guess is

$$Y_p(t) = A\cos(t) + B\sin(t) + C\cos(14t) + D\cos(14t) \tag{3.278}$$

(3) $g(t) = 7\sin(10t) - 5t^2 + 4t$
Here we have the sum of a trig function and a quadratic so the guess will be

$$Y_p(t) = A\cos(10t) + B\sin(10t) + Ct^2 + Dt + E. \tag{3.279}$$

(4) $g(t) = 9e^t + 3te^{-5t} - 5e^{-5t}$
This can be rewritten as $9e^t + (3t-5)e^{-5t}$. So our guess will be

$$Y_p(t) = Ae^t + (Bt + C)e^{-5t} \tag{3.280}$$

(5) $g(t) = t^2\sin(t) + 4\cos(t)$
So our guess will be

$$Y_p(t) = (At^2 + Bt + C)\cos(t) + (Dt^2 + Et + F)\sin(t). \tag{3.281}$$

(6) $g(t) = 3e^{-3t} + e^{-3t}\sin(3t) + \cos(3t)$
Our guess

$$Y_p(t) = Ae^{-3t} + e^{-3t}(B\cos(3t) + C\sin(3t)) + D\cos(3t) + E\sin(3t). \tag{3.282}$$

This seems simple, right? There is one problem which can arise you need to be aware of

■ **Example 3.28** Find a particular solution to

$$y'' - 4y' - 12y = e^{6t} \tag{3.283}$$

This seems straightforward, so try $Y_p(t) = Ae^{6t}$. If we differentiate and plug in

$$36Ae^{6t} - 24Ae^{6t} - 12Ae^{6t} = e^{6t} \tag{3.284}$$
$$0 = e^{6t} \tag{3.285}$$

Exponentials are never zero. So this cannot be possible. Did we make a mistake on our original guess? Yes, if we went through the normal process and found the complimentary solution in this case

$$y_c(t) = c_1e^{6t} + c_2e^{-2t}. \tag{3.286}$$

So our guess for the particular solution was actually part of the complimentary solution. So we need to find a different guess. Think back to repeated root solutions and try $Y_p(t) = Ate^{6t}$. Try it

$$(36Ate^{6t} + 12Ae^{6t}) - 4(6Ate^{6t} + Ae^{6t}) - 12Ate^{6t} = e^{6t} \tag{3.287}$$
$$(36A - 24A - 12A)te^{6t} + (12A - 4A)e^{6t} = e^{6t} \tag{3.288}$$
$$8Ae^{6t} = e^{6t} \tag{3.289}$$

Setting the coefficients equal, we conclude that $A = \frac{1}{8}$, so

$$Y_p(t) = \frac{1}{8}te^{6t}. \tag{3.290}$$

■

NOTE: If this situation arises when the complimentary solution has a repeated root and has the form

$$y_c(t) = c_1 e^{rt} + c_2 t e^{rt} \tag{3.291}$$

then our guess for the particular solution should be

$$Y_p(t) = At^2 e^{rt}. \tag{3.292}$$

■

3.5.6 Method of Undetermined Coefficients

Then we want to construct the general solution $y(t) = y_c(t) + Y_p(t)$ by following these steps:
(1) Find the general solution of the corresponding homogeneous equation.
(2) Make sure $g(t)$ belongs to a special set of basic functions we will define shortly.
(3) If $g(t) = g_1(t) + ... + g_n(t)$ is the sum of n terms, then form n subproblems each of which contains only one $g_i(t)$. Where the ith subproblem is

$$ay'' + by' + cy = g_i(t) \tag{3.293}$$

(4) For the ith subproblem assume a particular solution of the appropriate functions (exponential, sine, cosine, polynomial). If there is a duplication in $Y_i(t)$ with a solution to the homogeneous problem then multiply $Y_i(t)$ by t (or if necessary t^2).
(5) Find the particular solution $Y_i(t)$ for each subproblem. Then the sum of the Y_i is a particular solution for the full nonhomogeneous problem.
(6) Form the general solution by summing all the complimentary solutions from the homogeneous equation and the n particular solutions.
(7) Use the initial conditions to determine the values of the arbitrary constants remaining in the general solution.

Now for more examples, write down the guess for the particular solution:
(1) $y'' - 3y' - 28y = 6t + e^{-4t} - 2$
First we find the complimentary solution using the characteristic equation

$$y_c(t) = c_1 e^{7t} + c_2 e^{-4t} \tag{3.294}$$

Now look at the nonhomogeneous term which is a polynomial and exponential, $6t - 2 + e^{-4t}$. So our initial guess should be

$$At + B + Ce^{-4t} \tag{3.295}$$

The first two terms are fine, but the last term is in the complimentary solution. Since Cte^{-4t} does not show up in the complimentary solution our guess should be

$$Y_p(t) = At + B + Cte^{-4t}. \tag{3.296}$$

(2) $y'' - 64y = t^2 e^{8t} + \cos(t)$

The complimentary solution is

$$y_c(t) = c_1 e^{8t} + c_2 e^{-8t}. \tag{3.297}$$

Our initial guess for a particular solution is

$$(At^2 + Bt + C)e^{8t} + D\cos(t) + E\sin(t) \tag{3.298}$$

Again we have a Ce^{8t} term which is also in the complimentary solution. So we need to multiply the entire first term by t, so our final guess is

$$Y_p(t) = (At^3 + Bt^2 + Ct)e^{8t} + D\cos(t) + E\sin(t). \tag{3.299}$$

(3) $y'' + 4y' = e^{-t}\cos(2t) + t\sin(2t)$
The complimentary solution is

$$y_c(t) = c_1\cos(2t) + c_2\sin(2t) \tag{3.300}$$

Our first guess for a particular solution would be

$$e^{-t}(A\cos(2t) + B\sin(2t)) + (Ct + D)\cos(2t) + (Et + F)\sin(2t) \tag{3.301}$$

We notice the second and third terms contain parts of the complimentary solution so we need to multiply by t, so we have a our final guess

$$Y_p(t) = e^{-t}(A\cos(2t) + B\sin(2t)) + (Ct^2 + Dt)\cos(2t) + (Et^2 + Ft)\sin(2t). \tag{3.302}$$

(4) $y'' + 2y' + 5 = e^{-t}\cos(2t) + t\sin(2t)$
Notice the nonhomogeneous term in this example is the same as in the previous one, but the equation has changed. Now the complimentary solution is

$$y_c(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) \tag{3.303}$$

So our initial guess for the particular solution is the same as the last example

$$e^{-t}(A\cos(2t) + B\sin(2t)) + (Ct + D)\cos(2t) + (Et + F)\sin(2t) \tag{3.304}$$

This time the first term causes the problem, so multiply the first term by t to get the final guess

$$Y_p(t) = te^{-t}(A\cos(2t) + B\sin(2t)) + (Ct + D)\cos(2t) + (Et + F)\sin(2t) \tag{3.305}$$

So even though the nonhomogeneous parts are the same the guess also depends critically on the complimentary solution and the differential equation itself.
(5) $y'' + 4y' + 4y = t^2 e^{-2t} + 2e^{-2t}$
The complimentary solution is

$$y_c(t) = c_1 e^{-2t} + c_2 t e^{-2t} \tag{3.306}$$

Notice that we can factor out a e^{-2t} from out nonhomogeneous term, which becomes $(t^2 + 2)e^{-2t}$. This is the product of a polynomial and an exponential, so our initial guess is

$$(At^2 + Bt + C)e^{-2t} \tag{3.307}$$

But the Ce^{-2t} term is in $y_c(t)$. Also, Cte^{-2t} is in $y_c(t)$. So we must multiply by t^2 to get our final guess

$$Y_p(t) = (At^4 + Bt^3 + C)e^{-2t}. \tag{3.308}$$

3.6 Mechanical and Electrical Vibrations

Last Time: We studied the method of undetermined coefficients thoroughly, focusing mostly on determining guesses for particular solutions once we have solved for the complimentary solution.

3.6.1 Applications

The first application is mechanical vibrations. Consider an object of a given mass m hanging from a spring of natural length l, but there are a number of applications in engineering with the same general setup as this.

We will establish the convention that **always** the downward displacement and forces are **positive**, while upward displacements and forces are negative. BE CONSISTENT. We also measure all displacements from the equilibrium position. Thus if our displacement is $u(y)$, $u = 0$ corresponds to the center of gravity as it hangs at rest from a spring.

We need to develop a differential equation to model the displacement u of the object. Recall Newton's Second Law

$$F = ma \tag{3.309}$$

where m is the mass of the object. We want our equation to be for displacement, so we'll replace a by u'', and Newton's Second Law becomes

$$F(t,u,u') = mu''. \tag{3.310}$$

What are the various forces acting on the object? We will consider four different forces, some of which may or may not be present in a given situation.

(1) **Gravity**, F_g

The gravitational force always acts on an object. It is given by

$$F_g = mg \tag{3.311}$$

where g is the acceleration due to gravity. For simpler computations, you may take $g = 10$ m/s. Notice gravity is always positive since it acts downward.

(2) **Spring**, F_s

We attach an object to a spring, and the spring will exert a force on the object. Hooke's Law governs this force. The spring force is proportional to the displacement of the spring from its natural length. What is the displacement of the spring? When we attach an object to a spring, the spring gets stretched. The length of the stretched spring is L. Then the displacement from its natural length is $L+u$.

So the spring force is

$$F_s = -k(L+u) \tag{3.312}$$

where $k > 0$ is the **spring constant**. Why is it negative? It is to make sure the force is in the correct direction. If $u > -L$, i.e. the spring has been stretched beyond its natural length, then $u+L > 0$ and so $F_s < 0$, which is what we expect because the spring would pull upward on the object in this situation. If $u < -L$, so the spring is compressed, then the spring force would push the object back downwards and we expect to find $F_s > 0$.

(3) **Damping**, F_d

We will consider some situations where the system experiences damping. This will not always be present, but always notice if damping is involved. Dampers work to counteract motion (example: shocks on a car), so this will oppose the direction of the object's velocity.

In other words, if the object has downward velocity $u' > 0$, we would want the damping force to be acting in the upwards direction, so that $F_d < 0$. Similarly, if $u' < 0$, we want $F_d > 0$. Assume all damping is linear.

$$F_d = -\gamma u' \tag{3.313}$$

where $\gamma > 0$ is the **damping constant**.

(4) **External Force**, $F(t)$

This is encompasses all other forces present in a problem. An example is a spring hooked up to a piston that exerts an extra force upon it. We call $F(t)$ the **forcing function**, and it is just the sum of any of the external forces we have in a particular problem.

The most important part of any problem is identifying all the forces involved in the problem. Some may not be present. The forces will change depending on the particular situation. Let's consider the general form of our differential equation modeling a spring system. We have

$$F(t,u,u') = F_g + F_s + F_d + F(t) \tag{3.314}$$

so that Newton's Second Law becomes

$$mu'' = mg - k(L+u) - \gamma u' + F(t), \tag{3.315}$$

or upon reordering it becomes

$$mu'' + \gamma u' + ku = mg - kL + F(t). \tag{3.316}$$

What happens when the object is at rest. Equilibrium is $u = 0$, there are only two forces acting on the object: gravity and the spring force. Since the object is at rest, these two forces must balance to 0. So $F_g + F_s = 0$. In other words,

$$mg = kL. \tag{3.317}$$

So our equation simplifies to

$$mu'' + \gamma u' + ku = F(t), \tag{3.318}$$

and this is the most general form of our equation, with all forces present. We have the corresponding initial conditions

$$u(0) = u_0 \quad \text{Initial displacement from equilibrium position} \tag{3.319}$$
$$u'(0) = u'_0 \quad \text{Initial Velocity} \tag{3.320}$$

Before we discuss individual examples, we need to touch on how we might figure out the constants k and γ if they are not explicitly given. Consider the spring constant k. We know if the spring is attached to some object with mass m, the object stretches the spring by some length L when it is at rest. We know at equilibrium $mg = kL$. Thus, if we know how much some object with a known mass stretches the spring when it is at rest, we can compute

$$k = \frac{mg}{L}. \tag{3.321}$$

How do we compute γ? If we do not know the damping coefficient from the beginning, we may know how much force a damper exerts to oppose motion of a given speed. Then set $|F_d| = \gamma |u'|$, where $|F_d|$ is the magnitude of the damping force and $|u'|$ is the speed of motion. So we have $\gamma = \frac{F_d}{u'}$. We will see how to compute in examples on damped motion. Let's consider specific spring mass systems.

3.6.2 Free, Undamped Motion

Start with free systems with no damping or external forces. This is the simplest situation since $\gamma = 0$. Our differential equation is

$$mu'' + ku = 0, \tag{3.322}$$

where $m, k > 0$. Solve by considering the characteristic equation

$$mr^2 + k = 0, \tag{3.323}$$

which has roots

$$r_{1,2} = \pm i\sqrt{\frac{k}{m}}. \tag{3.324}$$

We'll write

$$r_{1,2} = \pm i\omega_0, \tag{3.325}$$

where we've substituted

$$\omega_0 = \sqrt{\frac{k}{m}}. \tag{3.326}$$

ω_0 is called the **natural frequency** of the system, for reasons that will be clear shortly.

Since the roots of our characteristic equation are imaginary, the form of our general solution is

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \tag{3.327}$$

This is why we called ω_0 the natural frequency of the system: it is the frequency of motion when the spring-mass system has no interference from dampers or external forces.

Given initial conditions we can solve for c_1 and c_2. This is not the ideal form of the solution though since it is not easy to read off critical information. After we solve for the constants rewrite as

$$u(t) = R\cos(\omega_0 t - \delta), \tag{3.328}$$

where $R > 0$ is the **amplitude of displacement** and δ is the **phase angle of displacement**, sometimes called the **phase shift**.

Before determining how to rewrite the general solution in this desired form lets compare the two forms. When we keep it as the general solution is it easier to find the constants c_1 and c_2. But the new form is easier to work with since we can immediately see the amplitude making it much easier to graph. So ideally we will find the general solution, solve for c_1 and c_2, and then convert to the final form.

Assume we have c_1 and c_2 how do we find R and δ? Consider Equation (3.328) we can use a trig identity to write it as

$$u(t) = R\cos(\delta)\cos(\omega_0 t) + R\sin(\delta)\sin(\omega_0 t). \tag{3.329}$$

Comparing this to the general solution, we see that

$$c_1 = R\cos(\delta), \quad c_2 = R\sin(\delta). \tag{3.330}$$

Notice

$$c_1^2 + c_2^2 = R^2(\cos^2(\delta) + \sin^2(\delta)) = R^2, \tag{3.331}$$

so that, assuming $R > 0$,

$$R = \sqrt{c_1^2 + c_2^2}. \tag{3.332}$$

Also,

$$\frac{c_2}{c_1} = \frac{\sin(\delta)}{\cos(\delta)} = \tan(\delta). \tag{3.333}$$

to find δ.

■ **Example 3.29** A 2kg object is attached to a spring, which it stretches by $\frac{5}{8}m$. The object is given an initial displacement of $1m$ upwards and given an initial downwards velocity of $4m/sec$. Assuming there are no other forces acting on the spring-mass system, find the displacement of the object at time t and express it as a single cosine.

The first step is to write down the initial value problem for this setup. We'll need to find an m and k. m is easy since we know the mass of the object is $2kg$. How about k? We know

$$k = \frac{mg}{L} = \frac{(2)(10)}{\frac{5}{8}} = 32. \tag{3.334}$$

So our differential equation is

$$2u'' + 32u = 0. \tag{3.335}$$

The initial conditions are given by

$$u(0) = -1, \quad u'(0) = 4. \tag{3.336}$$

The characteristic equation is

$$2r^2 + 32 = 0, \tag{3.337}$$

and this has roots $r_{1,2} = \pm 4i$. Hence $\omega_0 = 4$. Check: $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{32/2} = 4$. So our general solution is

$$u(t) = c_1 \cos(4t) + c_2 \sin(4t). \tag{3.338}$$

Using our initial conditions, we see

$$\begin{aligned} -1 &= u(0) = c_1 && (3.339)\\ 4 &= u'(0) = 4c_2 \Rightarrow c_2 = 1. && (3.340) \end{aligned}$$

So the solution is

$$u(t) = -\cos(4t) + \sin(4t). \tag{3.341}$$

We want to write this as a single cosine. Compute R

$$R = \sqrt{c_1^2 + c_2^2} = \sqrt{2}. \tag{3.342}$$

Now consider δ

$$\tan(\delta) = \frac{c_2}{c_1} = -1. \tag{3.343}$$

So δ is in Quadrants II or IV. To decide which look at the values of $\cos(\delta)$ and $\sin(\delta)$. We have

$$\begin{aligned} \sin(\delta) &= c_2 > 0 && (3.344)\\ \cos(\delta) &= c_1 < 0. && (3.345) \end{aligned}$$

So δ must be in Quadrant II, since there $\sin > 0$ and $\cos < 0$. If we take $\arctan(-1) = -\frac{\pi}{4}$, this has a value in Quadrant IV. Since tan is π-periodic, however, $-\frac{\pi}{4} + \pi = \frac{3\pi}{4}$ is in Quadrant II and also has a tangent of -1 Thus our desired phase angle is

$$\delta = \arctan(\frac{c_2}{c_1}) + \pi = \arctan(-1) + \pi = \frac{3\pi}{4} \tag{3.346}$$

and our solution has the final form

$$u(t) = \sqrt{2}\cos(4t - \frac{3\pi}{4}). \tag{3.347}$$

■

3.6.3 Free, Damped Motion

Now, let's consider what happens if we add a damper into the system with damping coefficient γ. We still consider free motion so $F(t) = 0$, and our differential equation becomes

$$mu'' + \gamma u' + ku = 0. \tag{3.348}$$

The characteristic equation is

$$mr^2 + \gamma r + k = 0, \tag{3.349}$$

and has solution

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}. \tag{3.350}$$

There are three different cases we need to consider, corresponding to the discriminant being positive, zero, or negative.

(1) $\gamma^2 - 4mk = 0$

This case gives a double root of $r = -\frac{\gamma}{2m}$, and so the general solution to our equation is

$$u(t) = c_1 e^{\frac{\gamma}{2m}} + c_2 t e^{-\frac{\gamma}{2m}} \tag{3.351}$$

Notice that $\lim_{t\to\infty} u(t) = 0$, which is good, since this signifies damping. This is called **critical damping** and occurs when

$$\gamma^2 - 4mk = 0 \tag{3.352}$$
$$\gamma = \sqrt{4mk} = 2\sqrt{mk} \tag{3.353}$$

This value of $\gamma - 2\sqrt{mk}$ is denoted by γ_{CR} and is called the **critical damping coefficient**. Since this case separates the other two it is generally useful to be able to calculate this coefficient for a given spring-mass system, which we can do using this formula. Critically damped systems may cross $u = 0$ once but will never cross more than that. No oscillation

(2) $\gamma^2 - 4mk > 0$

In this case, the discriminant is positive and so we will get two distinct real roots r_1 and r_2. Hence our general solution is

$$u(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \tag{3.354}$$

But what is the behavior of this solution? The solution should die out since we have damping. We need to check $\lim_{t\to\infty} u(t) = 0$. Rewrite the roots

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} \tag{3.355}$$
$$= \frac{-\gamma \pm \gamma(\sqrt{1 - \frac{4mk}{\gamma^2}})}{2m} \tag{3.356}$$
$$= -\frac{\gamma}{2m}(1 \pm \sqrt{1 - \frac{4mk}{\gamma^2}}) \tag{3.357}$$

By assumption, we have $\gamma^2 > 4mk$. Hence

$$1 - \frac{4mk}{\gamma^2} < 1 \tag{3.358}$$

and so

$$\sqrt{1 - \frac{4mk}{\gamma^2}} < 1. \tag{3.359}$$

so the quantity in parenthesis above is guaranteed to be positive, which means both of our roots are negative.

Thus the damping in this case has the desired effect, and the vibration will die out in the limit. This case, which occurs when $\gamma > \gamma_{CR}$, is called **overdamping**. The solution won't oscillate around equilibrium, but settles back into place. The overdamping kills all oscillation

(3) $\gamma^2 < 4mk$

The final case is when $\gamma < \gamma_{CR}$. In this case, the characteristic equation has complex roots

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = \alpha + i\beta. \tag{3.360}$$

The displacement is

$$\begin{aligned} u(t) &= c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t) && (3.361)\\ &= e^{\alpha t}(c_1\cos(\beta t) + c_2\sin(\beta t)). && (3.362) \end{aligned}$$

In analogy to the free undamped case we can rewrite as

$$u(t) = Re^{\alpha t}\cos(\beta t - \delta). \tag{3.363}$$

We know $\alpha < 0$. Hence the displacement will settle back to equilibrium. The difference is that solutions will oscillate even as the oscillations have smaller and smaller amplitude. This is called **overdamped**.

Notice that the solution $u(t)$ is not quite periodic. It has the form of a cosine, but the amplitude is not constant. A function $u(t)$ is called **quasi-periodic**, since it oscillates with a constant frequency but a varying amplitude. β is called the **quasi-frequency** of the oscillation.

So when we have free, damped vibrations we have one of these three cases. A good example to keep in mind when considering damping is car shocks. If the shocks are new its overdamping, when you hit a bump in the road the car settles back into place. As the shocks wear there is more of an initial bump but the car still settles does not bounce around. Eventually when your shocks where and you hit a bump, the car bounces up and down for a few minutes and then settles like underdamping. The critical point where the car goes from overdamped to underdamped is the critically damped case.

Another example is a washing machine. A new washing machine does not vibrate significantly due to the presence of good dampers. Old washing machines vibrate a lot.

In practice we want to avoid underdamping. We do not want cars to bounce around on the road or buildings to sway in the wind. With critical damping we have the right behavior, but its too hard to achieve this. If the dampers wear a little we are then underdamped. In practice we want to stay overdamped.

■ **Example 3.30** A $2kg$ object stretches a spring by $\frac{5}{8}m$. A damper is attached that exerts a resistive force of $48N$ when the speed is $3m/sec$. If the initial displacement is $1m$ upwards and the initial velocity is $2m/sec$ downwards, find the displacement $u(t)$ at any time t.

This is actually the example from the last class with damping added and different initial conditions. We already know $k = 32$. What is the damping coefficient γ? We know $|F_d| = 48$ when the speed is $|u'| = 3$. So the damping coefficients is given by

$$\gamma = \frac{|F_d|}{|u'|} = \frac{48}{3} = 16. \tag{3.364}$$

Thus the initial value problem is

$$2u'' + 16u' + 32u = 0, \quad u(0) = -1, \quad u'(0) = 2. \tag{3.365}$$

Before we solve it, see which case we're in. To do so, let's calculate the critical damping coefficient.

$$\gamma_{CR} = 2\sqrt{mk} = 2\sqrt{64} = 16. \tag{3.366}$$

So we are critically damped, since $\gamma = \gamma_{CR}$. This means we will get a double root. Solving the characteristic equation we get $r_1 = r_2 = -4$ and the general solution is

$$u(t) = c_1 e^{-4t} + c_2 t e^{-4t}. \tag{3.367}$$

The initial conditions give coefficients $c_1 = -1$ and $c_2 = -2$. So the solution is

$$u(t) = -e^{-4t} - 2te^{-4t} \tag{3.368}$$

Notice there is no oscillations in this case. ■

■ **Example 3.31** For the same spring-mass system as in the previous example, attach a damper that exerts a force of $40N$ when the speed is $2m/s$. Find the displacement at any time t.

the only difference from the previous example is the damping force. Lets compute γ

$$\gamma = \frac{|F_d|}{|u'|} = \frac{40}{2} = 20. \tag{3.369}$$

Since we computed $\gamma_{CR} = 16$, this means we are overdamped and the characteristic equation should give us distinct real roots. The IVP is

$$2u'' + 20u' + 32u = 0, \quad u(0) = -1, \quad u(0) = 2. \tag{3.370}$$

The characteristic equation has roots $r_1 = -8$ and $r_2 = -2$. So the general solution is

$$u(t) = c_1 e^{-8t} + c_2 e^{-2t} \tag{3.371}$$

The initial conditions give $c_1 = 0$ and $c_2 = -1$, so the displacement is

$$u(t) = -e^{-2t} \tag{3.372}$$

Notice here we do not actually have a "vibration" as we normally think of them. The damper is strong enough to force the vibrations to die out so quickly that we do not notice much if any of them. ■

■ **Example 3.32** For the same spring-mass system as in the previous two examples, add a damper that exerts a force of $16N$ when the speed is $2m/s$.

In this case, the damping coefficient is

$$\gamma = \frac{16}{2} = 8, \tag{3.373}$$

which tells us that this case is underdamped as $\gamma < \gamma_{CR} = 16$. We should expect complex roots of the characteristic equation. The IVP is

$$2u'' + 8u' + 32u = 0, \quad u(0) = -1, \quad u'(0) = 3. \tag{3.374}$$

The characteristic equation has roots

$$r_{1,2} = \frac{-8 \pm \sqrt{192}}{4} = -2 \pm i\sqrt{12}. \tag{3.375}$$

Thus our general solution is

$$u(t) = c_1 e^{-2t}\cos(\sqrt{12}t) + c_2 e^{2t}\sin(\sqrt{12}t) \tag{3.376}$$

The initial conditions give the constants $c_1 = 1$ and $c_2 = \frac{1}{\sqrt{12}}$, so we have

$$u(t) = -e^{-2t}\cos(\sqrt{12}t) + \frac{1}{\sqrt{12}}e^{2t}\sin(\sqrt{12}t). \tag{3.377}$$

Let's write this as a single cosine

$$R = \sqrt{(-1)^2 + (\frac{1}{\sqrt{12}})^2} = \sqrt{\frac{13}{12}} \tag{3.378}$$

$$\tan(\delta) = -\frac{1}{\sqrt{12}} \tag{3.379}$$

As in the undamped case, we look at the signs of c_1 and c_2 to figure out what quadrant δ is in. By doing so, we see that δ has negative cosine and positive sine, so it is in Quadrant II. Hence we need to take the arctangent and add π to it

$$\delta = \arctan(-\frac{1}{\sqrt{12}}) + \pi. \tag{3.380}$$

Thus our displacement is

$$u(t) = \sqrt{\frac{13}{12}}e^{-2t}\cos(\sqrt{12}t - \arctan(-\frac{1}{\sqrt{12}} - \pi). \tag{3.381}$$

In this case, we actually get a vibration, even though its amplitude steadily decreases until it is negligible. The vibration has quasi-frequency $\sqrt{12}$. ■

3.7 Forced Vibrations

Last Time: We studied non-forced vibrations with and without damping. We studied the four forces acting on an object gravity, spring force, damping, and external forces.

3.7.1 Forced, Undamped Motion

What happens when the external force $F(t)$ is allowed to act on our system. The function $F(t)$ is called the **forcing function**. We will consider the undamped case

$$mu'' + ku = F(t). \tag{3.382}$$

This is a nonhomogeneous equation, so we will need to find both the complimentary and particular solution.

$$u(t) = u_c(t) + U_p(t), \tag{3.383}$$

Recall that $u_c(t)$ is the solution to the associated homogeneous equation. We will use undetermined coefficients to find the particular solution $U_p(t)$ (if $F(t)$ has an appropriate form) or variation of parameters.

We restrict our attention to the case which appears frequently in applications

$$F(t) = F_0 \cos(\omega t) \quad \text{or} \quad F(t) = F_0 \sin(\omega t) \tag{3.384}$$

The force we are applying to our spring-mass system is a simple periodic function with frequency ω. For now we assume $F(t) = F_0 \cos(\omega t)$, but everything is analogous if it is a sine function. So consider

$$mu'' + ku' = F_0 \cos(\omega t). \tag{3.385}$$

Where the complimentary solution to the analogous free undamped equation is

$$u_c(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t), \tag{3.386}$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural frequency.

We can use the method of undetermined coefficients for this nonhomogeneous term $F(t)$. The initial guess for the particular solution is

$$U_p(t) = A\cos(\omega t) + B\sin(\omega t). \tag{3.387}$$

We need to be careful, note that we are okay since $\omega_0 \neq \omega$, but if the frequency of the forcing function is the same as the natural frequency, then this guess is the complimentary solution $u_c(t)$. Thus, if $\omega_0 = \omega$, we need to multiply by a factor of t. So there are two cases.

(1) $\omega \neq \omega_0$

In this case, our initial guess is not the complimentary solution, so the particular solution will be

$$U_p(t) = A\cos(\omega t) + B\sin(\omega t). \tag{3.388}$$

Differentiating and plugging in we get

$$m\omega^2(-A\cos(\omega t) - B\sin(\omega t)) + k(A\cos(\omega t) + B\sin(\omega t)) = F_0\cos(\omega t) \quad (3.389)$$
$$(-m\omega^2 A + kA)\cos(\omega t) + (-m\omega^2 B + kB)\sin(\omega t) = F_0\cos(\omega t). \quad (3.390)$$

Setting the coefficients equal, we get

$$\cos(\omega t): \quad (-m\omega^2 + k)A = F_0 \quad \Rightarrow \quad A = \frac{F_0}{k - m\omega^2} \quad (3.391)$$
$$\sin(\omega t): \quad (-m\omega^2 + k)B = 0 \quad \Rightarrow \quad B = 0. \quad (3.392)$$

So our particular solution is

$$U_p(t) = \frac{F_0}{k - m\omega^2}\cos(\omega t) \quad (3.393)$$
$$= \frac{F_0}{m(\frac{k}{m} - \omega^2)}\cos(\omega t) \quad (3.394)$$
$$= \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t). \quad (3.395)$$

Notice that the amplitude of the particular solution is dependent on the amplitude of the forcing function F_0 and the difference between the natural frequency and the forcing frequency.

We can write our displacement function in two forms, depending on which form we use for complimentary solution.

$$u(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t) \quad (3.396)$$
$$u(t) = R\cos(\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t) \quad (3.397)$$

Again, we get an analagous solution if the forcing function were $F(t) = F_0\sin(\omega t)$.

The key feature of this case can be seen in the second form. We have two cosine functions with different frequencies. These will interfere with each other causing the net oscillation to vary between great and small amplitude. This phenomena has a name "beats" derived from musical terminology. Thing of hitting a tuning fork after it has already been struck, the volume will increase and decrease randomly. One hears the waves created here in the exact form of our solution.

(2) $\omega = \omega_0$

If the frequency of the forcing function is the same as the natural frequency, so the guess for the particular solution is

$$U_p(t) = At\cos(\omega_0 t) + Bt\sin(\omega_0 t) \quad (3.398)$$

Differentiate and plug in

$$(-m\omega_0^2 + k)At\cos(\omega_0 t) + (-m\omega_0^2 + k)Bt\sin(\omega_0 t) + 2m\omega_0 B\cos(\omega_0 t) - 2m\omega_0 A\sin(\omega_0 t) = F_0\cos(\omega_0 t). \quad (3.399)$$

To begin simplification recall that $\omega_0^2 = \frac{k}{m}$, so $m\omega_0^2 = k$. this means the first two terms will vanish (expected since no analogous terms on right side), and we get

$$2m\omega_0 B\cos(\omega_0 t) - 2m\omega_0 A\sin(\omega_0 t) = F_0\cos(\omega_0 t). \tag{3.400}$$

Now set the coefficients equal

$$\cos(\omega_0 t): \quad 2m\omega_0 B = F_0 \quad B = \frac{F_0}{2m\omega_0} \tag{3.401}$$

$$\sin(\omega_0 t): \quad -2m\omega_0 A = 0 \quad A = 0 \tag{3.402}$$

Thus the particular solution is

$$U_p(t) = \frac{F_0}{2m\omega_0} t\sin(\omega_0 t) \tag{3.403}$$

and the displacement is

$$u(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{2m\omega_0} t\sin(\omega_0 t) \tag{3.404}$$

or

$$u(t) = R\cos(\omega_0 t - \delta) + \frac{F_0}{2m\omega_0} t\sin(\omega_0 t). \tag{3.405}$$

What stands out most about this equation? Notice that as $t \to \infty$, $u(t) \to \infty$ due to the form of the particular solution. Thus, in the case where the forcing frequency is the same as the natural frequency, the oscillation will have an amplitude that continues to increase for all time since the external force adds energy to the system in a way that reinforces the natural motion of the system.

This phenomenon is called **resonance**. Resonance is the phenomenon behind microwave ovens. The microwave radiation strikes the water molecules in what's being heated at their natural frequency, causing them to vibrate faster and faster, which generates heat. A similar trait is noticed in the Bay of Fundy, where tidal forces cause the ocean to resonate, yielding larger and larger tides. Resonance in the ear causes us to be able to distinguish between tones in sound.

A common example is the Tacoma Narrows Bridge. This is incorrect because the oscillation that led to the collapse of the bridge was from a far more complicated phenomenon than the simple resonance we're considering now. In general, for engineering purposes, resonance is something we would like to avoid unless we understand the situation and the effect on the system.

In summary when we drive our system at a different frequency than the natural frequency, the two frequencies interfere and we observe beats in motion. When the system is driven at a natural frequency, the natural motion of the system is reinforced, causing the amplitude of the motion to increase to infinity.

■ **Example 3.33** A $3kg$ object is attached to a spring, which it stretches by $40cm$. There is no damping, but the system is forced with the forcing function

$$F(t) = 10\cos(\omega t) \tag{3.406}$$

such that the system will experience resonance. If the object is initially displaced $20cm$ downward and given an initial upward velocity of $10cm/s$, find the displacement at any time t.

We need to be aware of the units, convert all lengths to meters. Find k

$$k = \frac{mg}{L} = \frac{(3)(10)}{.4} = 75 \tag{3.407}$$

Next, we are told the system experiences resonance. Thus the forcing frequency ω must be the natural frequency ω_0.

$$\omega = \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{75}{3}} = 5 \tag{3.408}$$

Thus our initial value problem is

$$3u'' + 75u = 10\cos(5t) \quad u(0) = .2, \quad u'(0) = -.1 \tag{3.409}$$

The complimentary solution is the general solution of the associated free, undamped case. Since we have computed the natural frequency already, the complimentary solution is just

$$u_c(t) = c_1 \cos(5t) + c_2 \sin(5t). \tag{3.410}$$

The particular solution (using formula derived above) is

$$\frac{1}{3} t \sin(5t), \tag{3.411}$$

and so the general solution is

$$u(t) = c_1 \cos(5t) + c_2 \sin(5t) + \frac{1}{3} t \sin(5t). \tag{3.412}$$

The initial conditions give $c_1 = \frac{1}{5}$ and $c_2 = -\frac{1}{50}$, so the displacement can be given as

$$u(t) = \frac{1}{5} \cos(5t) - \frac{1}{50} \sin(5t) + \frac{1}{3} t \sin(5t) \tag{3.413}$$

Let's convert the first two terms to a single cosine.

$$R = \sqrt{(\frac{1}{5})^2 + (-\frac{1}{50})^2} = \sqrt{\frac{101}{2500}} \tag{3.414}$$

$$\tan(\delta) = \frac{-\frac{1}{50}}{\frac{1}{5}} = -\frac{1}{10} \tag{3.415}$$

Looking at the signs of c_1 and c_2, we see that $\cos(\delta) > 0$ and $\sin(\delta) < 0$. Thus δ is in Quadrant IV, and so we can just take the arctangent.

$$\delta = \arctan(-\frac{1}{10}) \tag{3.416}$$

The displacement is then

$$u(t) = \sqrt{\frac{101}{2500}} \cos\left(5t - \arctan(-\frac{1}{10})\right) + \frac{1}{3} t \sin(5t) \tag{3.417}$$

■

4. Higher Order Linear Equations

4.1 General Theory for nth Order Linear Equations

Recall that an nth order linear differential equation is an equation of the form

$$P_n(t)\frac{d^n y}{dt^n} + P_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \ldots + P_1(t)\frac{dy}{dt} + P_0(t)y = g(t). \tag{4.1}$$

We will assume $P_i(t)$ and $g(t)$ are continuous on some interval $\alpha < t < \beta$ and $P_n(t)$ is nonzero on this interval. We can divide by $P_n(t)$ and consider the equation as a linear operator

$$L[y] = \frac{d^n y}{dt^n} + p_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \ldots + p_1(t)\frac{dy}{dt} + p_0(t)y = g(t). \tag{4.2}$$

where $p_i(t) = \frac{P_i(t)}{P_n(t)}$. The theory involved with an nth order equation is analogous to the theory for second order linear equations. To obtain a unique solution we need n initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \ldots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)} \tag{4.3}$$

we have the following theorem regarding uniqueness of a solution

Theorem 4.1.1 If $p_0, p_1, \ldots p_{n-1}$ and g are continuous on the open interval I, then there exists exactly one solution $y = \phi(t)$ of the differential equation that also satisfies the initial conditions. This solution exists on all of I. Note that the leading derivative (nth order) must have the coefficient 1.

■ **Example 4.1** Determine the intervals in which a solution is sure to exist

$$(t-2)y''' + \sqrt{t+3}\,y'' + \ln(t)y' + e^t y = \sin(t) \tag{4.4}$$

Ans: First put the equation in standard form

$$y''' + \frac{\sqrt{t+3}}{t-2}y'' + \frac{\ln(t)}{t-2}y' + \frac{e^t}{t-2}y = \frac{\sin(t)}{t-2} \tag{4.5}$$

By Theorem 1 we are looking for the intervals where all the coefficients are continuous. The coefficient for y'' is continuous on $[-3,2) \cup (2,\infty)$, The coefficient for y' is continuous on $(0,2) \cup (2,\infty)$, the coefficient of y is continuous on $(-\infty,2) \cup (2,\infty)$, and $g(t)$ is continuous on $(-\infty,2) \cup (2,\infty)$. So the intervals where a solution exists are

$$(0,2) \cup (2,\infty) \tag{4.6}$$

■

Theorem 4.1.2 If the functions $p_1, p_2, ..., p_n$ are continuous on the open interval I, the functions $y_1, y_2, ..., y_n$ are solutions of the homogeneous differential equation, and if $W(y_1, y_2, ..., y_n)(t) \neq 0$ for at least one point in I, then every solution on the differential equation can be expressed as a linear combination of the solutions $y_1, y_2, ..., y_n$ (Linear Superposition).

Theorem 4.1.3 If $y_1(t), y_2(t), ..., y_n(t)$ form a fundamental set of solutions of the homogeneous differential equation on an interval I, then $y_1(t), ..., y_n(t)$ are linearly independent solutions of the equation on I, then they form a fundamental set of solutions for the inhomogeneous equation on I.

■ **Example 4.2** Determine whether the given set of functions is linearly dependent or linearly independent.

$$t^2 - 1, t - 1, 2t + 1 \tag{4.7}$$

Consider

$$c_1(t^2 - 1) + c_2(t - 1) + c_3(2t + 1) = 0 \tag{4.8}$$

So we need to match coefficients

$$c_1 = 0 \tag{4.9}$$
$$c_2 + 2c_3 = 0 \tag{4.10}$$
$$-c_2 + c_3 = 0 \tag{4.11}$$

Rearrange the third equation $c_2 = c_3$ substitute into the second equation $3c_2 = 0$. Thus $c_1 = c_2 = c_3 = 0$. So the set of functions is linearly independent. ■

4.2 Homogeneous Equations with Constant Coefficients

Last Time: We studied the general theory of nth order linear equations. Now we want to find the general solution of an nth order equation using the methods developed for homogeneous equations with constant coefficients.

Consider the nth order linear homogeneous equation with constant coefficients

$$L[y] = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_n y = 0 \tag{4.12}$$

Just like in the second order case we have the characteristic equation

$$a_n r^n + \ldots + a_1 r + a_n = 0 \tag{4.13}$$

■ **Example 4.3** (Distinct Real Roots) Find the general solution of

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0, \quad y(0) = 1, y'(0) = 0, y''(0) = -2, y'''(0) = -1 \tag{4.14}$$

So the characteristic equation is

$$r^4 + r^3 - 7r^2 - r + 6 = 0. \tag{4.15}$$

Thus the roots are $1, -1, 2, -3$. So the general solution is

$$y = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-3t} \tag{4.16}$$

Using the initial conditions we find $c_1 = \frac{11}{8}, c_2 = \frac{5}{12}, c_3 = -\frac{2}{3}, c_4 = -\frac{1}{8}$. Therefore the solution to the IVP is

$$y = \frac{11}{8} e^t + \frac{5}{12} e^{-t} - \frac{2}{3} e^{2t} - \frac{1}{8} e^{-3t} \tag{4.17}$$

■

■ **Example 4.4** (Complex Roots) Find the general solution of

$$y^{(4)} - y = 0, \quad y(0) = 7/2, y'(0) = -4, y''(0) = 5/2, y'''(0) = -2 \tag{4.18}$$

The characteristic equation is

$$r^4 - 1 = (r^2 + 1)(r^2 - 1) = 0 \tag{4.19}$$

Therefore the roots are $1, -1, i, -i$. Thus the general solution is

$$y = c_1 e^t + c_2 e^{-t} + c_3 \cos(t) + c_4 \sin(t) \tag{4.20}$$

By imposing the initial conditions we get

$$c_1 = 0, c_2 = 3, c_3 = 1/2, c_4 = -1 \tag{4.21}$$

Thus the solution of the IVP is

$$y = 3e^{-t} + \frac{1}{2} \cos(t) - \sin(t) \tag{4.22}$$

■

■ **Example 4.5** (Repeated Roots) Find the general solution of

$$y^{(4)} + 2y'' + y = 0. \tag{4.23}$$

The characteristic equation is

$$r^4 + 2r^2 + 1 = (r^2 + 1)(r^2 + 1) = 0 \tag{4.24}$$

The roots are $i, i, -i, -i$. So the general solution is

$$y = c_1 \cos(t) + c_2 \sin(t) + c_3 t \cos(t) + c_4 t \sin(t) \tag{4.25}$$

■

IV

Solutions of ODE with Transforms

5. Laplace Transforms

5.1 Definition of the Laplace Transform

Last Time: We studied nth order linear differential equations and used the method of characteristics to solve them.

We have spent most of the course solving differential equations directly, but sometimes a transformation of the problem can make it much easier. One such example is the Laplace Transform.

5.1.1 The Definition

Definition 5.1.1 A function f is called **piecewise continuous** on an interval $[a,b]$, if $[a,b]$ can be broken into a finite number of subintervals $[a_n, b_n]$ such that f is continuous on each open subinterval (a_n, b_n) and has a finite limit at each endpoint a_n, b_n.

So a piecewise continuous function has only finitely many jumps and does not have any asymptotes where it blows up to infinity or minus infinity.

Definition 5.1.2 *(Laplace Transformation)* Suppose that $f(t)$ is a piecewise continuous function. The Laplace Transform of $f(t)$, denoted by $\mathscr{L}\{f(t)\}$, is given by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt. \tag{5.1}$$

REMARK: There is an alternate notation for the Laplace Transform that we will commonly use. Notice that the definition of $\mathscr{L}\{f(t)\}$ introduces a new variable, s, in the definite integral with respect to t. As a result, computing the transform yields a function which depends on s. Thus

$$\mathscr{L}\{f(t)\} = F(s) \tag{5.2}$$

It should also be noted that the integral definition of $\mathscr{L}\{f(t)\}$ is an improper integral.

In our first examples of computing Laplace Transforms, we will review how to handle them.

■ **Example 5.1** Compute $\mathscr{L}\{1\}$. Plugging $f(t) = 1$ into the definition we have

$$\mathscr{L}\{1\} = \int_0^\infty e^{-st} dt. \tag{5.3}$$

Recall we need to convert the improper integral into a limit

$$\begin{aligned}
&= \lim_{N\to\infty} \int_0^N e^{-st} dt && (5.4)\\
&= \lim_{N\to\infty} \big[-\frac{1}{s}e^{-st}\big]_0^N && (5.5)\\
&= \lim_{N\to\infty} \big(-\frac{1}{s}e^{-Ns} + \frac{1}{s}\big) && (5.6)
\end{aligned}$$

Note the value of s will affect our answer. If $s < 0$, the exponent of our exponential is positive, so the limit in question will diverge as the exponential goes to infinity. On the other hand, if $s > 0$, the exponential will go to 0 and the limit will converge.

Thus, we restrict our attention to the case where $s > 0$ and conclude that

$$\mathscr{L}\{1\} = \frac{1}{s} \quad \text{for} \quad s > 0 \tag{5.7}$$

■

Notice that we had to put a restriction on the domain of our Laplace Transform. This will always be the case: these integrals will not always converge for any s.

■ **Example 5.2** Compute $\mathscr{L}\{e^{at}\}$ for $a \neq 0$.

By definition

$$\begin{aligned}
\mathscr{L}\{e^{at}\} &= \int_0^\infty e^{-st} e^{at} dt && (5.8)\\
&= \int_0^\infty e^{(a-s)t} dt && (5.9)\\
&= \lim_{N\to\infty} \big[\frac{1}{a-s}e^{(a-s)t}\big]_0^N && (5.10)\\
&= \lim_{N\to\infty} \big(\frac{1}{a-s}e^{(a-s)N} - \frac{1}{a-s}\big) && (5.11)\\
&= \frac{1}{s-a} \quad \text{for} \quad s > a. && (5.12)
\end{aligned}$$

■

■ **Example 5.3** Compute $\mathscr{L}\{\sin(at)\}$.

$$\begin{aligned}
\mathscr{L}\{\sin(at)\} &= \int_0^\infty e^{-st}\sin(at) dt && (5.13)\\
&= \lim_{N\to\infty} \int_0^N e^{-st}\sin(at) dt && (5.14)
\end{aligned}$$

Integration By Parts (twice) yields

$$= \lim_{N\to\infty}\left(\frac{1}{a}(1-e^{-sN}\cos(aN)\right) - \frac{s}{a}\left(\frac{1}{a}e^{-sN}\sin(aN)+\frac{s}{a}\int_0^N e^{-st}\sin(at)dt\right). \quad (5.15)$$

After rewriting, we get

$$F(s) = \frac{1}{a} - \frac{s^2}{a^2}F(s) \quad (5.16)$$

$$\mathscr{L}\{\sin(at)\} = F(s) = \frac{a}{s^2+a^2} \quad \text{provided} \quad s>0. \quad (5.17)$$

■

■ **Example 5.4** If $f(t)$ is a piecewise continuous function with piecewise continuous derivative $f'(t)$, express $\mathscr{L}\{f'(t)\}$ in terms of $\mathscr{L}\{f(t)\}$.

We plug f' into the definition for the Laplace transform

$$\mathscr{L}\{f'\} = \int_0^\infty e^{-st}f'dt \quad (5.18)$$

$$= \lim_{N\to\infty}\int_0^N e^{-st}f'dt \quad (5.19)$$

The next step is to integrate by parts

$$= \lim_{N\to\infty}\left(e^{-st}f|_0^N + s\int_0^N e^{-st}fdt\right) \quad (5.20)$$

$$= \lim_{N\to\infty} e^{-sN}f(N) - f(0) + s\int_0^\infty e^{-st}fdt \quad (5.21)$$

$$= s\mathscr{L}\{f(t)\} - f(0) \quad \text{provided} \quad s>0 \quad (5.22)$$

■

Doing this repeatedly one finds

$$\mathscr{L}\{f^{(n)}(t)\} = s^n\mathscr{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0). \quad (5.23)$$

■ **Example 5.5** If $f(t)$ is a piecewise continuous function, express $\mathscr{L}\{e^{at}f(t)\}$ in terms of $\mathscr{L}\{e^{at}f(t)\}$

We begin by plugging into the definition

$$\mathscr{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}dt \quad (5.24)$$

$$= \int_0^\infty e^{(a-s)t}f(t)dt \quad (5.25)$$

This looks like the definition of $F(s)$, but its not the same, since the exponent is $a-s$. However, if we substitute $u = s-a$, we get

$$= \int_0^\infty e^{-ut}f(t)dt \quad (5.26)$$

$$= F(u) \quad (5.27)$$

$$= F(s-a) \quad (5.28)$$

Thus if we take the Laplace transform of a function multiplied by e^{at}, we'll get the Laplace Transform of the original function shifted by a. ■

5.1.2 Laplace Transforms

In general, we won't be using the definition we will be using a table of Laplace Transforms. I would make an effort to know all the common transforms above as well as the definition. From now on we will use a table, but be prepared on an exam to do a basic transform using the definition.

Note the Laplace Transform is linear

Theorem 5.1.1 Given piecewise continuous functions $f(t)$ and $g(t)$,

$$\mathscr{L}\{af(t)+bg(t)\} = a\mathscr{L}\{f(t)\}+b\mathscr{L}\{g(t)\} \tag{5.29}$$

for any constants a, b.

This follows from the linearity of integration. From a practical perspective we will not have to worry about constants or sums. We can decompose our function into individual pieces, transform them, and then put everything back together.

Example 5.6 Find the Laplace Transforms of the following functions
(i) $f(t) = 6e^{-5t} + e^{3t} + 5t^3 - 9$

$$\begin{aligned}
F(s) = \mathscr{L}\{f(t)\} &= 6\mathscr{L}\{e^{-5t}\}+\mathscr{L}\{e^{3t}\}+5\mathscr{L}\{t^3\}-9\mathscr{L}\{1\} && (5.30)\\
&= 6\frac{1}{s-(-5)}+\frac{1}{s-3}+5\frac{3!}{s^{3+1}}-9\frac{1}{s} && (5.31)\\
&= \frac{6}{s+5}+\frac{1}{s-3}+\frac{30}{s^4}-\frac{9}{s} && (5.32)
\end{aligned}$$

(ii) $g(t) = 4\cos(4t) - 2\sin(4t) - 3\cos(8t)$

$$\begin{aligned}
G(s) = \mathscr{L}\{g(t)\} &= 4\mathscr{L}\{\cos(4t)\}-2\mathscr{L}\{\sin(4t)\}-3\mathscr{L}\{\cos(10t)\} && (5.33)\\
&= 4\frac{s}{s^2+4^2}-2\frac{4}{s^2+4^2}-3\frac{s}{s^2+10^2} && (5.34)\\
&= \frac{4s-8}{s^2+16}-\frac{3s}{s^2+100} && (5.35)
\end{aligned}$$

(iii) $h(t) = e^{2t} + \cos(3t) + e^{2t}\cos(3t)$

$$\begin{aligned}
H(t) = \mathscr{L}\{h(t)\} &= \mathscr{L}\{e^{2t}\}+\mathscr{L}\{\cos(3t)\}-\mathscr{L}\{e^{2t}\cos(3t)\} && (5.36)\\
&= \frac{1}{s-2}+\frac{s}{s^2+3^2}-\frac{s-2}{(s-2)^2+3^2} && (5.37)\\
&= \frac{1}{s-2}+\frac{2}{s^2+9}-\frac{s-2}{(s-2)^2+9} && (5.38)
\end{aligned}$$

■

5.1.3 Initial Value Problems

We study Laplace Transforms to solve Initial Value Problems.

Example 5.7 Solve the following initial value problem using Laplace Transforms.

$$y'' - 6y' + 5y = 7t, \quad y(0) = -1, \quad y'(0) = 2. \tag{5.39}$$

The first step is using the Laplace Transform to solve an IVP is to transform both sides of the equation.

$$\begin{aligned}
\mathscr{L}\{y''\} - 6\mathscr{L}\{y'\} + 5\mathscr{L}\{y\} &= 7\mathscr{L}\{t\} && (5.40)\\
s^2Y(s) - sy(0) - y'(0) - 6\big(sY(s) - y(0)\big) + 5Y(s) &= \frac{7}{s^2} && (5.41)\\
s^2Y(s) + s - 2 - 6\big(sY(s) + 1\big) + 5Y(s) &= \frac{7}{s^2} && (5.42)
\end{aligned}$$

Now solve for $Y(s)$.

$$\begin{aligned}
(s^2 - 6s + 5)Y(s) + s - 8 &= \frac{7}{s^2} && (5.43)\\
Y(s) &= \frac{7}{s^2(s^2 - 6s + 5)} + \frac{8 - s}{s^2 - 6s + 5} && (5.44)
\end{aligned}$$

Now we want to solve for $y(t)$, but we have an expression for $Y(s) = \mathscr{L}\{y(t)\}$. Thus to finish solving this problem, we need the inverse Laplace Transform ■

5.1.4 Inverse Laplace Transform

In this section we have $F(s)$ and want to find $f(t)$. $f(t)$ is an inverse Laplace Transform of $F(s)$ with notation

$$f(t) = \mathscr{L}^{-1}\{F(s)\}. \tag{5.45}$$

Our starting point is that the inverse Laplace Transform is linear.

Theorem 5.1.2 Given two Laplace Transforms $F(s)$ and $G(s)$,

$$\mathscr{L}^{-1}\{aF(s) + bG(s)\} = a\mathscr{L}^{-1}\{F(s)\} + b\mathscr{L}^{-1}\{G(s)\} \tag{5.46}$$

for any constants a, b.

So we decompose our original transformed function into pieces, inverse transform, and then put everything back together. Using the table we want to look at the denominator, which will tell us what the original function will have to be, but sometimes we have to look at the numerator to distinguish between two potential inverses (i.e. $\sin(at)$ and $\cos(at)$).

■ **Example 5.8** Find the inverse transforms of the following

(i) $F(s) = \frac{6}{s} - \frac{1}{s-8} + \frac{4}{s-3}$

The denominator of the first term is s indicating that this will be the Laplace Transform of 1. Since $\mathscr{L}\{1\} = \frac{1}{s}$, we will factor out the 6 before taking the inverse transform. For the second term, this is just the Laplace Transform of e^{8t}, and there's nothing else to do with it. The third term is also an exponential, e^{3t}, and we'll need to factor out the 4 in the numerator before we take the inverse transform.

So we have

$$\begin{aligned}
\mathscr{L}^{-1}\{F(s)\} &= 6\mathscr{L}^{-1}\{\frac{1}{s}\} - \mathscr{L}^{-1}\{\frac{1}{s-8}\} + 4\mathscr{L}^{-1}\frac{1}{s-3} && (5.47)\\
f(t) &= 6(1) - e^{8t} + 4(e^{3t}) && (5.48)\\
&= 6 - e^{8t} + 4e^{3t} && (5.49)
\end{aligned}$$

(ii) $G(s) = \frac{12}{s+3} - \frac{1}{2s-4} + \frac{2}{s^4}$

The first term is just the transform of e^{-3t} multiplied by 12, which we will factor out before applying the inverse transform. The second term looks like it should be exponential, but it has a $2s$ instead of an s in the denominator, and transforms of exponentials should just have s. Fix this by factoring out the 2 in the denominator and then taking the inverse transform. The third term has s^4 as its denominator. This indicates that it will be related to the transform of t^3. The numerator is not correct since $\mathscr{L}\{t^3\} = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$. So we would need the numerator to be 6, and right now is 2. How do we fix this? We'll multiply by $\frac{3}{3}$, absorb the top 3 into the transform, with these fixes incorporated we have

$$\begin{aligned} G(s) &= 12\frac{1}{s-(-3)} - \frac{1}{2(s-2)} + \frac{3}{3}\frac{2}{s^4} && (5.50)\\ &= 12\frac{1}{s-(-3)} - \frac{1}{2}\frac{1}{(s-2)} + \frac{1}{3}\frac{6}{s^4} && (5.51)\end{aligned}$$

Now we can take the inverse transform.

$$g(t) = 12e^{-3t} - \frac{1}{2}e^{2t} + \frac{1}{3}t^3 \tag{5.52}$$

(iv) $H(s) = \frac{4s}{s^2+25} + \frac{3}{s^2+16}$

The denominator of the first term is, s^2+25, indicates that this should be the transform of either $\sin(5t)$ or $\cos(5t)$. The numerator is $4s$, which tells us that once we factor out the 4, it will be the transform of $\cos(5t)$. The second term's denominator is s^2+16, so it will be the transform of either $\sin(4t)$ or $\cos(4t)$. The numerator is a constant, 3, so it will be the transform of $\sin(4t)$. The only problem is that the numerator of $\mathscr{L}\{\sin(4t)\}$ should be 4, while here it is 3. We fix this by multiplying by $\frac{4}{4}$. Rewrite the transform

$$\begin{aligned} H(s) &= 4\frac{1}{s^2+5^2} + \frac{4}{4}\frac{3}{s^2+4^2} && (5.53)\\ &= 4\frac{1}{s^2+5^2} + \frac{3}{4}\frac{4}{s^2+4^2} && (5.54)\end{aligned}$$

Then take the inverse

$$h(t) = 4\cos(5t) + \frac{3}{4}\sin(4t) \tag{5.55}$$

■

Let's do some examples which require more work.

■ **Example 5.9** Find the inverse Laplace Transforms for each of the following.

(i) $F(s) = \frac{3s-7}{s^2+16}$

Looking at the denominator, we recognize it will be a sine or cosine, since it has the form s^2+a^2. It is not either because it has both s and a constant in the numerator, while a cosine just has s and the sine just has the constant. This is easy to compensate for, we split the fraction into the difference of two fractions, then fix them up as we did in the previous example, we will be able to then take the Inverse Laplace Transform

$$\begin{aligned} F(s) &= \frac{3s-7}{s^2+16} && (5.56)\\ &= \frac{3s}{s^2+16}-\frac{7}{s^2+16} && (5.57)\\ &= 3\frac{s}{s^2+16}-\frac{4}{4}\frac{7}{s^2+16} && (5.58)\\ &= 3\frac{s}{s^2+16}-\frac{7}{4}\frac{4}{s^2+16} && (5.59)\end{aligned}$$

Now each term is the correct form, and we can take the inverse transform.

$$f(t) = 3\cos(4t) - \frac{7}{4}\sin(4t) \tag{5.60}$$

(ii) $G(s) = \frac{1-3s}{s^2+2s+10}$

If we look at the table of Laplace Transforms, we might see that there are no denominators that look like a quadratic polynomial. Also, this polynomial does not factor nicely into linear terms. There are, however, some terms in the table with denominator of the form $(s-a)^2+b^2$. Those for $e^{at}\cos(bt)$ and $e^{at}\sin(bt)$. Put the denominator in this form by completing the square.

$$\begin{aligned} s^2+2s+10 &= s^2+2s+1-1+10 && (5.61)\\ &= s^2+2s+1+9 && (5.62)\\ &= (s+1)^2+9 && (5.63)\end{aligned}$$

Thus, our transformed function can be written as

$$G(s) = \frac{1-3s}{(s+1)^2+9} \tag{5.64}$$

We will not split this into two pieces yet. First, we need the s in the numerator to be $s+1$ so we can have the numerator of $e^{-t}\cos(3t)$. We do this by adding and subtracting 1 from the s. This will produce some other constant term, which we will combine with the already present constant and try to have the remaining terms look like the numerator for $e^{-t}\sin(3t)$.

$$\begin{aligned} G(s) &= \frac{1-3(s+1-1)}{(s+1)^2+9} && (5.65)\\ &= \frac{1-3(s+1)+3}{(s+1)^2+9} && (5.66)\\ &= \frac{-3(s+1)+4}{(s+1)^2+9} && (5.67)\end{aligned}$$

Now we can break our transform up into two pieces, one of which will correspond to the cosine and the other to the sine. At that point, fixing the numerators is the same as in the last few examples.

$$\begin{aligned} G(s) &= -3\frac{s+1}{(s+1)^2+9}+\frac{4}{3}\frac{3}{(s+1)^2+9} && (5.68)\\ g(s) &= -3e^{-t}\cos(3t)+\frac{4}{3}e^{-t}\sin(3t) && (5.69)\end{aligned}$$

(iii) $H(s) = \frac{s+2}{s^2-s-12}$

This seems identical to the last example, but there is a difference. We can immediately factor the denominator. This requires us to deal with the inverse transform differently. Factoring we see

$$H(s) = \frac{s+2}{(s+3)(s-4)}. \tag{5.70}$$

We know that if we have a linear denominator, that will correspond to an exponential. In this case we have two linear factors. This by itself is not the denominator of an particular Laplace Transform, but we know a method for turning certain rational functions with factored denominators into a sum of more simple radical functions with those factors in each denominator. This method is **Partial Fractions Decomposition**.

We start by writing

$$H(s) = \frac{s+2}{(s+3)(s-4)} = \frac{A}{s+3} + \frac{B}{s-4}. \tag{5.71}$$

Multiply through by $(s+3)(s-4)$:

$$s+2 = A(s-4) = B(s+3). \tag{5.72}$$

This must be true for any value of s. As a result, we have two methods for determining A and B.

Method 1: Match coefficients on functions of s just like in the method of undetermined coefficients

$$s: \quad 1 = A+B \tag{5.73}$$
$$1: \quad 2 = -4A+3B \tag{5.74}$$

Solving the system of two equations in two unknowns we get $A = \frac{1}{7}$ and $B = \frac{6}{7}$.

Method 2: Choose values of s (since it must hold for all s) and solve for A and B.

$$s=-3: \quad -1 = -7A \quad \Rightarrow \quad A = \frac{1}{7} \tag{5.75}$$
$$s=4: \quad 6 = 7B \quad \Rightarrow \quad B = \frac{6}{7} \tag{5.76}$$

Thus, our transform can be written as

$$H(s) = \frac{\frac{1}{7}}{s+3} + \frac{\frac{6}{7}}{s-4} \tag{5.77}$$

and taking the inverse transforms, we get

$$h(t) = \frac{1}{7}e^{-3t} + \frac{6}{7}e^{4t} \tag{5.78}$$

■

Factor in Denominator	Partial Fractions Term
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\ldots+\frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$

TABLE 25.1. Translation from factored denominator to partial fractions.

REMARK: We could have done the the last example by completing the square. However, this would have left us with expressions involving the hyperbolic sine, sinh, and the hyperbolic cosine, cosh. These are interesting functions which can be written in terms of exponentials, but it will be much easier for us to work directly with the exponentials. So we are better off just doing partial fractions even though it's slightly more work.

Partial Fractions and completing the square are a part of life when it comes to Laplace Transforms. Being good at this technique helps when solving IVPs, since most answers have sines, cosines, and exponentials.

Here is a quick review of partial fractions. The first step is to factor the denominator as much as possible. Then using the table above, we can find each of the terms for our partial fractions decomposition. This table is not exhaustive, but we will only have these factors in most if not all cases.

■ **Example 5.10** Find the inverse transform of each of the following.

(i) $F(s)=\frac{2s+1}{(s-2)(s+3)(s-1)}$

The form of the decomposition will be

$$G(s)=\frac{A}{s-2}+\frac{B}{s+3}+\frac{C}{s-1} \tag{5.79}$$

since all the factors in our denominator are linear. Putting the right hand side over a common denominator and setting numerators equal, we have

$$2s+1=A(s+3)(s-1)+B(s-2)(s-1)+C(s-2)(s+3) \tag{5.80}$$

We can again use one of the two methods in Partial Fractions, where we choose key values of s that will isolate the coefficients.

$$s=2: \quad 5 \;=\; A(5)(1) \quad \Rightarrow \quad A=1 \tag{5.81}$$

$$s=-3: \quad -5 \;=\; B(-5)(-4) \quad \Rightarrow \quad B=-\frac{1}{4} \tag{5.82}$$

$$s=1: \quad 3 \;=\; C(-1)(4) \quad \Rightarrow \quad C=-\frac{3}{4} \tag{5.83}$$

Thus, the partial fraction decomposition for this transform is

$$F(s)=\frac{1}{s-2}-\frac{\frac{1}{4}}{s+3}-\frac{\frac{3}{4}}{s-1}. \tag{5.84}$$

The inverse transform is

$$f(t)=e^{2t}-\frac{1}{4}e^{-3t}-\frac{3}{4}e^{t} \tag{5.85}$$

(ii) $G(s) = \frac{2-3s}{(s-2)(s^2+3)}$

Now we have a quadratic in the denominator. Looking at the table we see the form of the partial fractions decomposition will be

$$G(s) = \frac{A}{s-2} + \frac{Bs+C}{s^2+3}. \tag{5.86}$$

If we multiply through by $(s-2)(s^2+3)$, we get

$$2-3s = A(s^2+3) + (Bs+C)(s-2) \tag{5.87}$$

Notice that we cannot use the second method from the previous example, since there are 2 key values of s, but we have 3 constants. Thus we must compare coefficients

$$\begin{aligned} 2-3s &= A(s^2+3) + (Bs+C)(s-2) && (5.88)\\ &= As^2 + 3A + Bs^2 - 2Bs + Cs - 2C && (5.89)\\ &= (A+B)s^2 + (-2B+C)s + (3A-2C) && (5.90) \end{aligned}$$

We have the following system of equations to solve.

$$\begin{aligned} s^2: \quad A+B &= 0 && (5.91)\\ s: \quad -2B+C &= -3 \quad \Rightarrow \quad A = -\frac{8}{7} \;\; B = \frac{8}{7} \;\; C = -\frac{5}{7} && (5.92)\\ s^0 = 1: \quad 3A-2C &= 2 && (5.93) \end{aligned}$$

Thus the partial fraction decomposition is

$$\begin{aligned} G(s) &= -\frac{\frac{8}{7}}{s-2} + \frac{\frac{8}{7}s}{s^2+3} - \frac{\frac{5}{7}}{s^2+3} && (5.94)\\ &= -\frac{8}{7}\frac{1}{s-2} + \frac{8}{7}\frac{2}{s^2+3} - \frac{5}{7\sqrt{3}}\frac{\sqrt{3}}{s^2+3} && (5.95) \end{aligned}$$

and the inverse transform is

$$g(t) = -\frac{8}{7}e^{2t} + \frac{8}{7}\cos(\sqrt{3}t) - \frac{5}{7\sqrt{3}}\sin(\sqrt{3}t). \tag{5.96}$$

(iii) $H(s) = \frac{2}{s^3(s-1)}$

The partial fraction decomposition in this case is

$$H(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s-1}. \tag{5.97}$$

Multiplying through by the denominator of H gives

$$\begin{aligned} 2 &= As^2(s-1) + Bs(s-1) + C(s-1) + Ds^3 && (5.98)\\ &= As^3 - As^2 + Bs^2 - Bs + Cs - C + Ds^3 && (5.99) \end{aligned}$$

and we have to solve the system of equations

$$\begin{aligned} s^3: \quad A+D &= 0 && (5.100)\\ s^2: \quad -A+B &= 0 && (5.101)\\ s: \quad -B+C &= 0 \quad \Rightarrow \quad A=-2 \;\; B=-2 \;\; C=-2 \;\; D=2 && (5.102)\\ 1: \quad -C = 2 & && (5.103) \end{aligned}$$

Thus our partial fractions decomposition becomes

$$H(s) = -\frac{2}{s} - \frac{2}{s^2} - \frac{2}{s^3} + \frac{2}{s-1} \tag{5.104}$$
$$= -\frac{2}{s} - \frac{2}{s^2} - \frac{2}{2!}\frac{2!}{s^3} + \frac{2}{s-1} \tag{5.105}$$

and the inverse transform is

$$h(t) = -2 - 2t - t^2 + 2e^t \tag{5.106}$$

■

5.2 Laplace Transform for Derivatives

Last Time: We thoroughly studied Laplace and Inverse Laplace Transforms

Recall the following formula from a previous lecture

$$\mathscr{L}\{f^{(n)}\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0). \tag{5.107}$$

We will be dealing exclusively with second order differential equations, so we will need to remember

$$\mathscr{L}\{y'\} = sY(s) - y(0) \tag{5.108}$$
$$\mathscr{L}\{y''\} = s^2 Y(s) - sy(0) - y'(0) \tag{5.109}$$

You should be familiar with the general formula from lecture 6.1.

REMARK: Notice that we must have our initial conditions at $t = 0$ to use Laplace Transforms.

■ **Example 5.11** Solve the following IVP using Laplace Transforms

$$y'' - 5y' - 6y = 5t \quad y(0) = -1 \quad y'(0) = 2 \tag{5.110}$$

We begin by transforming both sides of the equation:

$$\mathscr{L}\{y''\} - 5\mathscr{L}\{y'\} - 6\mathscr{L}\{y\} = 5\mathscr{L}\{t\} \tag{5.111}$$
$$s^2 Y(s) - sy(0) - y'(0) - 5sY(s) + 5y(0) - 6Y(s) = \frac{5}{s^2} \tag{5.112}$$
$$(s^2 - 5s - 6)Y(s) + s - 2 - 5 = \frac{5}{s^2}. \tag{5.113}$$

As we have already begun doing, now we solve for $Y(s)$.

$$Y(s) = \frac{5}{s^2(s^2 - 5s - 6)} + \frac{7-s}{s^2 - 5s - 6} \tag{5.114}$$
$$= \frac{5}{s^2(s-6)(s+1)} + \frac{7-s}{(s-6)(s+1)} \tag{5.115}$$
$$= \frac{5 + 7s^2 - s^3}{s^2(s-6)(s+1)} \tag{5.116}$$

We now have an expression for $Y(s)$, which is the Laplace Transform of the solution $y(t)$ to the initial value problem. We have simplified as much as possible now we need partial fractions decomposition to take the inverse transform

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-6} + \frac{D}{s+1}. \tag{5.117}$$

Multiplying through by $s^2(s-6)(s+1)$ gives

$$6 + 7s^2 + s^3 = As(s-6)(s+1) + B(s-6)(s+1) + Cs^2(s+1) + Ds^2(s-6) \tag{5.118}$$

We can find the constant by choosing key values of s

$$s = 0: \quad 6 = -6B \quad \Rightarrow \quad B = -1 \tag{5.119}$$

$$s - 6: \quad 42 = 252C \quad \Rightarrow \quad C = \frac{1}{6} \tag{5.120}$$

$$s = -1: \quad 14 = -7D \quad \Rightarrow \quad A = \frac{1}{12} \tag{5.121}$$

So

$$Y(s) = \frac{1}{12}\frac{1}{s} - \frac{1}{s^2} + \frac{1}{6}\frac{1}{s-6} - \frac{1}{2}\frac{1}{s+1} \tag{5.122}$$

$$y(t) = \frac{1}{12} - t + \frac{1}{6}e^{6t} - \frac{1}{2}e^{-t}. \tag{5.123}$$

■

EXERCISE: Solve the IVP in the previous example using the Method of Undetermined Coefficients. Do you get the same thing? Which method took less work?

■ **Example 5.12** Solve the following initial value problem

$$y'' + 2y' + 5y = \cos(t) - 10t, \quad y(0) = 0, \quad y'(0) = 1 \tag{5.124}$$

We begin by transforming the entire equation and solving for $Y(s)$.

$$\mathscr{L}\{y''\} + 2\mathscr{L}\{y'\} + 5\mathscr{L}\{y\} = \mathscr{L}\{\cos(t)\} - 10\mathscr{L}\{t\} \tag{5.125}$$

$$s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + 5Y(s) = \frac{s}{s^2+1} - \frac{10}{s^2} \tag{5.126}$$

$$(s^2 + 2s + 5)Y(s) - 1 = \frac{s}{s^2+1} - \frac{10}{s^2} \tag{5.127}$$

So we have

$$Y(s) = \frac{s}{(s^2+1)(s^2+2s+5)} - \frac{10}{s^2(s^2+2s+5)} + \frac{1}{s^2+2s+5} \tag{5.128}$$

$$= Y_1(s) + Y_2(s) + Y_3(s). \tag{5.129}$$

Now we will have to take inverse transforms. This will require doing partial fractions on the first two pieces.

Let's start with the first one

$$Y_1(s) = \frac{s}{(s^2+1)(s^2+2s+5)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+5} \tag{5.130}$$

Multiply through by $(s^2+1)(s^2+2s+5)$

$$\begin{aligned}2 &= As(s^2+2s+5)+B(s^2+2s+5)+Cs(s^2+1)+D(s^2+1) && (5.131)\\ &= (A+C)s^3+(2A+B+D)s^2+(5A+2B+C)s+(5B+D) && (5.132)\end{aligned}$$

This gives us the following system of equations, which we solve

$$\begin{aligned}A+C &= 0 && (5.133)\\ 2A+B+D &= 0 && (5.134)\\ 5A+2B+C &= 1 \quad \Rightarrow \quad A=\frac{1}{5}\ B=\frac{1}{10}\ C=-\frac{1}{5}\ D=-\frac{1}{2} && (5.135)\\ 5B+D &= 0 && (5.136)\end{aligned}$$

Thus our first term becomes

$$Y_1(s)=\frac{1}{5}\frac{s}{s^2+1}+\frac{1}{10}\frac{1}{s^2+1}-\frac{1}{5}\frac{s}{s^2+2s+5}-\frac{1}{2}\frac{1}{s^2+2s+5}. \tag{5.137}$$

We will hold off on taking the inverse transform until we solve for Y_2.

$$Y_2(s)=-\frac{10}{s^2(s^2+2s+5)}=\frac{A}{s}+\frac{B}{s^2}+\frac{Cs+D}{s^2+2s+5} \tag{5.138}$$

We multiply through by $s^2(s^2+2s+5)$

$$\begin{aligned}-10 &= As(s^2+2s+5)+B(s^2+2s+5)+Cs^3+Ds^2 && (5.139)\\ &= (A+C)s^3+(2A+B+D)s^2+(5A+2B)s+5B && (5.140)\end{aligned}$$

This gives the following system of equations

$$\begin{aligned}A+C &= 0 && (5.141)\\ 2A+B+D &= 0 && (5.142)\\ 5A+2B &= 0 \quad \Rightarrow \quad A=\frac{4}{5}\ B=-2\ C=-\frac{4}{5}\ D=\frac{2}{5} && (5.143)\\ 5B &= -10 && (5.144)\end{aligned}$$

Thus we have

$$Y_2(s)=\frac{4}{5}\frac{1}{s}-\frac{2}{s^2}-\frac{4}{5}\frac{s}{s^2+2s+5}+\frac{2}{5}\frac{1}{s^2+2s+5}. \tag{5.145}$$

Let's return to our original function

$$\begin{aligned}Y(s) &= Y_1(s)+Y_2(s)+Y_3(s) && (5.146)\\ &= \frac{1}{5}\frac{s}{s^2+1}+\frac{1}{10}\frac{1}{s^2+1}+\frac{4}{5}\frac{1}{s}-\frac{2}{s^2}+(-\frac{1}{5}-\frac{4}{5})\frac{s}{s^2+2s+5}+(-\frac{1}{2}+\frac{2}{5}+1)\frac{1}{s^2+2s+5} && (5.147)\\ &= \frac{1}{5}\frac{s}{s^2+1}+\frac{1}{10}\frac{1}{s^2+1}+\frac{4}{5}\frac{1}{s}-\frac{2}{s^2}-\frac{s}{(s+1)^2+4}+\frac{9}{10}\frac{1}{(s+1)^2+4} && (5.148)\end{aligned}$$

Now we have to adjust the last two terms to make them suitable for the inverse transform. Namely, we need to have $s+1$ in the numerator of the second to last term, and 2 in the numerator of the last term.

$$= \frac{1}{5}\frac{s}{s^2+1}+\frac{1}{10}\frac{1}{s^2+1}+\frac{4}{5}\frac{1}{s}-\frac{2}{s^2}-\frac{s+1-1}{(s+1)^2+4}+\frac{9}{10}\frac{1}{(s+1)^2+4} \quad (5.149)$$

$$= \frac{1}{5}\frac{s}{s^2+1}+\frac{1}{10}\frac{1}{s^2+1}+\frac{4}{5}\frac{1}{s}-\frac{2}{s^2}-\frac{s+1}{(s+1)^2+4}+\frac{19}{10}\frac{1}{(s+1)^2+4} \quad (5.150)$$

$$= \frac{1}{5}\frac{s}{s^2+1}+\frac{1}{10}\frac{1}{s^2+1}+\frac{4}{5}\frac{1}{s}-\frac{2}{s^2}-\frac{s+1}{(s+1)^2+4}+\frac{19}{20}\frac{2}{(s+1)^2+4} \quad (5.151)$$

$$(5.152)$$

So our solution is

$$y(t)=\frac{1}{5}\cos(t)+\frac{1}{10}\sin(t)+\frac{4}{5}-2t-e^{-t}\cos(2t)+\frac{19}{20}e^{-t}\sin(2t). \quad (5.153)$$

■

We could have done both the preceding examples using the method of Undetermined Coefficients. In fact, it would have been a lot less work.

The following examples are to be done after learning Section 6.3. Let's do some involving step functions, which is where Laplace Transforms work great.

■ **Example 5.13** Solve the following initial value problem.

$$y''-y'+6y=2-u_2(t)e^{2t-4} \quad y(0)=0 \quad y'(0)=0 \quad (5.154)$$

As before, we begin by transforming everything. Before we do that, however, we need to write the coefficient function $u_2(t)$ as a function evaluated at $t-2$.

$$y''-5y'+6y=2-u_2(t)e^{2(t-2)} \quad (5.155)$$

Now we can transform

$$\begin{aligned}
\mathscr{L}\{y''\}-5\mathscr{L}\{y'\}+6\mathscr{L}\{y\} &= 2\mathscr{L}\{1\}-\mathscr{L}\{u_2(t)e^{2(t-2)}\} && (5.156)\\
s^2Y(s)-sy(0)-y'(0)-5sY(s)+5y(0)-6Y(s) &= \frac{2}{s}-e^{-2s}\mathscr{L}\{e^{2t}\} && (5.157)\\
(s^2-5s+6)Y(s) &= \frac{2}{s}-e^{-2s}\frac{1}{s-2} && (5.158)
\end{aligned}$$

So we end up with

$$\begin{aligned}
Y(s) &= \frac{2}{s(s-3)(s-2)}-e^{-2s}\frac{1}{(s-3)(s-2)^2} && (5.159)\\
&= Y_1(s)+e^{-2s}Y_2(s). && (5.160)
\end{aligned}$$

Since one of the terms has an exponential, we will need to deal with each term separately. I'll leave it to you to check the partial fractions.

$$\begin{aligned}
Y_1(s) &= \frac{1}{3}\frac{1}{s}+\frac{2}{3}\frac{1}{s-3}-\frac{1}{s-2} && (5.161)\\
Y_2(s) &= -\frac{1}{s-3}+\frac{1}{s-2}+\frac{1}{(s-2)^2} && (5.162)
\end{aligned}$$

Thus we have

$$Y(s) = \frac{1}{2}\frac{1}{s} + \frac{2}{3}\frac{1}{s-3} - \frac{1}{s-2} + e^{-2s}\left(-\frac{1}{s-3} + \frac{1}{s-2} + \frac{1}{(s-2)^2}\right) \tag{5.163}$$

and taking the inverse transforms

$$\begin{aligned} y(t) &= \frac{1}{2} + \frac{2}{3}e^{3t} - e^{2t} + u_2(t)\left(-e^{3(t-2)} + e^{2(t-2)} + (t-2)e^{2(t-2)}\right) && (5.164)\\ &= \frac{1}{2} + \frac{2}{3}e^{3t} - e^{2t} + u_2(t)\left(-e^{3t-6} - e^{2t-4} + te^{2t-4}\right) && (5.165)\end{aligned}$$

once we observe that $\mathscr{L}\{\frac{1}{(s-a)^2}\} = te^{at}$. ■

■ **Example 5.14** Solve the following initial value problem

$$y'' + 4y = 8 + tu_4(t), \quad y(0) = 0, \quad y'(0) = 0 \tag{5.166}$$

We need to first write the coefficient function of $u_4(t)$ in the form $h(t-4)$ for some function $h(t)$. So we write $h(t-4) = t = t-4+4$ and conclude $h(t) = t+4$. So our equation is

$$y'' + 4y = 8 + \big((t-4)+4\big)u_4(t). \tag{5.167}$$

Now, we want to Laplace Transform everything.

$$\begin{aligned} \mathscr{L}\{y''\} + 4\mathscr{L}\{y\} &= 8\mathscr{L}\{1\} + \mathscr{L}\{(t-4)+4)u_4(t)\} && (5.168)\\ s^2Y(s) - sy(0) - y'(0) + 4Y(s) &= \frac{8}{s} + e^{-4s}\mathscr{L}\{t+4\} && (5.169)\\ (s^2+4)Y(s) &= \frac{8}{s} + e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right) && (5.170)\end{aligned}$$

So we have

$$\begin{aligned} Y(s) &= \frac{8}{s(s^2+4)} + e^{-4s}\left(\frac{1}{s^2(s^2+4)} + \frac{4}{s(s^2+4)}\right) && (5.171)\\ &= \frac{8}{s(s^2+4)} + e^{-4s}\frac{1+4s}{s^2(s^2+4)} && (5.172)\\ &= Y_1(s) + e^{-4s}Y_2(s). && (5.173)\end{aligned}$$

where we have consolidated the two fractions being multiplied by the exponential to reduce the number of partial fraction decompositions we need to compute. After doing partial fractions (leaving the details for you), we have

$$Y_1(s) = \frac{2}{s} - \frac{2s}{s^2+4} \tag{5.174}$$

and

$$Y_2(s) = \frac{1}{s} + \frac{1}{4}\frac{1}{s^2} - \frac{s}{s^2+4} - \frac{1}{4}\frac{1}{s^2+4}. \tag{5.175}$$

so

$$Y(s) = \frac{2}{s} - \frac{2s}{s^2+4} + e^{-4s}\left(\frac{1}{s} + \frac{1}{4}\frac{1}{s^2} - \frac{s}{s^2+4} - \frac{1}{4}\frac{1}{s^2+4}\right) \quad (5.176)$$

$$= \frac{2}{s} - 2\frac{s}{s^2+4} + e^{-4s}\left(\frac{1}{s} + \frac{1}{4}\frac{1}{s^2} - \frac{s}{s^2+4} - \frac{1}{8}\frac{2}{s^2+4}\right) \quad (5.177)$$

$$(5.178)$$

and the solution is

$$y(t) = 2 - 2\cos(2t) + u_4(t)\left(1 - \frac{1}{4}(t-4) - \cos(2(t-4)) - \frac{1}{8}\sin(2(t-4))\right) \quad (5.179)$$

$$= 2 - 2\cos(2t) + u_4(t)\left(\frac{1}{4}t - \cos(2t-8) - \frac{1}{8}\sin(2t-8)\right). \quad (5.180)$$

■

5.3 Step Functions

Last Time: We thoroughly studied the solution process for IVPs using Laplace Transform Methods.

When we originally defined the Laplace Transform we said we could use it for piecewise continuous functions, but so far we have only used it for continuous functions. We would like to solve differential equations with forcing functions that were not continuous, but had isolated points of discontinuity where the forcing function jumped from one value to another abruptly. An example is a mechanical vibration where we add some extra force later on in the process. Without Laplace Transforms we would have to split these IVPs into several different problems, with initial conditions derived from previous solutions.

5.3.1 Step Functions

Consider the following function

$$u(t-c) = u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases} \quad (5.181)$$

This function is called the **step** or **Heavyside function** at c. It represents a jump at $t = c$ from zero to one at $t = c$. One can think of a step function as a switch that turns on its coefficient at a specific time. The step function takes only values of 0 or 1, but is easy enough to give it any value desired by changing its coefficient. $4u_c(t)$ has the value 4 at $t = c$ and beyond. We can produce a switch that turns off at $t = c$,

$$1 - u_c(t) = \begin{cases} 1-0 = 1 & \text{if } t < c \\ 1-1 = 0 & \text{if } t \geq c \end{cases} \quad (5.182)$$

it will exhibit the desired behavior. This function will get any contribution we want prior to $t = c$. This is the key to writing a piecewise continuous function as a single expression instead of a system of cases.

■ **Example 5.15** Write the following function in terms of step functions.

$$f(t)=\begin{cases}9 & \text{if } t<2\\ -6 & \text{if } 2\le t<6\\ 25 & \text{if } 6\le t<9\\ 7 & \text{if } 9\le t\end{cases} \tag{5.183}$$

There are three jumps in this function at $t=2$, $t=6$, and $t=9$. So we will need a total of three step functions, each of which will correspond to one of these jumps. In terms of step functions,

$$f(t)=9-15u(t-2)+31u(t-6)-18u(t-9) \tag{5.184}$$

How did we come up with this?

When $t<2$, all of the step functions have the value 0. So the only contributing term in our expression for $f(t)$ is 9, and on this region $f(t)=9$.

On the next interval, $2\le t<6$, we want $f(t)=-6$. The first step function $u(t-2)$ is on, while the others are off. Notice that 9 is still present and as a result, the coefficient of $u(t-2)$ will need to cause the sum of it and 9 to equal -6. So the coefficient must be -15.

On the third interval, $6\le t<9$, we now have two functions turned on while the last one is off. The first two terms contribute $9-15u(t-2)$. Thus the coefficient of $u(t-6)$ will need to combine with these to give our desired value of $f(t)=25$. So it must be 31.

The last term, we have the interval $9\le t$. Now all the switches are turned on and the coefficient of $u(t-9)$, the step function corresponding to the final jump, should move us from our previous value of 25 to our new value, $f(t)=7$. As a result, it must be -18. ■

We are not just interested in situations where our forcing function takes constant values on intervals. In the case of mechanical vibrations of the sort we considered earlier, we might want to add in a new external force which is sinusoidal.

So consider the following piecewise continuous function: $g(t)=u(t-c)f(t-c)$, where $f(t)$ is some function. We shift it by c, the starting point of the step function, to indicate that we want it to start working at $t=c$ instead of $t=0$, which it would normally. Think of this graphically, to get the graph of $g(t)$, what we want to do is take the graph of $f(t)$, starting at $t=0$, and push it to start at $t=c$ with the value of 0 prior to this time. This requires shifting the argument of f by c.

5.3.2 Laplace Transform

What is the Laplace Transform $\mathscr{L}\{g(t)\}$?

$$\begin{aligned}\mathscr{L}\{g(t)\} &= \mathscr{L}\{u(t-c)f(t-c)\} && (5.185)\\ &= \int_0^\infty u(t-c)e^{-st}f(t-c)dt && (5.186)\\ &= \int_c^\infty e^{-st}f(t-c) \quad \text{using the definition of the step function} && (5.187)\end{aligned}$$

Now this looks almost like a Laplace Transform, except that the integral starts at $t=c$ instead of $t=0$. So we introduce a new variable $y=t-c$ to shift the integral back to starting at 0.

$$G(s) = \int_0^\infty e^{-s(u+c)} f(u)du \tag{5.188}$$

$$= e^{-sc} \int_0^\infty e^{-su} f(u)du \tag{5.189}$$

$$= e^{-sc} F(s) \quad \text{using the notation} F(s) = \mathscr{L}\{f(u)\}. \tag{5.190}$$

Notice that the Laplace Transform we end up with is the Laplace Transform of the **original** function $f(t)$ multiplied by an exponential related to the step function's "on" time, even though we had shifted the function by c to begin with. Summarizing, we have the formula

$$\mathscr{L}\{u_c(t)f(t-c)\} = e^{-sc}F(s) = e^{-sc}\mathscr{L}\{f(t)\}. \tag{5.191}$$

It is **critical** that we write the function to be transformed in the correct form, as a different function shifted by c, before we transform it using the above equation. We compute the transform of $f(t)$ NOT the shifted function $f(t-c)$. This is the most common mistake initially.

We can get a formula for a step function by itself. To do so, we consider a step function multiplied by the constant function $f(t) = 1$. In this case $f(t-c) = 1$. So

$$\mathscr{L}\{u_c(t)\} = \mathscr{L}\{u_c(t) \cdot 1\} = e^{-cs}\mathscr{L}\{1\} = \frac{1}{s}e^{-cs}. \tag{5.192}$$

■ **Example 5.16** Find the Laplace Transforms of each of the following
(i) $f(t) = 10u_6(t) + 3(t-4)^2u_4(t) - (1+e^{10-2t})u_5(t)$

Recall that we must write each piece in the form $u_c(t)h(t-c)$ before we take the transform. If it is not in that form, we have to put it in that form first. There are three terms in $f(t)$. We will use the linearity of the Laplace Transform to treat them separately, then add them together in the end. Write

$$f(t) = f_1(t) + f_2(t) + f_3(t) \tag{5.193}$$

$f_1(t) = 10u_6(t)$, so it is just a constant multiplied by a step function. We can thus us Equation (5.192) to determine its Laplace Transform

$$\mathscr{L}\{f_1(t)\} = 10\mathscr{L}\{u_6(t)\} = \frac{10e^{-6s}}{s} \tag{5.194}$$

$f_2(t) = 3(t-4)^2u_4(t)$, so we have to do two things: 1. Write it as a function shifted by 4 (if not in that form already) and 2. Isolate the function that was shifted and transform it. In this case, we are good: we can write $f_2(t) = h(t-4)u_4(t)$, with $h(t) = 3t^2$. Thus

$$\mathscr{L}\{f_2(t)\} = e^{-4s}\mathscr{L}\{3t^2\} = 3e^{-4s}\frac{2}{s^3} = \frac{6e^{-4s}}{s^3}. \tag{5.195}$$

Finally, we have $f_3(t) = -(1+e^{10-2t})u_5(t)$. Again, we have to express it as a function shifted by 5 and then identify the unshifted function so that we may transform it. This can be accomplished by rewriting

$$f_3(t) = -(1+e^{-2(t-5)})u_5(t), \tag{5.196}$$

so, writing $g_3(t) = h(t-5)u_5(t)$, we have $h(t) = -(1+e^{-2t})$ as the unshifted coefficient function. Thus

$$\mathscr{L}\{f_3(t)\} = e^{-5s}\mathscr{L}\{-(1+e^{-2t})\} = -e^{-5s}\left(\frac{1}{s}+\frac{1}{s+2}\right). \tag{5.197}$$

Putting it all together,

$$F(s) = \frac{10e^{-6s}}{s} + \frac{6e^{-4s}}{s^3} - e^{-5s}\left(\frac{1}{s}+\frac{1}{s+2}\right). \tag{5.198}$$

(ii) $g(t) = t^2u_2(t) - \cos(t)u_7(t)$

In the last example, it turned out that all of the coefficient functions were pre-shifted (the most we had to do was pull out a constant to see that). In this example, that is definitely not the case. So what we want to do is to write each of our coefficient functions as the shift (by whichever constant is appropriate for that step function) of a different function. The idea is that we add and subtract the desired quantity, then simplify, keeping the correct shifted term.

So, let's write $g(t) = g_1(t) + g_2(t)$ where

$$\begin{aligned} g_1(t) &= t^2u_2(t) && (5.199)\\ g_1(t) &= (t-2+2)^2u_2(t) && (5.200)\end{aligned}$$

This is not quite there yet, use the Associative Property of Addition

$$\begin{aligned} g_1(t) &= ((t-2)+2)^2u_2(t) && (5.201)\\ &= ((t-2)^2+4(t-2)+4)u_2(t). && (5.202)\end{aligned}$$

Now we can see that $g_1(t) = h(t-2)u_2(t)$, where $h(t) = t^2+4t+4$.

$$\mathscr{L}\{g_1(t)\} = e^{-2s}\mathscr{L}\{t^2+4t+4\} = e^{-2s}(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}) \tag{5.203}$$

The second term is similar. We start with

$$g_2(t) = -\cos(t)u_7(t) = -\cos((t-7)+7)u_7(t) \tag{5.204}$$

Here we need to use the trig identity

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b). \tag{5.205}$$

This yields

$$g_2(t) = -(\cos(t-7)\cos(7) - \sin(t-7)\sin(7))u_7(t) \tag{5.206}$$

Since $\cos(7)$ and $\sin(7)$ are just constants we get (after using the linearity of the Laplace Transform)

$$\begin{aligned} \mathscr{L}\{g_2(t)\} &= -e^{-7s}(\cos(7)\mathscr{L}\{\cos(t)\} - \sin(7)\mathscr{L}\{\sin(t)\}) && (5.207)\\ &= -e^{-7s}(\frac{s\cos(7)}{s^2+1} - \frac{\sin(7)}{s^2+1}). && (5.208)\end{aligned}$$

Piecing back together we get

$$G(s) = e^{-2s}(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}) - e^{-7s}(\frac{s\cos(7)-\sin(7)}{s^2+1}). \tag{5.209}$$

(iii) $f(t) = \begin{cases} t^3 & \text{if } t < 4 \\ t^3 + 2\sin(\frac{t}{12}-\frac{1}{3}) & \text{if } 4 \leq t \end{cases}$

The first step is to write $f(t)$ as a single equation using step functions.

$$f(t) = t^3 + 2\sin(\frac{t}{12}-\frac{1}{3})u_4(t) \tag{5.210}$$

Next, we want to write the coefficients of $u_4(t)$ as another function shifted by 4.

$$f(t) = t^3 + 2\sin(\frac{1}{12}(t-4))u_4(t). \tag{5.211}$$

Since everything is shifted, we have

$$\begin{aligned} F(s) &= \mathscr{L}\{t^3\} + 2e^{-4s}\mathscr{L}\{\sin(\frac{1}{12}t)\} && (5.212)\\ &= \frac{3!}{s^4} + 2e^{-4s}\frac{\frac{1}{12}}{s^2+(\frac{1}{12})^2} && (5.213)\\ &= \frac{6}{s^4} + \frac{e^{-4s}}{12(s^2+\frac{1}{144})} && (5.214)\\ &= \frac{6}{s^4} + \frac{e^{-4s}}{12s^2+\frac{1}{12}}. && (5.215) \end{aligned}$$

(iv) $g(t) = \begin{cases} t & \text{if } t < 2 \\ 3+(t-2)^2 & \text{if } 2 \leq t \end{cases}$

First, we need to write $g(t)$ using step functions.

$$g(t) = t + (-t+3+(t-2)^2)u_2(t). \tag{5.216}$$

Notice that we had to subtract t from the coefficient of $u_2(t)$ in order to make $g(t)$ have the correct value when $t \geq 2$. However, this means that the coefficient function of $u_2(t)$ is no longer properly shifted. As a result, we need to add and subtract 2 from that t to make it have the proper form.

$$\begin{aligned} g(t) &= t + (-(t-2+2)+3+(t-2)^2)u_2(t) && (5.217)\\ &= t + (-(t-2)-2+3+(t-2)^2)u_2(t) && (5.218)\\ &= t + (-(t-2)+1+(t-2)^2)u_2(t) && (5.219) \end{aligned}$$

So, we have

$$\begin{aligned} G(s) &= \mathscr{L}\{t\} + e^{-2s}(\mathscr{L}\{-t\}+\mathscr{L}\{1\}+\mathscr{L}\{t^2\}) && (5.220)\\ &= \frac{1}{s^2} + e^{-2s}(-\frac{1}{s^2}+\frac{1}{s}+\frac{2}{s^3}). && (5.221) \end{aligned}$$

■

As you can see, taking the Laplace Transforms of the functions involving step functions can be a bit more complicated than taking Laplace Transforms of original functions. It still is not too bad, we have to make sure our coefficient functions are appropriately shifted.

5.4 Differential Equations With Discontinuous Forcing Functions

Last Time: We considered the Laplace Transforms of Step Functions.

$$\mathscr{L}\{u(t-c)f(t-c)\} = e^{-cs}\mathscr{L}\{f(t)\} \tag{5.222}$$

where $f(t-c)$ is the coefficient function of $u(t-c)$.

5.4.1 Inverse Step Functions

Now let's look at some inverse transforms. The previous formula's associated inverse transform is

$$\mathscr{L}^{-1}\{e^{-cs}F(s)\} = u(t-c)f(t-c), \tag{5.223}$$

where $f(t) = \mathscr{L}^{-1}\{F(s)\}$. So we need to be careful about the shifting. This time we shift at the end of the calculation, after finding the inverse transform of the coefficient of the exponential.

■ **Example 5.17** Find the inverse Laplace Transform of the following.

(i) $F(s) = \frac{se^{-2s}}{(s-4)(s+3)}$

Whenever we do this, it is a good idea to ignore the exponential and determine the inverse transform of what is left over first. In this case we cannot split anything up, since there is only one exponential and no terms without an exponential. So we pull out the exponential and ignore it for the time being

$$F(s) = e^{-2s}\frac{s}{(s-4)(s+3)} = e^{-2s}H(s). \tag{5.224}$$

We want to determine $h(t) = \mathscr{L}^{-1}\{H(s)\}$. Once we have that, the definition of the inverse transform tells us that the inverse will be

$$f(t) = h(t-2)u(t-2). \tag{5.225}$$

Now we need to use partial fractions on $H(s)$ so that we can take its inverse transform. The form of the decomposition is

$$H(s) = \frac{A}{s-4} + \frac{B}{s+3} \tag{5.226}$$

So

$$s = A(s+3) + B(s-4) \tag{5.227}$$

We can use the quick method of picking key values of s

$$s = 4: \quad 4 \quad = \quad 7A \quad \Rightarrow \quad A = \frac{4}{7} \tag{5.228}$$

$$s = -3: \quad -3 \quad = \quad -7B \quad \Rightarrow \quad B = \frac{3}{7} \tag{5.229}$$

So the partial fraction decomposition is

$$H(s) = \frac{4}{7}\frac{1}{s-4} + \frac{3}{7}\frac{1}{s+3} \tag{5.230}$$

$$= \frac{4}{7}\frac{1}{s-4} + \frac{3}{7}\frac{1}{s+3}. \tag{5.231}$$

So we have

$$h(t) = \frac{4}{7}e^{4t} + \frac{3}{7}e^{-3t}. \tag{5.232}$$

Thus, since $f(t) = h(t-2)u(t-2)$,

$$\begin{aligned} f(t) &= u(t-2)(\frac{4}{7}e^{4(t-2)} + \frac{3}{7}e^{-3(t-2)}) && (5.233)\\ &= u(t-2)(\frac{4}{7}e^{4t-8} + \frac{3}{7}e^{-3t+6}). && (5.234)\end{aligned}$$

(ii) $G(s) = \frac{2e^{-3s}+3e^{-7s}}{(s-3)(s^2+9)}$
We write

$$G(s) = (2e^{-3s} + 3e^{-7s})\frac{1}{(s-3)(s^2+9)} = (2e^{-3s} + 3e^{-7s})H(s). \tag{5.235}$$

We want to find the inverse transform of

$$H(s) = \frac{1}{(s-3)(s^2+9)}. \tag{5.236}$$

The partial fraction decomposition is

$$H(s) = \frac{A}{s-3} + \frac{Bs+C}{s^2+9} = \frac{A(s^2+9) + (Bs+C)(s-3)}{(s-3)(s^2+9)} \tag{5.237}$$

So we have the following

$$\begin{aligned} 1 &= A(s^2+0) + (Bs+C)(s-3) && (5.238)\\ &= (A+B)s^2 + (-3B+C)s + (9A-3C) && (5.239)\end{aligned}$$

Setting the coefficients equal and solving gives

$$\begin{aligned} s^2: \quad A+B &= 0 && (5.240)\\ s: \quad -3B+C &= 0 \quad \Rightarrow \quad A = \frac{1}{18} \;\; B = -\frac{1}{18} \;\; C = -\frac{3}{18} && (5.241)\\ s^0: \quad 9A-3C &= 1 && (5.242)\end{aligned}$$

Substituting back into the transform, we get

$$\begin{aligned} H(s) &= \frac{1}{18}(\frac{1}{s-3} + \frac{-s-3}{s^2+9}) && (5.243)\\ &= \frac{1}{18}(\frac{1}{s-3} - \frac{s}{s^2+9} - \frac{3}{s^2+9}). && (5.244)\end{aligned}$$

Now, if we take the inverse transform, we get

$$h(t) = \frac{1}{18}(e^{3t} - \cos(3t) - \sin(3t)). \tag{5.245}$$

Returning to the original problem, we had

$$\begin{aligned} G(s) &= (2e^{-3s}+3e^{-7s})H(s) && (5.246)\\ &= 2e^{-3s}H(s)+3e^{-7s}H(s). && (5.247)\end{aligned}$$

We had to distribute $H(s)$ through the parenthesis and use the definition, since we must end up with each term containing one step function and one coefficient function. So

$$\begin{aligned} g(t) &= 2h(t-3)u(t-3)+3h(t-7)u(t-7)) && (5.248)\\ &= \frac{u(t-3)}{9}(e^{3(t-3)}-\cos(3(t-3))-\sin(3(t-3))) && (5.249)\\ &+ \frac{3u(t-7)}{18}(e^{3(t-7)}-\cos(3(t-7))-\sin(3(t-7))) && (5.250)\end{aligned}$$

(iii) $F(s)=\frac{e^{-4s}}{s^2(s+1)}$
We write

$$\begin{aligned} F(s) &= e^{-4s}\frac{1}{s^2(s+1)} = e^{-4s}H(s) && (5.251)\\ H(s) &= \frac{A}{s^2}+\frac{B}{s}+\frac{C}{s+1} && (5.252)\\ 1 &= A(s+1)+Bs(s+1)+Cs^2 && (5.253)\\ &= (B+C)s^2+(A+B)s+A && (5.254)\end{aligned}$$

So we have

$$\begin{aligned} B+C &= 0 && (5.255)\\ A+B &= 0 \quad \Rightarrow \quad A=1 \;\; B=-1 \;\; C=1 && (5.256)\\ A &= 1 && (5.257)\end{aligned}$$

Thus $H(s)$ and its inverse transform are

$$H(s)=\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s+1} \tag{5.258}$$

$$h(t)=t-1+e^{-t} \tag{5.259}$$

Our original transform function was

$$F(s)=e^{-4s}H(s). \tag{5.260}$$

By the definition of the inverse transform, it will be

$$\begin{aligned} f(t) &= h(t-4)u(t-4) && (5.261)\\ &= ((t-4)-1+e^{-(t-4)})u(t-4) && (5.262)\\ &= (t-5+e^{4-t})u(t-4) && (5.263)\end{aligned}$$

(iv) $G(s)=\frac{s-e^{-2s}}{s^2+2s+5}$

In this case, we won't have to do partial fractions, since the denominator does not factor. Instead, we will have to complete the square and we get

$$G(s) = \frac{s - e^{-2s}}{(s+1)^2+4} = \frac{s}{(s+1)^2+4} - e^{-2s}\frac{1}{(s+1)^2+4} = G_1(s) - e^{-2s}G_2(s) \quad (5.264)$$

We need to treat $G_1(s)$ and $G_2(s)$ separately. $G_1(t)$ is almost fine, but we need the numerator to contain $s+1$ instead of s. We do this by adding and subtracting 1 from the numerator

$$\begin{aligned} G_1(s) &= \frac{s+1-1}{(s+1)^2+4} && (5.265)\\ &= \frac{s+1}{(s+1)^2+4} - \frac{1}{(s+1)^2+4} && (5.266)\\ &= \frac{s+1}{(s+1)^2+4} - \frac{1}{2}\frac{2}{(s+1)^2+4} && (5.267)\\ g_1(t) &= e^{-t}\cos(2t) - \frac{1}{2}e^{-t}\sin(2t). && (5.268) \end{aligned}$$

For $G_2(t)$, all we have to of is to make the numerator 2 instead of 1.

$$\begin{aligned} G_2(s) &= \frac{1}{(s+1)^2+4} && (5.269)\\ &= \frac{1}{2}\frac{2}{(s+1)^2+4} && (5.270)\\ g_2(t) &= \frac{1}{2}e^{-t}\sin(2t) && (5.271) \end{aligned}$$

Our original transform was

$$G(s) = G_1(s) - e^{-2s}G_2(s) \quad (5.272)$$

By the definition of the inverse transform

$$\begin{aligned} g(t) &= g_1(t) - g_2(t-2)u(t-2) && (5.273)\\ &= e^{-t}\cos(2t) - \frac{1}{2}e^{-t}\sin(2t) - \frac{1}{2}e^{-(t-2)}\sin(2(t-2))u(t-2) && (5.274)\\ &= e^{-t}\cos(2t) - \frac{1}{2}e^{-t}\sin(2t) - \frac{1}{2}e^{2-t)}\sin(2t-4))u(t-2) && (5.275) \end{aligned}$$

■

HW 6.3 # 19,21

5.4.2 Solving IVPs with Discontinuous Forcing Functions

We want to now actually solve IVPs by using the Laplace Transform and our experience with step functions.

■ **Example 5.18** Find the solution to the initial value problem

$$y'' + y = g(t), \quad y(0) = 0, \quad y'(0) = 1 \tag{5.276}$$

where

$$g(t) = \begin{cases} 2t & \text{if } 0 \leq t < 1 \\ 2 & \text{if } 1 \leq t < \infty \end{cases} \tag{5.277}$$

The Laplace Transform of the left side and using ICs we get

$$\mathscr{L}\{y'' + y\} = s^2 Y(s) - sy(0) - y'(0) + Y(s) = (s^2+1)Y(s) - 1 \tag{5.278}$$

From techniques in the previous section we know

$$\mathscr{L}\{g(s)\} = G(s) = \frac{(2 - 2e^{-s})}{s^2} \tag{5.279}$$

So combining these two

$$(s^2+1)Y(s) - 1 = \frac{2 - 2e^{-s}}{s^2} \tag{5.280}$$

Solving for $Y(s)$ we get

$$Y(s) = \frac{2 - 2e^{-s}}{s^2(s^2+1)} + \frac{1}{s^2+1} \tag{5.281}$$

Note that if we use partial fractions decomposition

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1} \tag{5.282}$$

So the equation becomes

$$Y(s) = \left(\frac{2 - 2e^{-s}}{s^2} - \frac{2 - 2e^{-s}}{s^2+1}\right) + \frac{1}{s^2+1} \tag{5.283}$$

$$= \frac{2}{s^2} - \frac{2e^{-s}}{s^2} + \frac{2e^{-s}}{s^2+1} - \frac{1}{s^2+1} \tag{5.284}$$

From our table we know the inverse of $1/s^2$ is the function t, and the inverse transform of $1/(s^2+1)$ is $\sin(t)$. Thus

$$y(t) = 2t - 2(t-1)u(t-1) + 2u(t-1)\sin(t-1) - \sin(t) \tag{5.285}$$

The function $y(t)$ can also be written as

$$y(t) = \begin{cases} 2t - \sin(t) & \text{if } 0 \leq t < 1 \\ 2 + 2\sin(t-1) - \sin(t) & \text{if } 1 \leq t < \infty \end{cases} \tag{5.286}$$

■

Example 5.19 Actually #6 in Homework Section of book. Solve the following IVP

$$y'' + 3y' + 2y = u_2(t), \quad y(0) = 0, \quad y'(0) = 1 \tag{5.287}$$

Take the Laplace Transform of both sides of the ODE

$$s^2Y(s) - sy(0) - y'(0) + 3[sY(s) - y(0)] + 2Y(s) = \frac{e^{-2s}}{s} \tag{5.288}$$

Applying the initial conditions,

$$s^2Y(s) + 3sY(s) + 2Y(s) - 1 = \frac{e^{-2s}}{s} \tag{5.289}$$

Solving for the transform

$$Y(s) = \frac{1}{s^2+3s+2} + \frac{e^{-2s}}{s(s^2+3s+2)}. \tag{5.290}$$

Using Partial Fractions Decomposition

$$\frac{1}{s^2+3s+2} = \frac{1}{s+1} - \frac{1}{s+2} \tag{5.291}$$

and

$$\frac{1}{s(s^2+3s+2)} = \frac{1}{2}\left[\frac{1}{s} + \frac{1}{s+2} - \frac{2}{s+1}\right] \tag{5.292}$$

Taking the inverse transform term by term the solution is

$$y(t) = e^{-t} - e^{-2t} + \left[\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\right]u_2(t). \tag{5.293}$$

■

Example 5.20 #5 in the Homework Section of the book. Solve the following IVP

$$y'' + 3y' + 2y = f(t), \quad y(0) = 0, \quad y'(0) = 0 \tag{5.294}$$

where

$$f(t) = \begin{cases} 1 & \text{if } 0 \le t < 10 \\ 0 & \text{if } 10 \le t < \infty \end{cases} \tag{5.295}$$

Finding the Laplace Transform of both sides of the ODE we have

$$s^2Y(s) - sy(0) - y'(0) + 3[sY(s) - y(0)] + 2Y(s) = \mathscr{L}\{f(t)\} \tag{5.296}$$

Applying the Initial Conditions

$$s^2Y(s) + 3sY(s) + 2Y(s) = \mathscr{L}\{f(t)\} \tag{5.297}$$

The transform of the forcing function is

$$\mathscr{L}\{f(t)\} = \frac{1}{s} - \frac{e^{-10s}}{s}. \tag{5.298}$$

Solving for the transform gives

$$Y(s) = \frac{1}{s(s^2+3s+2)} - \frac{e^{-10s}}{s(s^2+3s+2)}. \tag{5.299}$$

Using Partial Fractions Decomposition

$$\frac{1}{s(s^2+3s+2)} = \frac{1}{2}\left[\frac{1}{s} + \frac{1}{s+2} - \frac{2}{s+1}\right]. \tag{5.300}$$

Thus

$$\mathscr{L}^{-1}\left[\frac{1}{s(s^2+3s+2)}\right] = \frac{1}{2} + \frac{e^{-2t}}{2} - e^{-t}. \tag{5.301}$$

So using the inverse theorem for Step Functions

$$\mathscr{L}^{-1}\left[\frac{e^{-10s}}{s(s^2+3s+2)}\right] = \left[\frac{1}{2} + \frac{e^{-2t}}{2} - e^{-t}\right]u_{10}(t). \tag{5.302}$$

Hence the solution to the IVP is

$$y(t) = \frac{1}{2}[1-u_{10}(t)] + \frac{e^{-2t}}{2} - e^{-t} - \frac{1}{2}[e^{-(2t-20)} - 2e^{-(t-10)}]u_{10}(t). \tag{5.303}$$

■

5.5 Dirac Delta and the Laplace Transform

Last Time: We studied differential equations with discontinuous forcing functions. Now we want to look at when that forcing function is a **Dirac Delta**. When we considered step functions we considered these to be like switches which get turned on after a given amount of time. In applications this represented a new external force applied at a certain time. What if instead we wanted to apply a large force over a short period of time? In applications this represents a hammer striking an object once or a force applied only at one instant. We want to introduce the Dirac Delta "function" to accomplish this goal.

5.5.1 The Dirac Delta

Paul Dirac (1902 - 1984) was a British Physicist who studied quantum Mechanics. Famous Quote on Poetry: "In Science one tries to tell people, in such a way as to be understood by everyone, something that no one ever knew before. But in poetry, it's exactly the opposite."

There are several ways to define the Dirac delta, but we will define it so it satisfies the following properties:

Definition 5.5.1 *(Dirac Delta)* The Dirac delta at $t = c$, denoted $\delta(t-c)$, satisfies the following:
(1) $\delta(t-c) = 0$, when $t \neq c$
(2) $\int_{c-\varepsilon}^{c+\varepsilon} \delta(t-c)dt = 1$, for any $\varepsilon > 0$
(3) $\int_{c-\varepsilon}^{c+\varepsilon} f(t)\delta(t-c)dt = f(c)$, for any $\varepsilon > 0$.

We can think of $\delta(t-c)$ as having an "infinite" value at $t = c$, so that its total energy is 1, all concentrated at a single point. So think of the Dirac delta as an instantaneous impulse at time $t = c$. The second and third properties will not work when the limits are the endpoints of any interval including $t = c$. This function is zero everywhere but one point, and yet the integral is 1. This is why it is a "function", the Dirac delta is not really a function, but in higher mathematics it is referred to as a **generalized function** or a **distribution**.

5.5.2 Laplace Transform of the Dirac Delta

Before we try solving IVPs with Dirac Deltas, we will need to know its Laplace Transform. By definition

$$\mathscr{L}\{\delta(t-c)\} = \int_0^\infty e^{-st}\delta(t-c)dt = e^{-cs} \tag{5.304}$$

by the third property of the Dirac delta. Notice that this requires $c > 0$ or the integral would just vanish. Now let's try solving some IVPs involving Dirac deltas

■ **Example 5.21** Solve the following initial value problem

$$y'' + 3y' - 10y = 4\delta(t-2), \quad y(0) = 2, \quad y'(0) = -3 \tag{5.305}$$

We begin by taking the Laplace Transform of both sides of the equation.

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + 3(sY(s) - y(0)) - 10Y(s) &= 4e^{-2s} && (5.306) \\ (s^2 + 3s - 10)Y(s) - 2s - 3 &= 4e^{-2s} && (5.307) \end{aligned}$$

So

$$\begin{aligned} Y(s) &= \frac{4e^{-2s}}{(s+5)(s-2)} + \frac{2s+3}{(s+5)(s-2)} && (5.308) \\ &= Y_1(s)e^{-2s} + Y_2(s) && (5.309) \end{aligned}$$

By partial fractions decomposition we have

$$\begin{aligned} Y_1(s) &= \frac{4}{(s+5)(s-2)} = \frac{4}{7}\frac{1}{s-2} - \frac{4}{7}\frac{1}{s+5} && (5.310) \\ Y_2(s) &= \frac{2s+3}{(s+5)(s-2)} = \frac{1}{s-2} + \frac{1}{s+5} && (5.311) \end{aligned}$$

Take inverse Laplace Transforms to get

$$\begin{aligned} y_1(t) &= \frac{4}{7}e^{2t} - \frac{4}{7}e^{-5t} && (5.312) \\ y_2(t) &= e^{2t} + e^{-5t} && (5.313) \end{aligned}$$

and the solution is then

$$\begin{aligned} y(t) &= y_1(t-2)u_2(t)+y_2(t) && (5.314)\\ &= u_2(t)\left(\frac{4}{7}e^{2(t-2)}-\frac{4}{7}e^{-5(t-2)}\right)+e^{2t}+e^{-5t} && (5.315)\\ &= u_2(t)\left(\frac{4}{7}e^{2t-4}-\frac{4}{7}e^{-5t-10}\right)+e^{2t}+e^{-5t} && (5.316) \end{aligned}$$

■

Notice that even though the exponential in the transform $Y(s)$ came originally from the delta, once we inverse the transform the corresponding term becomes a step function. This is generally the case, because there is a relationship between the step function $u_c(t)$ and the delta $\delta(t-c)$.

We begin with the integral

$$\begin{aligned} \int_{-\infty}^{t}\delta(u-c)du &= \begin{cases} 0, & t<c \\ 1, & t>c \end{cases} && (5.317)\\ &= u_c(t) && (5.318) \end{aligned}$$

The Fundamental Theorem of Calculus says

$$u_c'(t) = \frac{d}{dt}\left(\int_{-\infty}^{t}\delta(u-c)du\right) = \delta(t-c) \qquad (5.319)$$

Thus the Dirac delta at $t=c$ is the derivative of the step function at $t=c$, which we can think of geometrically by remembering that the graph of $u_c(t)$ is horizonatal at every $t \neq c$, hence at those points $u_c'(t)=0$ and it has a jump of one at $t=c$.

■ **Example 5.22** Solve the following initial value problem.

$$y''+4y'+9y = 2\delta(t-1)+e^t, \quad y(0)=0, \quad y'(0)=-1 \qquad (5.320)$$

First we Laplace Transform both sides and solve for $Y(s)$.

$$\begin{aligned} s^2Y(s)-sy(0)-y'(0)+4(sY(s)-y(0))+9Y(s) &= 2e^{-s}+\frac{1}{s-1} && (5.321)\\ (s^2+4s+9)Y(s)+1 &= 2e^{-s}+\frac{1}{s-1} && (5.322) \end{aligned}$$

Thus

$$\begin{aligned} Y(s) &= \frac{2e^{-s}}{s^2+4s+9}+\frac{1}{(s-1)(s^2+4s+9)}-\frac{1}{s^2+4s+9} && (5.323)\\ &= Y_1(s)e^{-s}+Y_2(s)=Y_3(s) && (5.324) \end{aligned}$$

Next, we have to prepare $Y(s)$ for the inverse transform. This will require completing the square for $Y_1(s)$ and $Y_3(s)$, while we will need to first use partial fractions decomposition on $Y_2(s)$. I leave the details to you to verify, so we now have everything in the correct form for the inverse transform

$$\begin{aligned}
Y_1(s) &= \frac{2}{s^2+4s+9} = \frac{2}{(s+2)^2+5} \\
&= \frac{2}{\sqrt{5}}\frac{\sqrt{5}}{(s+2)^2+5} \\
Y_2 &= \frac{1}{(s-1)(s^2+4s+9)} = \frac{1}{14}\left(\frac{1}{s-1} - \frac{s+5}{(s+2)^2+5}\right) \\
&= \frac{1}{14}\left(\frac{1}{s-1} - \frac{(s+2-2)+5}{(s+2)^2+5}\right) \\
&= \frac{1}{14}\left(\frac{1}{s-1} - \frac{s+2}{(s+2)^2+5} - \frac{3}{(s+2)^2+5}\right) \\
&= \frac{1}{14}\left(\frac{1}{s-1} - \frac{s+2}{(s+2)^2+5} - \frac{3}{\sqrt{5}}\frac{\sqrt{5}}{(s+2)^2+5}\right) \\
Y_3(s) &= \frac{1}{(s^2+4s+9)} = \frac{1}{(s+2)^2+5} \\
&= \frac{1}{\sqrt{5}}\frac{\sqrt{5}}{(s+2)^2+5}
\end{aligned}$$

So their inverse transforms are

$$y_1(t) = \frac{2}{\sqrt{5}}3^{-2t}\sin(\sqrt{5}t) \tag{5.325}$$

$$y_2(t) = \frac{1}{14}(e^t - e^{-2t}\cos(\sqrt{5}t) - \frac{3}{\sqrt{5}}e^{-2t}\sin(\sqrt{5}t)) \tag{5.326}$$

$$y_3(t) = \frac{1}{\sqrt{5}}e^{-2t}\sin(\sqrt{5}t). \tag{5.327}$$

Thus, since our original transformed function was

$$Y(s) = Y_1(s)e^{-s} + Y_2(s) - Y_3(s) \tag{5.328}$$

we obtain

$$y(t) = u_1(t)y_1(t-1) + y_2(t) - y_3(t) \tag{5.329}$$

$$= u_1(t)\left(\frac{2}{\sqrt{5}}e^{-2t+2}\sin(\sqrt{5}t - \sqrt{5})\right) \tag{5.330}$$

$$+ \frac{1}{14}\left(e^t - e^{-2t}\cos(\sqrt{5}t) - \frac{3}{\sqrt{5}}e^{-2t}\sin(\sqrt{5}t)\right) \tag{5.331}$$

$$- \frac{1}{\sqrt{5}}e^{-2t}\sin(\sqrt{5}t). \tag{5.332}$$

■

■ **Example 5.23** Solve the following initial value problem.

$$y' + 16y = 2u_3(t) + 5\delta(t-1), \quad y(0) = 1, \quad y'(0) = 2 \tag{5.333}$$

Again, we begin by taking the Laplace Transform of the entire equation and applying our initial conditions, we get

$$s^2Y(s) - sy(0) - y'(0) + 16Y(s) = \frac{2e^{-3s}}{s} + 5e^{-s} \tag{5.334}$$
$$(s^2+16)Y(s) - s - 2 = \frac{2e^{-3s}}{s} + 5e^{-s} \tag{5.335}$$

Solving for $Y(s)$

$$Y(s) = \frac{2e^{-3s}}{s(s^2+16)} + \frac{5e^{-s}}{s^2+16} + \frac{s+2}{s^2+16} \tag{5.336}$$
$$= Y_1(s)e^{-3s} + Y_2(s)e^{-s} + Y_3(s) \tag{5.337}$$

The only one of these we need to use partial fractions on is the first one. The rest can be dealt with directly, all they need is a little modification.

$$Y_1(s) = \frac{2}{s(s^2+16)} = \frac{1}{8}\frac{1}{s} - \frac{1}{8}\frac{s}{s^2+16} \tag{5.338}$$
$$Y_2(s) = \frac{5}{s^2+16} = \frac{5}{4}\frac{4}{s^2+16} \tag{5.339}$$
$$Y_3(s) = \frac{s+2}{s^2+16} = \frac{s}{s^2+16} + \frac{1}{2}\frac{4}{s^2+16} \tag{5.340}$$

and so the associated inverse transforms are

$$y_1(t) = \frac{1}{8} - \frac{1}{8}\cos(4t) \tag{5.341}$$
$$y_2(t) = \frac{5}{4}\sin(4t) \tag{5.342}$$
$$y_3(t) = \cos(4t) + \frac{1}{2}\sin(4t). \tag{5.343}$$

Our solution is the inverse transform of

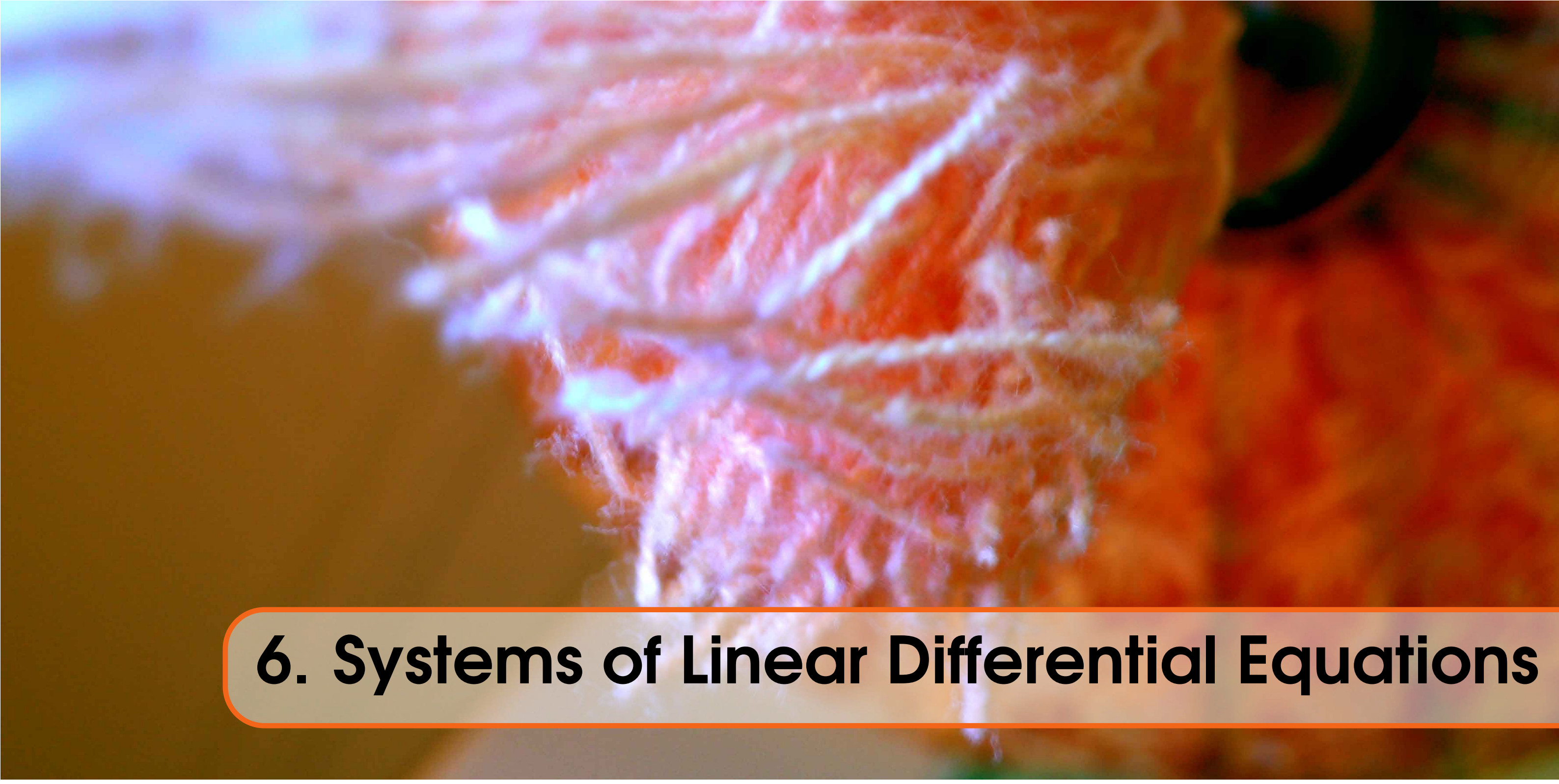
$$Y(s) = Y_1(s)e^{-3s} + Y_2(s)e^{-s} + Y_3(s) \tag{5.344}$$

and this will be

$$y(t) = u_3(t)y_1(t-3) + u_1(t)y_2(t-1) + y_3(t) \tag{5.345}$$
$$= u_3(t)\left(\frac{1}{8} - \frac{1}{8}\cos(4(t-3))\right) + \frac{5}{4}u_1(t)\sin(4(t-1)) + \cos(4t) + \frac{1}{2}\sin(4t) \tag{5.346}$$
$$= u_3(t)\left(\frac{1}{8} - \frac{1}{8}\cos(4t-12)\right) + \frac{5}{4}u_1(t)\sin(4t-4) + \cos(4t) + \frac{1}{2}\sin(4t) \tag{5.347}$$

■

V Systems of Differential Equations



6. Systems of Linear Differential Equations

6.1 Systems of Differential Equations

To this point we have focused on solving a single equation, but may real world systems are given as a system of differential equations. An example is Population Dynamics, Normally the death rate of a species is not a constant but depends on the population of predators. An example of a system of first order linear equations is

$$x_1' = 3x_1 + x_2 \tag{6.1}$$
$$x_2' = 2x_1 - 4x_2 \tag{6.2}$$

We call a system like this **coupled** because we need to know what x_1 is to know what x_2 is and vice versa. It is important to note that there will be a lot of similarities between our discussion and the previous sections on second and higher order linear equations. This is because any higher order linear equation can be written as a system of first order linear differential equations.

■ **Example 6.1** Write the following second order differential equation as a system of first order linear differential equations

$$y'' + 4y' - y = 0, \quad y(0) = 2, \quad y'(0) = -2 \tag{6.3}$$

All that is required to rewrite this equation as a first order system is a very simple change of variables. In fact, this is **ALWAYS** the change of variables to use for a problem like this. We set

$$x_1(t) = y(t) \tag{6.4}$$
$$x_2(t) = y'(t) \tag{6.5}$$

Then we have

$$x_1' = y' = x_2 \tag{6.6}$$
$$x_2' = y'' = y - 4y' = x_1 - 4x_2 \tag{6.7}$$

Notice how we used the original differential equation to obtain the second equation. The first equation, $x_1' = x_2$, is always something you should expect to see when doing this. All we have left to do is to convert the initial conditions.

$$\begin{aligned} x_1(0) &= y(0) = 2 && (6.8) \\ x_2(0) &= y'(0) = -2 && (6.9) \end{aligned}$$

Thus our original initial value problem has been transformed into the system

$$\begin{aligned} x_1' &= x_2, \quad x_1(0) = 2 && (6.10) \\ x_2' &= x_1 - 4x_2, \quad x_2(0) = -2 && (6.11) \end{aligned}$$

■

Let's do an example for higher order linear equations.

■ **Example 6.2**

$$y^{(4)} + ty''' - 2y'' - 3y' - y = 0 \tag{6.12}$$

as a system of first order differential equations.

We want to use a similar change of variables as the previous example. The only difference is that since our equation in this example is fourth order we will need four new variables instead of two.

$$\begin{aligned} x_1 &= y && (6.13) \\ x_2 &= y' && (6.14) \\ x_3 &= y'' && (6.15) \\ x_4 &= y''' && (6.16) \end{aligned}$$

Then we have

$$\begin{aligned} x_1' &= y' = x_2 && (6.17) \\ x_2' &= y'' = x_3 && (6.18) \\ x_3' &= y''' = x_4 && (6.19) \\ x_4' &= y^{(4)} = y + 3y' + 2y'' - ty''' = x_1 + 3x_2 + 2x_3 - tx_4 && (6.20) \end{aligned}$$

as our system of equations. To be able to solve these, we need to review some facts about systems of equations and linear algebra. ■

6.2 Review of Matrices

6.2.1 Systems of Equations

In this section we will restrict our attention only to the linear algebra that might come up when studying systems of differential equations. This is far from a complete treatment, so if you're curious, taking a linear algebra course is either mandatory for your major or a good idea to be more well-rounded.

Suppose we start with a system of n equations with n unknowns $x_1, x_2, \ldots, x_n$.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 && (6.21)\\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 && (6.22)\\ . \quad . &\quad . && (6.23)\\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n && (6.24)\\ & && (6.25) \end{aligned}$$

Here's the basic fact about linear systems of equations with the same number of unknowns as equations.

Theorem 6.2.1 Given a system of n equations with n unknowns, there are three possibilities for the number of solutions:
(1) No Solutions
(2) Exactly One Solution
(3) Infinitely Many Solutions

Definition 6.2.1 A system of equations is called **nonhomogeneous** if at least one $b_i \neq 0$. If every $b_i = 0$, the system is called **homogeneous**. A homogeneous system has the following form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 && (6.26)\\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 && (6.27)\\ . \quad . &\quad . && (6.28)\\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= 0 && (6.29)\\ & && (6.30) \end{aligned}$$

Notice that there is always at least one solution given by

$$x_1 = x_2 = \ldots = x_n = 0 \qquad (6.31)$$

This solution is called the **trivial solution**. This means that it is impossible for a homogeneous system to have zero solutions, and Theorem 1 can be modified as follows

Theorem 6.2.2 Given a homogeneous system of n equations with n unknowns, there are two possibilities for the number of solutions:
(1) Exactly one solution, the trivial solution
(2) Infinitely many non-zero Solutions in addition to the trivial solution.

6.2.2 Linear Algebra

While we could solve the homogeneous and nonhomogeneous systems directly, we have very powerful tools available to us. The main objects of study in linear algebra are **Matrices** and **Vectors**.

An $n \times n$ matrix (referred to as an n-dimensional matrix) is an array of numbers with n rows and n columns. It is possible to consider matrices with different numbers of rows and columns, but this is more general than we need for this course. An $n \times n$ matrix has the form

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \tag{6.32}$$

There is one special matrix we will need to be familiar with. This is the n-dimensional **Identity Matrix**.

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \tag{6.33}$$

We will focus on 2×2 matrices in this class. The principles we discuss extend to higher dimensions, but computationally 2×2 matrices are much easier.

Matrix addition and subtraction are fairly straightforward. Do everything component-wise. If we multiply by a constant, **Scalar Multiplication**, we multiply each component by a constant.

■ **Example 6.3** Given the matrices

$$A = \begin{pmatrix} 3 & 1 \\ -2 & 5 \end{pmatrix} \quad B = \begin{pmatrix} -2 & 0 \\ 1 & 4 \end{pmatrix} \tag{6.34}$$

compute $A - 2B$.

The first thing to do is compute $2B$

$$2B = 2\begin{pmatrix} -2 & 0 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 2 & 8 \end{pmatrix} \tag{6.35}$$

Then we have

$$A - 2B = \begin{pmatrix} 3 & 1 \\ -2 & 5 \end{pmatrix} - \begin{pmatrix} -4 & 0 \\ 2 & 8 \end{pmatrix} \tag{6.36}$$

$$= \begin{pmatrix} 7 & 1 \\ -4 & -3 \end{pmatrix}. \tag{6.37}$$

■

Notice that these operations require the dimensions of the matrices to be equal. A **vector** is a one-dimensional array of numbers. For example

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{6.38}$$

is a vector of n unknowns. We can think of a vector as a $1 \times n$ or an $n \times 1$ dimensional matrix with regards to matrix operations.

We can multiply two matrices A and B together by "multiplying" each row in A by each column of B. That is, to find the element in the ith row and the jth column, we multiply the corresponding elements in the ith row of the first matrix and jth column of the second matrix and add these products together.

■ **Example 6.4** Compute AB, where

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 2 & -3 \end{pmatrix}. \tag{6.39}$$

So

$$AB = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & -3 \end{pmatrix} \tag{6.40}$$

$$= \begin{pmatrix} 1(0)+2(2) & 1(1)+2(-3) \\ -1(0)+3(2) & -1(1)+3(-3) \end{pmatrix} \tag{6.41}$$

$$= \begin{pmatrix} 4 & -5 \\ 6 & -10 \end{pmatrix} \tag{6.42}$$

■

Notice that $AB \neq BA$ in general. Matrix Multiplication is **NOT** commutative. We must pay special attention to the dimensions of the matrices being multiplied. If the number of columns of A do not match the number of rows of B, we cannot compute AB. Also, the identity matrix I_n is the identity for matrix multiplication, i.e. $I_nA = AI_n = A$ for any matrix A.

In particular, we can multiply an n-dimensional matrix over a vector with n-components together as in the following example

■ **Example 6.5** Compute

$$\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \end{pmatrix} \tag{6.43}$$

We proceed by "multiplying" each row in the matrix by the vector.

$$\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2(-1)+-1(4) \\ 3(-1)+2(4) \end{pmatrix} \tag{6.44}$$

$$= \begin{pmatrix} -6 \\ 5 \end{pmatrix} \tag{6.45}$$

■

R Multiplication of a matrix with a vector yields another vector. We then have an interpretation of a matrix A as a linear function on vectors.

Definition 6.2.2 *(Determinants)* Every square $(n \times n)$ matrix has a number associated to it, called the **determinant**. We will not learn how to compute determinants for $n > 2$, as the process gets more and more complicated as n increases. The standard notation for the determinant of a matrix is

$$\det(A) = |A| \tag{6.46}$$

For a 2×2 matrix, the determinant is computed using the following formula

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \tag{6.47}$$

that is, the determinant is the product of the main diagonal minus the product of the off diagonal.

■ **Example 6.6** Compute the determinants of

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \tag{6.48}$$

There is not much to do here but us the definition.

$$\det(A) = 2(2) - 3(1) = 4 - 3 = 1 \tag{6.49}$$

$$\det(B) = 1(4) - 2(2) = 4 - 4 = 0 \tag{6.50}$$

■

We call a matrix A **singular** if $\det(A) = 0$ and **nonsingular** otherwise. In the previous example, the first matrix was nonsingular while the second was singular.

Determinants give us important information about the existence of an inverse for a given matrix. The inverse of a matrix A, denoted A^{-1}, satisfies

$$AA^{-1} = A^{-1}A = I_n \tag{6.51}$$

Inverses do not necessarily exist for a given matrix.

Theorem 6.2.3 Given a matrix A,
(1) If A is nonsingular an inverse, A^{-1}, will exist.
(2) If A is singular, no inverse, A^{-1}, will exist.

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{6.52}$$

Definition 6.2.3 The **Transpose** of a matrix is switching the rows and columns so that $a_{ij} = a^T_{ji}$.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \tag{6.53}$$

6.3 Linear Independence, Eigenvalues and Eigenvectors

We return our attention now to the system of equations

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \tag{6.54}$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \tag{6.55}$$

$$\vdots \tag{6.56}$$

$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \tag{6.57}$$

$$\tag{6.58}$$

To express this system of equations in matrix form, we start by writing both sides as vectors.

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n \end{pmatrix} \tag{6.59}$$

Notice that the left side of the equation can be rewritten as a matrix-vector product.

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \tag{6.60}$$

We can simplify this notation by writing

$$A\mathbf{x} = b \tag{6.61}$$

where $\mathbf{x}$ is the vector whose entries are variables in the system, A is the matrix of coefficients of the system (called the **coefficient matrix**), and $\mathbf{b}$ is the vector whose entries are the right-hand side of the equations. We call Equation (6.61) the **matrix form** of the system of equations.

We know that the system of equations has zero one or infinitely many solutions. Suppose $\det(A) \neq 0$, i.e. A is nonsingular. Then Equation (6.61) has only one solution

$$\mathbf{x} = A^{-1}\mathbf{b}. \tag{6.62}$$

So we can rewrite our earlier Theorem from last lecture as

Theorem 6.3.1 Given the system of equations (6.61),
(1) If $\det(A) \neq 0$, there is exactly one solution,
(2) If $\det(A) = 0$, there are either zero or infinitely many solutions.

Recall that if the system were homogeneous, each $b_i = 0$, we always have the trivial solution $x_i = 0$. Denoting the vector with entries all 0 by $\mathbf{0}$, the matrix form of a homogeneous system is

$$A\mathbf{x} = \mathbf{0} \tag{6.63}$$

Thus we can express the earlier Theorem 2 from last lecture as

Theorem 6.3.2 Given the homogeneous system of equations,
(1) If $\det(A) \neq 0$, there is exactly one solution $\mathbf{x} = \mathbf{0}$,
(2) If $\det(A) = 0$, there will be infinitely many nonzero solutions.

6.3.1 Eigenvalues and Eigenvectors

The following is probably the most important aspect of linear algebra. We have already observed if we multiply a vector by a matrix, we get another vector, i.e.,

$$A\eta = \mathbf{y} \tag{6.64}$$

A natural question to ask is when $\mathbf{y}$ is just a scalar multiple of η. In other words, for what vectors η is multiplication by A equivalent to scaling η, or

$$A\eta = \lambda\eta \tag{6.65}$$

If (6.65) is satisfied for some constant λ and some vector η, we call η an **eigenvector** of A with **eigenvalue** λ. We first notice if $\eta = 0$, (6.65) will be satisfied for any λ. We are not interested in that case, so in general we will assume $\eta \neq 0$.

So how can we find solutions to (6.65)? Start by rewriting it, recalling that I is the 2×2 identity matrix.

$$A\eta = \lambda\eta \tag{6.66}$$

$$A\eta - \lambda I\eta = \mathbf{0} \tag{6.67}$$

$$(A-\lambda I)\eta = \mathbf{0} \tag{6.68}$$

We had to multiply λ by the identity I before we could factor it out. This is because we cannot subtract a constant from a matrix. The last equation is the matrix form for a homogeneous system of equations. By Theorem 3 form last lecture, if $A-\lambda I$ is nonsingular ($\det(A)\neq 0$), the only solution is the trivial solution $\eta = \mathbf{0}$, which we have already said we are not interested in. On the other hand, if $A-\lambda I$ is singular, we will have infinitely many nonzero solutions. Thus the condition that we will need to find any eigenvalues and eigenvectors that may exist for A is for

$$\det(A-\lambda I) = 0 \tag{6.69}$$

It is a basic fact that this equation is an nth degree polynomial if A is an $n\times n$ matrix. This is called the **characteristic equation** of the matrix A.

As a result, the Fundamental Theorem of Algebra tells us that an $n\times n$ matrix A has n eigenvalues, counting multiplicities. To find them, all we have to do is to find the roots of an nth degree polynomial, which is no problem for small n. Suppose we have found these eigenvalues. What can we conclude about their associated eigenvectors?

Definition 6.3.1 We call k vectors $\mathbf{x_1},\mathbf{x_2},\ldots,\mathbf{x_k}$ **linearly independent** if the only constants $c_1,c_2,\ldots,c_k$ satisfying

$$c_1\mathbf{x_1}+c_2\mathbf{x_2}+\ldots+c_k\mathbf{x_k} = 0 \tag{6.70}$$

are $c_1 = c_2 = \ldots = c_k = 0$. This definition should look familiar. This is an identical definition to our earlier definition of linear independence for functions.

Theorem 6.3.3 If $\lambda_1,\lambda_2,\ldots,\lambda_n$ is the complete list of eigenvalues of A, including multiplicities, then
(1) If λ occurs only once in the list it is called **simple**
(2) If λ occurs $k>1$ times it has multiplicity k
(3) If $\lambda_1,\lambda_2,\ldots,\lambda_k (k\leq n)$ are the simple eigenvalues of A with corresponding eigenvectors $\eta^{(\mathbf{1})},\eta^{(\mathbf{2})},\ldots,\eta^{(\mathbf{k})}$, then these eigenvectors $\eta^{(\mathbf{i})}$ are linearly independent.
(4) If λ is an eigenvalue with multiplicity k, then λ will have anywhere from 1 to k linearly independent eigenvectors.

This fact should look familiar from our discussion of second and higher order equations. This theorem tells us when we have linearly independent eigenvectors, which is useful when trying to solve systems of differential equations. Now once we have the eigenvalues, how do we find the associated eigenvectors?

■ **Example 6.7** Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}. \tag{6.71}$$

The first thing we need to do is to find the roots of the characteristic equation of the matrix

$$A - \lambda I = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3-\lambda & 4 \\ 2 & 1-\lambda \end{pmatrix} \tag{6.72}$$

This is

$$0 = \det(A - \lambda I) = (3-\lambda)(1-\lambda) - 8 = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1). \tag{6.73}$$

This the two eigenvalues of A are $\lambda_1 = 5$ and $\lambda_2 = -1$. Now to find the eigenvectors we need to plug each eigenvalue into $(A - \lambda I)\eta = 0$ and solve for η.

(1) $\lambda_1 = 5$

In this case, we have the following system

$$\begin{pmatrix} -2 & 4 \\ 2 & -4 \end{pmatrix} \eta = \mathbf{0} \tag{6.74}$$

Next, we will write out components of the two vectors and multiply through

$$\begin{pmatrix} -2 & 4 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6.75}$$

$$\begin{pmatrix} -2\eta_1 + 4\eta_2 \\ 2\eta_1 - 4\eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6.76}$$

For this vector equation to hold, the components must match up. So we have got to find a solution to the system

$$-2\eta_1 + 4\eta_2 = 0 \tag{6.77}$$

$$2\eta_1 - 4\eta_2 = 0. \tag{6.78}$$

Notice that these are the same equation, but differ by a constant, in this case -1. This will always be the case if we have found our eigenvalues correctly, since we know that $A - \lambda I$ is singular and so our system should have infinitely many solutions.

Since the equations are basically the same we need to choose one and obtain a relation between eigenvector components η_1 and η_2. Let's choose the first. This gives

$$2\eta_1 = 4\eta_2. \tag{6.79}$$

and so we have $\eta_1 = 2\eta_2$. As a result, any eigenvector corresponding to $\lambda_1 = 5$ has the form

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 2\eta_2 \\ \eta_2 \end{pmatrix}. \tag{6.80}$$

There are infinitely many vectors of this form, we need only one. We can select one by choosing a value for η_2. The only restriction is we do not want to pick $\eta_2 = 0$, since then $\eta = \mathbf{0}$, which we want to avoid. We may choose, for example, $\eta_2 = 1$, and then we have

$$\eta^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \tag{6.81}$$

(2) $\lambda_2 = -1$

In the previous case we went into more detail than we will in future examples. The process is the same. Plugging λ_2 into $(A - \lambda I)\eta = 0$ gives the system

$$\begin{aligned} \begin{pmatrix} 4 & 4 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} && (6.82) \\ \begin{pmatrix} 4\eta_1 + 4\eta_2 \\ 2\eta_1 + 2\eta_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} && (6.83) \end{aligned}$$

The two equations corresponding to this vector equation are

$$\begin{aligned} 4\eta_1 + 4\eta_2 &= 0 && (6.84) \\ 2\eta_1 + 2\eta_2 &= 0. && (6.85) \end{aligned}$$

Once again, these differ by a constant factor. Solving the first equation we find

$$\eta_1 = -\eta_2 \tag{6.86}$$

and so any eigenvector has the form

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\eta_2 \\ \eta_2 \end{pmatrix}. \tag{6.87}$$

We can choose $\eta_2 = 1$, giving us a second eigenvector of

$$\eta^{(\mathbf{2})} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{6.88}$$

Summarizing the eigenvalue/eigenvector pairs of A are

$$\begin{aligned} \lambda_1 &= 5 \qquad \eta^{(\mathbf{1})} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} && (6.89) \\ \lambda_2 &= -1 \qquad \eta^{(\mathbf{2})} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. && (6.90) \end{aligned}$$

■

REMARK: We could have ended up with any number of different values for our eigenvectors $\eta^{(\mathbf{1})}$ and $\eta^{(\mathbf{2})}$, depending on the choices we made at the end. However, they would have only differed by a multiplicative constant.

■ **Example 6.8** Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 2 & -1 \\ 5 & 4 \end{pmatrix}. \tag{6.91}$$

The characteristic equation for this matrix is

$$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} && (6.92) \\ &= (2-\lambda)(4-\lambda)+5 && (6.93) \\ &= \lambda^2 - 6\lambda + 13 && (6.94) \end{aligned}$$

By completing the square (or quadratic formula), we see that the roots are $r_{1,2} = 3 \pm 2i$. If we get complex eigenvalues, to find the eigenvectors we proceed as we did in the previous example.

(1) $\lambda_1 = 3+2i$

Here the matrix equation

$$(A - \lambda I)\eta = 0 \tag{6.95}$$

becomes

$$\begin{aligned} \begin{pmatrix} -1-2i & -1 \\ 5 & 1-2i \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} && (6.96) \\ \begin{pmatrix} (-1-2i)\eta_1 - \eta_2 \\ 5\eta_1 + (1-2i)\eta_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} && (6.97) \end{aligned}$$

So the pair of equations we get are

$$\begin{aligned} (-1-2i)\eta_1 - \eta_2 &= 0 && (6.98) \\ 5\eta_1 + (1-2i)\eta_2 &= 0. && (6.99) \end{aligned}$$

It is not as obvious as the last example, but these two equations are scalar multiples. If we multiply the first equation by $-1+2i$, we recover the second. Now we choose one of these equations to work with. Let's use the first. This gives us that $\eta_2 = (-1-2i)\eta_1$, so any vector has the form

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ (-1-2i)\eta_1 \end{pmatrix}. \tag{6.100}$$

Choosing $\eta_1 = 1$ gives a first eigenvector of

$$\eta^{(\mathbf{1})} = \begin{pmatrix} 1 \\ -1-2i \end{pmatrix} \tag{6.101}$$

(2) $\lambda_1 = 3-2i$

Here the matrix equation

$$(A - \lambda I)\eta = \mathbf{0} \tag{6.102}$$

becomes

$$\begin{pmatrix} -1+2i & -1 \\ 5 & 1+2i \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6.103}$$

$$\begin{pmatrix} (-1+2i)\eta_1 - \eta_2 \\ 5\eta_1 + (1+2i)\eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6.104}$$

So the pair of equations we get are

$$(-1+2i)\eta_1 - \eta_2 = 0 \tag{6.105}$$

$$5\eta_1 + (1+2i)\eta_2 = 0 \tag{6.106}$$

Let's use the first equation again. This gives us that $\eta_2 = (-1+2i)\eta_1$, so any eigenvector has the form

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ (-1+2i)\eta_1 \end{pmatrix}. \tag{6.107}$$

Choosing $\eta_1 = 1$ gives a second eigenvector of

$$\eta^{(2)} = \begin{pmatrix} 1 \\ -1+2i \end{pmatrix}. \tag{6.108}$$

To summarize, A has the following eigenvalue/eigenvector pairs

$$\lambda_1 = 3-2i \qquad \begin{pmatrix} 1 \\ -1-2i \end{pmatrix} \tag{6.109}$$

$$\lambda_2 = 3+2i \qquad \begin{pmatrix} 1 \\ -1+2i \end{pmatrix} \tag{6.110}$$

■

R Notice that the eigenvalues came in **complex conjugate** pairs, i.e. in the form $a \pm bi$. This is always the case for complex roots, as we can easily see from the quadratic formula. Moreover, the complex entries in the eigenvectors were also complex conjugates, and the real entries were the same up to multiplication by a constant. This is always the case as long as A does not have any complex entries.

6.4 Homogeneous Linear Systems with Constant Coefficients

Last Time: We studied linear independence, eigenvalues, and eigenvectors.

6.4.1 Solutions to Systems of Differential Equations

A two-dimensional equation has the form

$$x' = ax + by \tag{6.111}$$

$$y' = cx + dy \tag{6.112}$$

Suppose we have got our system written in matrix form

$$x' = Ax \tag{6.113}$$

How do we solve this equation? If A were a 1×1 matrix, i.e. a constant, and x were a vector with 1 component, the differential equation would be the separable equation

$$x' = ax \tag{6.114}$$

We know this is solved by

$$x(t) = ce^{at}. \tag{6.115}$$

One might guess, then, that in the $n \times n$ case, instead of a we have some other constant in the exponential, and instead of the constant of integration c we have some constant vector η. So our guess for the solution will be

$$x(t) = \eta e^{rt}. \tag{6.116}$$

Plugging the guess into the differential equation gives

$$\begin{aligned} r\eta e^{rt} &= A\eta e^{rt} && (6.117) \\ (A\eta - r\eta)e^{rt} &= 0 && (6.118) \\ (A - rI)\eta e^{rt} &= 0. && (6.119) \end{aligned}$$

Since $e^{rt} \neq 0$, we end up with the requirement that

$$(A - rI)\eta = 0 \tag{6.120}$$

This should seem familiar, it is the condition for η to be an eigenvector of A with eigenvalue r. Thus, we conclude that for (6.116) to be a solution of the original differential equation, we must have η an eigenvalue of A with eigenvalue r.

That tells us how to get some solutions to systems of differential equations, we find the eigenvalues and vectors of the coefficient matrix A, then form solutions using (6.116). But how will we form the general solution?

Thinking back to the second/higher order linear case, we need enough linearly independent solutions to form a fundamental set. As we noticed last lecture, if we have all simple eigenvalues, then all the eigenvectors are linearly independent, and so the solutions formed will be as well. We will handle the case of repeated eigenvalues later.

So we will find the fundamental solutions of the form (6.116), then take their linear combinations to get our general solution.

6.4.2 The Phase Plane

We are going to rely on qualitatively understanding what solutions to a linear system of differential equations look like, this will be important when considering nonlinear equations. We know the trivial solution $x = 0$ is always a solution to our homogeneous system $x' = Ax$. $x = 0$ is an example of an **equilibrium solution**, i.e. it satisfies

$$x' = Ax = 0 \tag{6.121}$$

and is a constant solution. We will assume our coefficient matrix A is nonsingular ($\det(A) \neq 0$), thus $x = 0$ is the only equilibrium solution.

The question we want to ask is whether other solutions move towards or away from this constant solution as $t \to \pm\infty$, so that we can understand the long term behavior of the system. This is no different than what we did when we classified equilibrium solutions for first order autonomous equations, we will generalize the ideas to systems of equations.

When we drew solution spaces then, we did so on the ty-plane. To do something analogous we would require three dimensions, since we would have to sketch both x_1 and x_2 vs. t. Instead, what we do is ignore t and think of our solutions as trajectories on the x_1x_2-plane. Then our equilibrium solution is the origin. The x_1x_2-plane is called the **phase plane**. We will see examples where we sketch solutions, called **phase portraits**.

6.4.3 Real, Distinct Eigenvalues

Lets get back to the equation $x' = Ax$. We know if λ_1 and λ_2 are real and distinct eigenvalues of the 2×2 coefficient matrix A associated with eigenvectors $\eta^{(1)}$ and $\eta^{(2)}$, respectively. We know from above $\eta^{(1)}$ and $\eta^{(2)}$ are linearly independent, as λ_1 and λ_2 are simple. Thus the solutions obtained from them using (6.116) will also be linearly independent, and in fact will form a fundamental set of solutions. The general solution is

$$x(t) = c_1 e^{\lambda_1 t}\eta^{(1)} + c_2 e^{\lambda_2 t}\eta^{(2)} \tag{6.122}$$

So if we have real, distinct eigenvalues, all that we have to do is find the eigenvectors, form the general solution as above, and use any initial conditions that may exist.

■ **Example 6.9** Solve the following initial value problem

$$x' = \begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix} x \quad x(0) = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{6.123}$$

The first thing we need to do is to find the eigenvalues of the coefficient matrix.

$$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{vmatrix} -2-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} && (6.124) \\ &= \lambda^2 + \lambda - 6 && (6.125) \\ &= (\lambda - 2)(\lambda + 3) && (6.126) \end{aligned}$$

So the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -3$. Next we need the eigenvectors.

(1) $\lambda_1 = 2$

$$\begin{aligned} (A - 2I)\eta &= 0 && (6.127) \\ \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} && (6.128) \end{aligned}$$

So we will want to find solutions to the system

$$\begin{aligned} -4\eta_1 + 2\eta_2 &= 0 && (6.129) \\ 2\eta_1 - \eta_2 &= 0. && (6.130) \end{aligned}$$

Using either equation we find $\eta_2 = 2\eta_1$, and so any eigenvector has the form

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ 2\eta_1 \end{pmatrix} \tag{6.131}$$

Choosing $\eta_1 = 1$ we obtain the first eigenvector

$$\eta^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{6.132}$$

(2) $\lambda_2 = -3$

$$(A+3I)\eta = 0 \tag{6.133}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6.134}$$

So we will want to find solutions to the system

$$\eta_1 + 2\eta_2 = 0 \tag{6.135}$$

$$2\eta_1 + 4\eta_2 = 0. \tag{6.136}$$

Using either equation we find $\eta_1 = -2\eta_2$, and so any eigenvector has the form

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -2\eta_2 \\ \eta_2 \end{pmatrix}. \tag{6.137}$$

Choosing $\eta_2 = 1$ we obtain the second eigenvector

$$\eta^{(2)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}. \tag{6.138}$$

Thus our general solution is

$$x(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}. \tag{6.139}$$

Now let's use the initial condition to solve for c_1 and c_2. The condition says

$$\begin{pmatrix} 5 \\ 0 \end{pmatrix} = x(0) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} -2 \\ 1 \end{pmatrix}. \tag{6.140}$$

All that's left is to write out is the matrix equation as a system of equations and then solve.

$$c_1 - 2c_2 = 5 \tag{6.141}$$

$$2c_1 + c_2 = 0 \quad \Rightarrow c_1 = 1, c_2 = -2 \tag{6.142}$$

Thus the particular solution is

$$x(t) = e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2e^{-3t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}. \tag{6.143}$$

■

■ **Example 6.10** Sketch the phase portrait of the system from Example 1.

In the last example we saw that the eigenvalue/eigenvector pairs for the coefficient matrix were

$$\lambda_1 = 2 \qquad \eta^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{6.144}$$

$$\lambda_2 = -3 \qquad \eta^{(2)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}. \tag{6.145}$$

The starting point for the phase portrait involves sketching solutions corresponding to the eigenvectors (i.e. with c_1 or $c_2 = 0$). We know that if $x(t)$ is one of these solutions

$$x'(t) = Ac_i e^{\lambda_i t}\eta^{(i)} = c_i\lambda_i e^{\lambda_i t}\eta^{(i)}. \tag{6.146}$$

This is just, for any t, a constant times the eigenvector, which indicates that lines in the direction of the eigenvector are these solutions to the system. There are called **eigensolutions** of the system.

Next, we need to consider the direction that these solutions move in. Let's start with the first eigensolution, which corresponds to the solution with $c_2 = 0$. The first eigenvalue is $\lambda_1 = 2 > 0$. This indicates that this eigensolution will grow exponentially, as the exponential in the solution has a positive exponent. The second eigensolution corresponds to $\lambda_2 = -3 < 0$, so the exponential in the appropriate solution is negative. Hence this solution will decay and move towards the origin.

What does the typical trajectory do (i.e. a trajectory where both $c_1, c_2 \neq 0$)? The general solution is

$$x(t) = c_1 e^{2t}\eta^{(1)} + c_2 e^{-3t}\eta^{(2)}. \tag{6.147}$$

Thus as $t \to \infty$, this solution will approach the positive eigensolution, as the component corresponding to the negative eigensolution will decay away. On the other hand, as $t \to -\infty$, the trajectory will asymptotically reach the negative eigensolution, as the positive eigensolution component will be tiny. The end result is the phase portrait as in Figure 1. When the phase portrait looks like this (which happens in all cases with eigenvalues of mixed signs), the equilibrium solution at the origin is classified as a **saddle point** and is **unstable**.

■

■ **Example 6.11** Solve the following initial value problem.

$$x_1' = 4x_1 + x_2 \quad x_1(0) = 6 \tag{6.148}$$

$$x_2' = 3x_1 + 2x_2 \quad x_2(0) = 2 \tag{6.149}$$

Before we can solve anything, we need to convert this system into matrix form. Doing so converts the initial value problem to

$$x' = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} x \quad x(0) = \begin{pmatrix} 6 \\ 2 \end{pmatrix}. \tag{6.150}$$

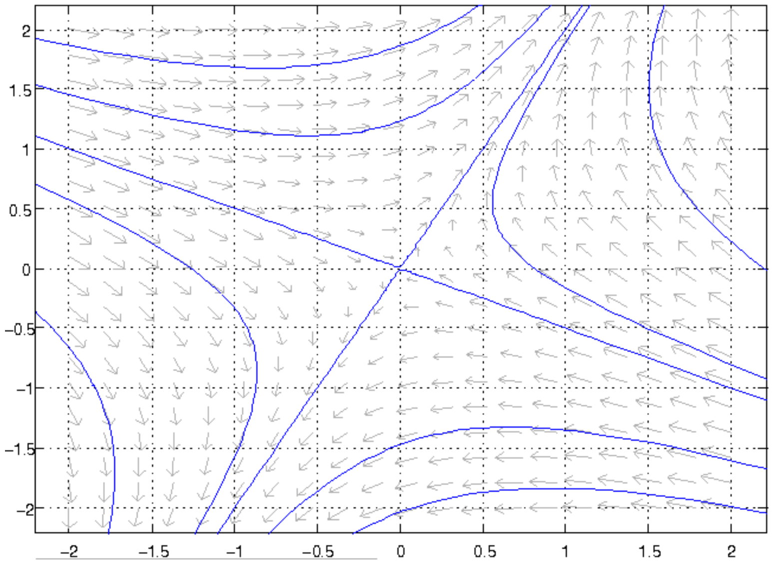


Figure 6.1: Phase Portrait of the saddle point in Example 1

To solve, the first thing we need to do is to find the eigenvalues of the coefficient matrix.

$$\begin{aligned} 0 = \det(A-\lambda I) &= \begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} && (6.151)\\ &= \lambda^2 - 6\lambda + 5 && (6.152)\\ &= (\lambda-1)(\lambda-5) && (6.153)\end{aligned}$$

So the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 5$. Next, we find the eigenvectors.

(1) $\lambda_1 = 1$

$$(A - I)\eta = 0 \qquad (6.154)$$

$$\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (6.155)$$

So we will want to find solutions to the system

$$\begin{aligned} 3\eta_1 + \eta_2 &= 0 && (6.156)\\ 3\eta_1 + \eta_2 &= 0. && (6.157)\end{aligned}$$

Using either equation we find $\eta_2 = -3\eta_1$, and so any eigenvector has the form

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ -3\eta_1 \end{pmatrix} \qquad (6.158)$$

Choosing $\eta_1 = 1$ we obtain the first eigenvector

$$\eta^{(1)} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}. \qquad (6.159)$$

(2) $\lambda_2 = 5$

$$
\begin{aligned}
(A-5I)\eta &= 0 && (6.160)\\
\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} && (6.161)
\end{aligned}
$$

So we will want to find solutions to the system

$$
\begin{aligned}
-\eta_1+\eta_2 &= 0 && (6.162)\\
3\eta_1-3\eta_2 &= 0. && (6.163)
\end{aligned}
$$

Using either equation we find $\eta_1 = \eta_2$, and so any eigenvector has the form

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_2 \\ \eta_2 \end{pmatrix}. \tag{6.164}$$

Choosing $\eta_2 = 1$ we obtain the second eigenvector

$$\eta^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{6.165}$$

Thus our general solution is

$$x(t) = c_1 e^t \begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{6.166}$$

Now using our initial conditions we solve for c_1 and c_2. The condition gives

$$\begin{pmatrix} 6 \\ 2 \end{pmatrix} = x(0) = c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{6.167}$$

All that is left is to write out this matrix equation as a system of equations and then solve

$$
\begin{aligned}
c_1+c_2 &= 6 && (6.168)\\
-3c_1+c_2 &= 2 \Rightarrow c_1 = 1, c_2 = 5 && (6.169)
\end{aligned}
$$

Thus the particular solution is

$$x(t) = e^t \begin{pmatrix} 1 \\ -3 \end{pmatrix} + 5e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{6.170}$$

■

■ **Example 6.12** Sketch the phase portrait of the system from Example 3.

In the last example, we saw that the eigenvalue/eigenvector pairs for the coefficient matrix were

$$
\begin{aligned}
\lambda_1 &= 1 \qquad \eta^{(1)} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}. && (6.171)\\
\lambda_2 &= 5 \qquad \eta^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. && (6.172)
\end{aligned}
$$

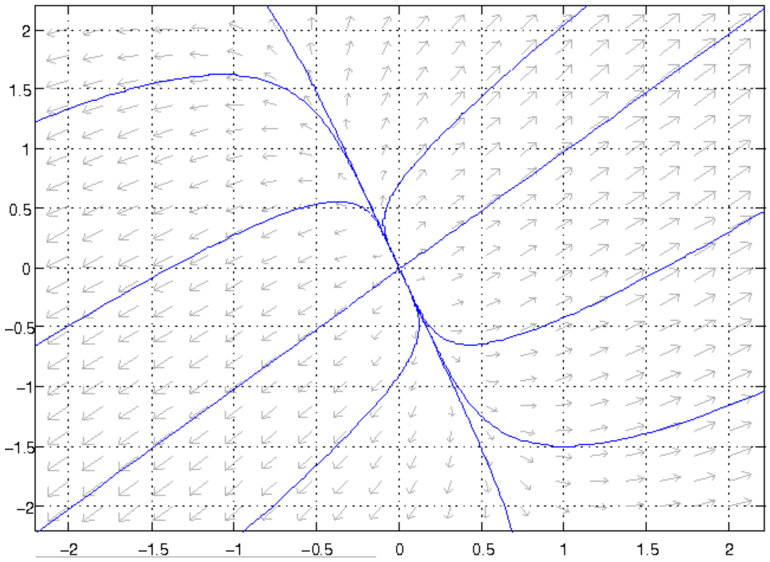


Figure 6.2: Phase Portrait of the unstable node in Example 2

We begin by sketching the eigensolutions (these are straight lines in the directions of the eigenvectors). Both of these trajectories move away from the origin, though, as the eigenvalues are both positive.

Since $|\lambda_2| > |\lambda_1|$, we call the second eigensolution the **fast eigensolution** and the first one the **slow eigensolution**. The term comes from the fact that the eigensolution corresponds to the eigenvalue with larger magnitude will either grow or decay more quickly than the other one.

As both grow in forward time, asymptotically, as $t \to \infty$, the fast eigensolution will dominate the typical trajectory, as it gets larger much more quickly than the slow eigensolution does. So in forward time, other trajectories will get closer and closer to the eigensolution corresponding to $\eta^{(2)}$. On the other hand, as $t \to -\infty$, the fast eigensolution will decay more quickly than the slow one, and so the eigensolution corresponding to $\eta^{(1)}$ will dominate in backwards time.

Thus the phase portrait will look like Figure 2. Whenever we have two positive eigenvalues, every solution moves away from the origin. We call the equilibrium solution at the origin, in this case, a **node** and classify it as being **unstable**.

■

■ **Example 6.13** Solve the following initial value problem.

$$x_1' = -5x_1 + x_2 \quad x_1(0) = 2 \tag{6.173}$$
$$x_2' = 2x_1 - 4x_2 \quad x_2(0) = -1 \tag{6.174}$$

We convert this system into matrix form.

$$x' = \begin{pmatrix} -5 & 1 \\ 2 & -4 \end{pmatrix} x \quad x(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}. \tag{6.175}$$

To solve, the first thing we need to do is to find the eigenvalues of the coefficient matrix.

$$\begin{aligned} 0=\det(A-\lambda I) &= \begin{vmatrix} -5-\lambda & 1 \\ 2 & -4-\lambda \end{vmatrix} && (6.176)\\ &= \lambda^2+9\lambda+18 && (6.177)\\ &= (\lambda+3)(\lambda+6) && (6.178)\end{aligned}$$

So the eigenvalues are $\lambda_1=-3$ and $\lambda_2=-6$. Next, we find the eigenvectors.

(1) $\lambda_1=-3$

$$(A+3I)\eta=0 \qquad (6.179)$$

$$\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (6.180)$$

So we will want to find solutions to the system

$$\begin{aligned} -2\eta_1+\eta_2 &= 0 && (6.181)\\ 2\eta_1-\eta_2 &= 0. && (6.182)\end{aligned}$$

Using either equation we find $\eta_2=2\eta_1$, and so any eigenvector has the form

$$\eta=\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}=\begin{pmatrix} \eta_1 \\ 2\eta_1 \end{pmatrix} \qquad (6.183)$$

Choosing $\eta_1=1$ we obtain the first eigenvector

$$\eta^{(1)}=\begin{pmatrix} 1 \\ 2 \end{pmatrix}. \qquad (6.184)$$

(2) $\lambda_2=-6$

$$(A+6I)\eta=0 \qquad (6.185)$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (6.186)$$

So we will want to find solutions to the system

$$\begin{aligned} \eta_1+\eta_2 &= 0 && (6.187)\\ 2\eta_1+2\eta_2 &= 0. && (6.188)\end{aligned}$$

Using either equation we find $\eta_1=-\eta_2$, and so any eigenvector has the form

$$\eta=\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}=\begin{pmatrix} -\eta_2 \\ \eta_2 \end{pmatrix}. \qquad (6.189)$$

Choosing $\eta_2=1$ we obtain the second eigenvector

$$\eta^{(2)}=\begin{pmatrix} -1 \\ 1 \end{pmatrix}. \qquad (6.190)$$

Thus our general solution is

$$x(t) = c_1 e^{-3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-6t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{6.191}$$

Now using our initial conditions we solve for c_1 and c_2. The condition gives

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} = x(0) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{6.192}$$

All that is left is to write out this matrix equation as a system of equations and then solve

$$c_1 - c_2 = 2 \tag{6.193}$$
$$2c_1 + c_2 = -1 \Rightarrow c_1 = \frac{1}{3}, c_2 = -\frac{5}{3} \tag{6.194}$$

Thus the particular solution is

$$x(t) = \frac{1}{3} e^{-3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{5}{3} e^{-6t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{6.195}$$

■

■ **Example 6.14** Sketch the phase portrait of the system from Example 5.

In the last example, we saw that the eigenvalue/eigenvector pairs for the coefficient matrix were

$$\lambda_1 = -3 \qquad \eta^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{6.196}$$
$$\lambda_2 = -6 \qquad \eta^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{6.197}$$

We begin by sketching the eigensolutions. Both of these trajectories decay towards the origin, since both eigenvalues are negative. Since $|\lambda_2| > |\lambda_1|$, the second eigensolution is the fast eigensolution and the first one the slow eigensolution. In the general solution, both exponentials are negative and so every solution will decay and move towards the origin. Asymptotically, as $t \to \infty$ the trajectory gets closer and closer to the origin, the slow eigensolution will dominate the typical trajectory, as it dies out less quickly than the fast eigensolution. So in forward time, other trajectories will get closer and closer to the eigensolution corresponding to $\eta^{(1)}$. On the other hand, as $t \to -\infty$, the fast solution will grow more quickly than the slow one, and so the eigensolution corresponding to $\eta^{(2)}$ will dominate in backwards time.

Thus the phase portrait will look like Figure 3. Whenever we have two negative eigenvalues, every solution moves toward the origin. We call the equilibrium solution at the origin, in this case, a **node** and classify it as being **asymptotically stable**.

■

6.5 Complex Eigenvalues

Last Time: We studied phase portraits and systems of differential equations with real eigenvalues.

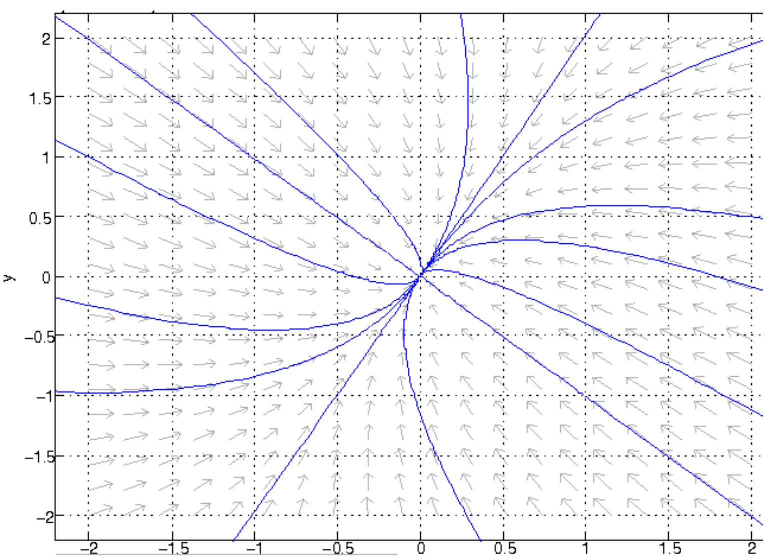


Figure 6.3: Phase Portrait of the Stable Node in Example 3

We are looking for solutions to the equation $x' = Ax$. What happens when the eigenvalues are complex?

We still have solutions of the form

$$x = \eta e^{\lambda t} \tag{6.198}$$

where η is an eigenvector of A with eigenvalue λ. However, we want real-valued solutions, which we will not have if they remain in this form.

Our strategy will be similar in this case: we'll use Euler's formula to rewrite

$$e^{(a+ib)t} = e^{at}\cos(bt) + e^{at} i \sin(bt) \tag{6.199}$$

then we will write out one of our solutions fully into real and imaginary parts. It will turn out that each of these parts gives us a solution, and in fact they will also form a fundamental set of solutions.

■ **Example 6.15** Solve the following initial value problem.

$$x' = \begin{pmatrix} 3 & 6 \\ -2 & -3 \end{pmatrix} x \quad x(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \tag{6.200}$$

The first thing we need to do is to find the eigenvalues of the coefficient matrix.

$$0 = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 6 \\ -2 & -3-\lambda \end{vmatrix} \tag{6.201}$$

$$= \lambda^2 + 3 \tag{6.202}$$

$$\tag{6.203}$$

So the eigenvalues are $\lambda_1 = \sqrt{3}i$ and $\lambda_2 = -\sqrt{3}i$. Next we need the eigenvectors. It turns out we will only need one. Consider $\lambda_1 = \sqrt{3}i$.

$$(A - \sqrt{3}iI)\eta^* = 0 \tag{6.204}$$

$$\begin{pmatrix} 3-\sqrt{3}i & 6 \\ -2 & -3-\sqrt{3}i \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6.205}$$

The system of equations to solve is

$$
\begin{aligned}
(3-\sqrt{3}i)\eta_1 + 6\eta_2 &= 0 && (6.206)\\
-2\eta_1 + (-3-\sqrt{3}i)\eta_2 &= 0. && (6.207)
\end{aligned}
$$

We can use either equation to find solutions, but lets solve the second one. This gives $\eta_1 = \frac{1}{2}(-3-\sqrt{3}i)\eta_2$. Thus any eigenvector has the form

$$\eta = \begin{pmatrix} \frac{1}{2}(-3-\sqrt{3}i)\eta_2 \\ \eta_2 \end{pmatrix} \tag{6.208}$$

and choosing $\eta_2 = 2$ yields the first eigenvector

$$\eta^{(1)} = \begin{pmatrix} -3-\sqrt{3}i \\ 2 \end{pmatrix}. \tag{6.209}$$

Thus we have a solution

$$x_1(t) = e^{\sqrt{3}it} \begin{pmatrix} -3-\sqrt{3}i \\ 2 \end{pmatrix} \tag{6.210}$$

Unfortunately, this is complex-valued, and we would like a real-valued solution. We he had a similar problem in the chapter on second order equations. What did we do then? We use Euler's formula to expand this imaginary exponential into sine and cosine terms, then split the solution into real and imaginary parts. This gave two fundamental solutions we needed.

We will do the same thing here. Using Euler's Formula to expand

$$e^{\sqrt{3}it} = \cos(\sqrt{3}t) + i\sin(\sqrt{3}t). \tag{6.211}$$

then multiply it through the eigenvector. After separating into real and complex parts using the basic matrix arithmetic operations, it will turn out that each of these parts is a solution. They are linearly independent and give us a fundamental set of solutions.

$$
\begin{aligned}
x_1(t) &= (\cos(\sqrt{3}t) + i\sin(\sqrt{3}t)) \begin{pmatrix} -3-\sqrt{3}i \\ 2 \end{pmatrix} && (6.212)\\
&= \begin{pmatrix} (-3\cos(\sqrt{3}t) - 3i\sin(\sqrt{3}t) - \sqrt{3}i\cos(\sqrt{3}t) + \sqrt{3}\sin(\sqrt{3}t) \\ 2\cos(\sqrt{3}t) + 2i\sin(\sqrt{3}t) \end{pmatrix} && (6.213)\\
&= \left(\begin{pmatrix} -3\cos(\sqrt{3}t) + \sqrt{3}\sin(\sqrt{3}t) \\ 2\cos(\sqrt{3}t) \end{pmatrix} + i \begin{pmatrix} -3\sin(\sqrt{3}t) - \sqrt{3}\cos(\sqrt{3}t) \\ 2\sin(\sqrt{3}t) \end{pmatrix} \right) && (6.214)\\
&= u(t) + iv(t) && (6.215)
\end{aligned}
$$

Both $u(t)$ and $v(t)$ are real-valued solutions to the differential equation. Moreover, they are linearly independent. Our general solution is then

$$
\begin{aligned}
x(t) &= c_1 u(t) + c_2 v(t) && (6.216)\\
&= c_1 \begin{pmatrix} -3\cos(\sqrt{3}t) + \sqrt{3}\sin(\sqrt{3}t) \\ 2\cos(\sqrt{3}t) \end{pmatrix} + c_2 \begin{pmatrix} -3\sin(\sqrt{3}t) - \sqrt{3}\cos(\sqrt{3}t) \\ 2\sin(\sqrt{3}t) \end{pmatrix} && (6.217)
\end{aligned}
$$

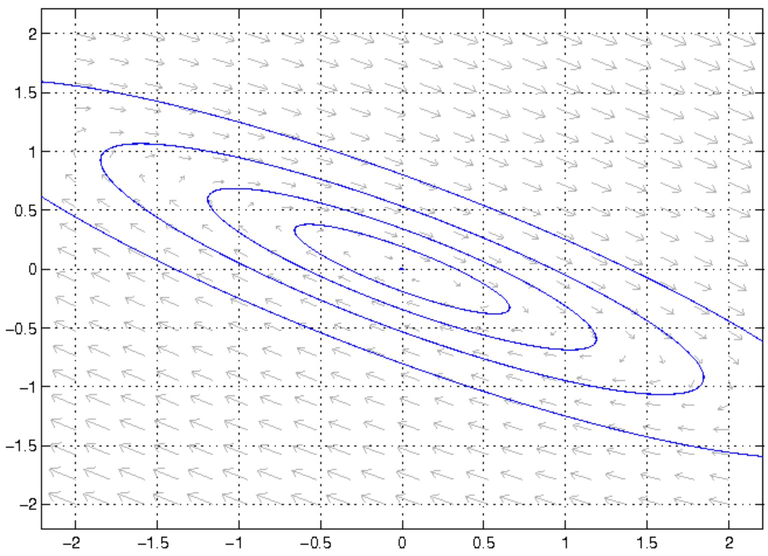


Figure 6.4: Phase Portrait of the center point in Example 1

Finally, we need to use the initial condition to get c_1 and c_2. It says

$$\begin{pmatrix} -2 \\ 4 \end{pmatrix} = x(0) = c_1 \begin{pmatrix} -3 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -\sqrt{3} \\ 0 \end{pmatrix}. \tag{6.218}$$

This translates into the system

$$-3c_1 - \sqrt{3}c_2 = -2 \tag{6.219}$$

$$2c_1 = 4 \quad \Rightarrow c_1 = 2 \quad c_2 = -\frac{4}{\sqrt{3}}. \tag{6.220}$$

Hence our particular solution is

$$x(t) = 2 \begin{pmatrix} -3\cos(\sqrt{3}t) + \sqrt{3}\sin(\sqrt{3}t) \\ 2\cos(\sqrt{3}t) \end{pmatrix} - \frac{4}{\sqrt{3}} \begin{pmatrix} -3\sin(\sqrt{3}t) - \sqrt{3}\cos(\sqrt{3}t) \\ 2\sin(\sqrt{3}t) \end{pmatrix} \tag{6.221}$$

■

■ **Example 6.16** Sketch the phase portrait of the system in Example 1.
The general solution to the system in Example 1 is

$$x(t) = c_1 \left(\begin{pmatrix} -3\cos(\sqrt{3}t) + \sqrt{3}\sin(\sqrt{3}t) \\ 2\cos(\sqrt{3}t) \end{pmatrix} + c_2 \begin{pmatrix} -3\sin(\sqrt{3}t) - \sqrt{3}\cos(\sqrt{3}t) \\ 2\sin(\sqrt{3}t) \end{pmatrix} \right) \tag{6.222}$$

Every term in this solution is periodic, we have $\cos(\sqrt{3}t)$ and $\sin(\sqrt{3}t)$. Thus both x_1 and x_2 are periodic functions for any initial conditions. On the phase plane, this translates to trajectories which are closed, that is they form circles or ellipses. As a result, the phase portrait looks like Figure 1.

This is always the case when we have purely imaginary eigenvalues, as the exponentials turn into a combination of sines and cosines. In this case, the equilibrium solution is called a **center** and is **neutrally stable** or just **stable**, note that it is not asymptotically stable.

The only work left to do in these cases is to figure out the eccentricity and direction that the trajectory traveled. The eccentricity is difficult, and we usually do not care that much about it. The direction traveled is easier to find. We can determine whether the trajectories orbit the origin in a clockwise or counterclockwise direction by calculating the tangent vector x' at a single point. For example, at the point $(1,0)$ in the previous example, we have

$$x' = \begin{pmatrix} 3 & 6 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{6.223}$$

Thus at $(1,0)$, the tangent vector points down and to the right. This can only happen is the trajectories circle to origin in a clockwise direction. ∎

∎ **Example 6.17** Solve the following initial value problem.

$$x' = \begin{pmatrix} 6 & -4 \\ 8 & -2 \end{pmatrix} x \quad x(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{6.224}$$

The first thing we need to do is to find the eigenvalues of the coefficient matrix.

$$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{vmatrix} 6-\lambda & -4 \\ 8 & -2-\lambda \end{vmatrix} && (6.225) \\ &= \lambda^2 - 4\lambda + 20 && (6.226) \\ & && (6.227) \end{aligned}$$

So the eigenvalues, using the Quadratic Formula are $\lambda_{1,2} = 2 \pm 4i$. Next we need the eigenvectors. It turns out we will only need one. Consider $\lambda_1 = 2+4i$.

$$(A - (2+4i)I)\eta^* = 0 \tag{6.228}$$

$$\begin{pmatrix} 4-4i & -4 \\ 8 & -4-4i \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6.229}$$

The system of equations to solve is

$$\begin{aligned} (4-4i)\eta_1 - 4\eta_2 &= 0 && (6.230) \\ 8\eta_1 + (-4-4i)\eta_2 &= 0. && (6.231) \end{aligned}$$

We can use either equation to find solutions, but lets solve the first one. This gives $\eta_2 = (1-i)\eta_1$. Thus any eigenvector has the form

$$\eta = \begin{pmatrix} \eta_1 \\ (1-i)\eta_1 \end{pmatrix} \tag{6.232}$$

and choosing $\eta_1 = 1$ yields the first eigenvector

$$\eta^{(1)} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}. \tag{6.233}$$

Thus we have a solution

$$x_1(t) = e^{(2+4i)t}\begin{pmatrix} 1 \\ 1-i \end{pmatrix} \tag{6.234}$$

Using Euler's Formula to expand

$$\begin{aligned}
&= e^{2i}e^{4it}\begin{pmatrix} 1 \\ 1-i \end{pmatrix} && (6.235)\\
&= e^{2t}(\cos(4t)+i\sin(4t))\begin{pmatrix} 1 \\ 1-i \end{pmatrix} && (6.236)\\
&= e^{2t}\begin{pmatrix} \cos(4t)+i\sin(4t) \\ \cos(4t)+i\sin(4t)-i\cos(4t)+\sin(4t) \end{pmatrix} && (6.237)\\
&= \left(\begin{pmatrix} \cos(4t) \\ \cos(4t)+\sin(4t) \end{pmatrix} + i\begin{pmatrix} \sin(4t) \\ \sin(4t)-\cos(4t) \end{pmatrix}\right) && (6.238)\\
&= u(t)+iv(t) && (6.239)
\end{aligned}$$

Our general solution is then

$$\begin{aligned}
x(t) &= c_1u(t)+c_2v(t) && (6.240)\\
&= c_1e^{2t}\begin{pmatrix} \cos(4t) \\ \cos(4t)+\sin(4t) \end{pmatrix} + c_2e^{2t}\begin{pmatrix} \sin(4t) \\ \sin(4t)-\cos(4t) \end{pmatrix} && (6.241)
\end{aligned}$$

Finally, we need to use the initial condition to get c_1 and c_2. It says

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} = x(0) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ -1 \end{pmatrix}. \tag{6.242}$$

This translates into the system

$$\begin{aligned}
c_1 &= 1 && (6.243)\\
c_1 - c_2 &= 3 \quad \Rightarrow c_1 = 1 \quad c_2 = -2. && (6.244)
\end{aligned}$$

Hence our particular solution is

$$x(t) = e^{2t}\begin{pmatrix} \cos(4t) \\ \cos(4t)+\sin(4t) \end{pmatrix} - 2e^{2t}\begin{pmatrix} \sin(4t) \\ \sin(4t)-\cos(4t) \end{pmatrix} \tag{6.245}$$

■

■ **Example 6.18** Sketch the phase portrait of the system in Example 3.

The only difference between the general solution to this example

$$x(t) = c_1e^{2t}\begin{pmatrix} \cos(4t) \\ \cos(4t)+\sin(4t) \end{pmatrix} + c_2e^{2t}\begin{pmatrix} \sin(4t) \\ \sin(4t)-\cos(4t) \end{pmatrix} \tag{6.246}$$

and the one in Example 1 is the exponential sitting out front of the periodic terms. This will make the solution quasi-periodic, rather than actually periodic. The exponential, having a positive exponent, will cause the solution to grow as $t \to \infty$ away from the origin. The solution will still rotate, however, as the trig terms will cause the oscillation. Thus, rather than forming closed circles or ellipses, the trajectories will spiral out of the origin.

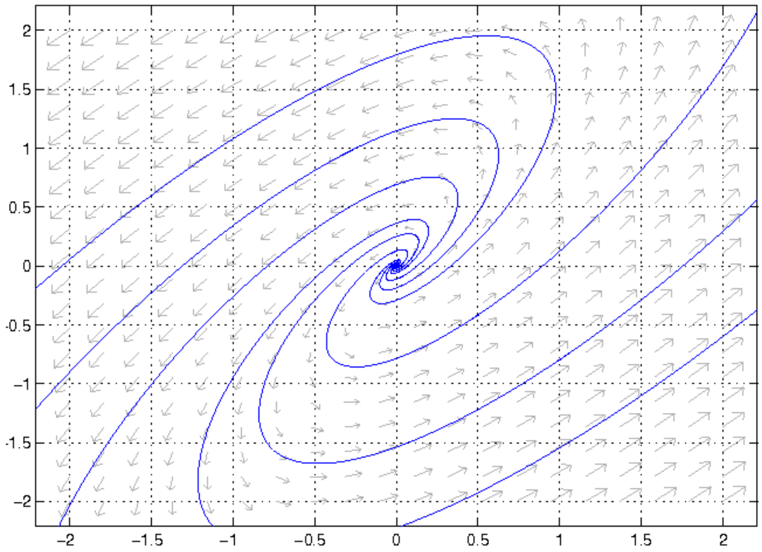


Figure 6.5: Phase Portrait of the unstable spiral in Example 3

As a result, when we have complex eigenvalues $\lambda_{1,2} = a \pm bi$, we call the solution **spiral**. In this case, as the real part a (which affects the exponent) is positive, and the solution grows, the equilibrium at the center is unstable. If a is negative, then spiral would decay into the origin, and the equilibrium would have been asymptotically stable.

So what is there to calculate if we recognize we have a stable/unstable spiral? We still need to know the direction of rotation. This requires, as with the center, that we calculate the tangent vector at a point or two. In this case, the tangent vector at the point $(1,0)$ is

$$x' = \begin{pmatrix} 6 & -4 \\ 7 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix}. \tag{6.247}$$

Thus the tangent vector at $(1,0)$ points up and to the right. Combined with the knowledge that the solution is leaving the origin, this can only happen if the direction of rotation of the spiral is counterclockwise. We obtain a picture as in Figure 2.

■

6.6 Repeated Eigenvalues

Last Time: We studied phase portraits and systems of differential equations with complex eigenvalues.

In the previous cases we had distinct eigenvalues which led to linearly independent solutions. Thus, all we had to do was calculate those eigenvectors and write down solutions of the form

$$x_i(t) = \eta^{(i)} e^{\lambda_i t}. \tag{6.248}$$

When we have an eigenvalue of multiplicity 2, however, Theorem 3 from a previous lecture tells us that we could have either one or two eigenvectors up to linear independence. If we have two, we are ok, if not then we have more work to do.

6.6.1 A Complete Eigenvalue

We call a repeated eigenvalue **complete** if it has two distinct (linearly independent) eigenvectors. Suppose our repeated eigenvalue λ has two linearly independent eigenvectors $\eta^{(1)}$ and $\eta^{(2)}$. Then we can proceed as before and our general solution is

$$x(t) = c_1 e^{\lambda t}\eta^{(1)} + c_2 e^{\lambda t}\eta^{(2)} \tag{6.249}$$
$$= e^{\lambda t}(c_1\eta^{(1)} + c_2\eta^{(2)}). \tag{6.250}$$

It is a basic fact from linear algebra that given two linearly independent vectors such as $\eta^{(1)}$ and $\eta^{(2)}$, we can form any other two-dimensional vector out of a linear combination of these two. So any vector function of the form $x(t) = e^{\lambda t}\eta$ is a solution. As discussed earlier, this can only happen if η is an eigenvector of the coefficient matrix with eigenvalue λ. The conclusion, is that if λ has two linearly independent eigenvectors, every vector is an eigenvector.

This only happens if the coefficient matrix A is a scalar multiple of the identity matrix, since we need

$$A\eta = \lambda\eta = \lambda I\eta \tag{6.251}$$

for every vector η. Thus, this case only arises when

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \tag{6.252}$$

or when the original system is

$$x_1 = \lambda x_1 \tag{6.253}$$
$$x_2 = \lambda x_2 \tag{6.254}$$

What does the phase portrait look like in this case? Since every vector is an eigenvector, every trajectory that is not a constant solution at the origin is an eigensolution and hence a straight line. We call such the equilibrium solution, in this case a **star node** and its stability is determined by the sign of λ.
(1) If $\lambda > 0$ all the eigensolutions grow away from the origin and the origin is unstable.
(2) If $\lambda < 0$ every eigensolution decays to the origin and the origin is asymptotically stable. We get Figure 8.5.1.

This is a fairly degenerate situation that will not come up in any further discussion, but is important to keep in mind when it happens.

6.6.2 A Defective Eigenvalue

The other possibility is that λ only has a single eigenvector η up to linear independence. In this case, to form the general solution we need two linearly independent solutions, but we only have one.

$$x_1(t) = e^{\lambda t}\eta \tag{6.255}$$

In this case, we say that λ is **defective** or **incomplete**. What should we do in this case?

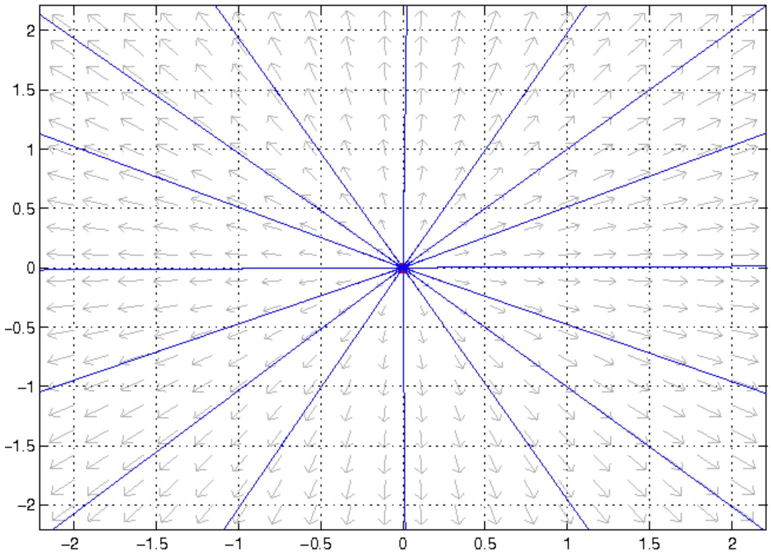


Figure 6.6: Phase Portrait of a star node

We had a similar problem in the second order linear case. When we ran into this situation there, we were able to work around it by multiplying the solution be t. What if we try that here?

$$x(t) = te^{\lambda t}\eta \tag{6.256}$$

is a solution

$$\begin{aligned} x' &= Ax && (6.257) \\ \eta e^{\lambda t} + \lambda \eta t e^{\lambda t} &= A\eta t e^{\lambda t} && (6.258) \end{aligned}$$

Matching coefficients, in order for this guess to be a solution we require

$$\begin{aligned} \eta &= 0 \quad \Rightarrow \quad \eta = 0 && (6.259) \\ \lambda \eta e^{\lambda t} &= A\eta e^{\lambda t} \quad \Rightarrow \quad (A - \lambda I)\eta = 0 && (6.260) \end{aligned}$$

Thus we need η to be an eigenvector of A, which we knew it was, but we also need $\eta = 0$, which cannot be since an eigenvector by definition is nonzero. We need another approach.

The problem with the last attempt is we ended up with a term that did not have a t, but rather just an exponential in it, and this term caused us to require $\eta = 0$. A possible fix might be to add in another term in our guess that only involves an exponential and some other vector ρ. Let's guess that the form of the solution is

$$x(t) = te^{\lambda t}\eta + e^{\lambda t}\rho \tag{6.261}$$

and see what conditions on ρ we can derive.

$$\begin{aligned} x' &= Ax && (6.262) \\ \lambda \eta t e^{\lambda t} + \eta e^{\lambda t} + \lambda \rho e^{\lambda t} &= A(\eta t e^{\lambda t} + \rho e^{\lambda t}) && (6.263) \\ (\eta + \lambda \rho + \lambda \eta t)e^{\lambda t} &= A\eta t e^{\lambda t} + A\rho e^{\lambda t} && (6.264) \end{aligned}$$

Thus, setting the coefficients equal and we have

$$\begin{aligned} A\eta &= \lambda\eta \quad \Rightarrow \quad (A-\lambda I)\eta = 0 && (6.265)\\ \eta+\lambda\rho &= A\rho \quad \Rightarrow \quad (A-\lambda I)\rho = \eta && (6.266) \end{aligned}$$

The first condition only tells us that η is an eigenvector of A, which we knew. But the second condition is more useful. It tells us that $(A-\lambda I)\rho = \eta$, then

$$x(t) = \eta t e^{\lambda t} + \rho e^{\lambda t} \tag{6.267}$$

will be a solution to the differential equation.

A vector ρ satisfying

$$(A-\lambda I)\rho = \eta \tag{6.268}$$

is called a **generalized eigenvector**, while $(A-\lambda I)\rho \neq 0$, it is not hard to verify

$$(A-\lambda I)^2\rho = 0 \tag{6.269}$$

So as long as we can reproduce a generalized eigenvector ρ, this formula will give a second solution and we can form a general solution.

■ **Example 6.19** Find the general solution to the following problem.

$$x' = \begin{pmatrix} -6 & -5 \\ 5 & 4 \end{pmatrix} x \tag{6.270}$$

The first thing we need to do is to find the eigenvalues of the coefficient matrix.

$$\begin{aligned} 0 = \det(A-\lambda I) &= \begin{vmatrix} -6-\lambda & -5 \\ 5 & 4-\lambda \end{vmatrix} && (6.271)\\ &= \lambda^2+2\lambda+1 && (6.272)\\ &= (\lambda+1)^2 && (6.273) \end{aligned}$$

So we have a repeated eigenvalue of $\lambda = -1$. Due to the form of the matrix, we can also figure out from our previous discussion that λ will only have one eigenvector up to linear independence. Let's calculate it by solving $(A+I)\eta = 0$.

$$\begin{pmatrix} -5 & -5 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6.274}$$

So the system of equations we want to solve is

$$\begin{aligned} -5\eta_1 - 5\eta_2 &= 0 && (6.275)\\ 5\eta_1 + 5\eta_2 &= 0. && (6.276) \end{aligned}$$

This is solved by anything of the form $\eta_1 = -\eta_2$. So if we choose $\eta_2 = 1$, we get the eigenvector of

$$\eta = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{6.277}$$

This isn't enough. We also need to find a generalized eigenvector ρ. So we need to solve $(A+I)\rho = \eta$, or

$$\begin{pmatrix} -5 & -5 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \rho_1 = \frac{1}{5} - \rho_2 \tag{6.278}$$

So our generalized eigenvector has the form

$$\rho = \begin{pmatrix} \frac{1}{5} - \rho_2 \\ \rho_2 \end{pmatrix}. \tag{6.279}$$

Choose $\rho_2 = 0$ gives

$$\rho = \begin{pmatrix} \frac{1}{5} \\ 0 \end{pmatrix}. \tag{6.280}$$

Thus our general solution is

$$\begin{aligned} x(t) &= c_1 e^{\lambda t}\eta + c_2(te^{\lambda t}\eta + e^{\lambda t}\rho) && (6.281) \\ &= c_1 e^{-t}\eta \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \left[te^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^{-t} \begin{pmatrix} \frac{1}{5} \\ 0 \end{pmatrix} \right]. && (6.282) \end{aligned}$$

■

■ **Example 6.20** Sketch the Phase Portrait for the system in Example 1.

We begin by drawing our eigensolution. Note that in this case, we only have one, unlike the case where we had a (nondegenerate) node. The eigensolution is the straight line in the direction $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, as indicated in Figure 8.5.1. As the eigenvalue is negative, the solution will decay towards the origin.

But what happens to the other trajectories? First, let's just consider the general solution

$$x(t) = c_1 e^{-t}\eta \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \left[te^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^{-t} \begin{pmatrix} \frac{1}{5} \\ 0 \end{pmatrix} \right] \tag{6.283}$$

with $c_2 \neq 0$. All three terms have the same exponential, but as $t \to \pm\infty$ the te^{-t} term will have a larger magnitude than the other two. Thus in both the forward and backward time, we would get that the trajectories will become parallel to the single eigensolution.

Now, since $\lambda < 0$ and exponentials decay faster than polynomials grow, we can see that as $t \to \infty$, every solution will decay to the origin. So the origin will be asymptotically stable. We also call the origin a **degenerate node** in this case, since it behaves like the a node, but has a single eigensolution.

Consider a node with two close eigenvalues. Then try to imagine what happens to the general solution as we bring the eigenvalues together. The eigensolutions will collapse together, but the non-eigensolution trajectories would keep their asymptotic behavior with regard to this collapsed eigensolution.

Notice that, as illustrated in Figure 8.5.1 we end up with a large degree of rotation of the solution. The solution has to turn around to be able to be asymptotic to the solution in

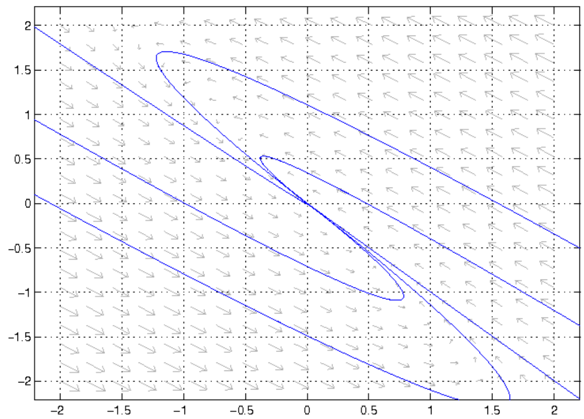


Figure 6.7: Phase Portrait for the asymptotically stable degenerate node in Example 1

both forward and backward time. This is because degenerate nodes are the borderline case between nodes and spirals. Suppose our characteristic equation is

$$0 = \lambda^2 + b\lambda + c. \tag{6.284}$$

The eigenvalues are then, by the quadratic formula

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4c}}{2a} \tag{6.285}$$

The discriminant of this equation is positive in the node case and negative in the spiral/center cases. We get degenerate nodes when the solutions transition between these two cases and the discriminant becomes zero. So for degenerate nodes the solutions are trying to wind around in a spiral, but they do not quite make it due to the lack of complexity of the eigenvalue.

But how do we know the direction of rotation? We do the same thing we did in the spiral case, compute the tangent vector at a point. Combined with our knowledge of the stability of the origin, will tell us how the non-eigensolutions must turn.

Let's start by considering the point $(1,0)$. At this point,

$$\begin{pmatrix} -6 & -5 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -6 \\ 5 \end{pmatrix}. \tag{6.286}$$

Since we know our solution is asymptotically stable, the tangent vector can only point up and to the left if the solution rotates counterclockwise as they start to approach the origin.

■

■ **Example 6.21** Find the general solution to the following system.

$$x' = \begin{pmatrix} 12 & 4 \\ -16 & -4 \end{pmatrix} x \tag{6.287}$$

The first thing we need to do is to find the eigenvalues of the coefficient matrix.

$$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{vmatrix} 12-\lambda & 4 \\ -16 & -4-\lambda \end{vmatrix} && (6.288) \\ &= \lambda^2 - 8\lambda + 16 && (6.289) \\ &= (\lambda - 4)^2 && (6.290) \end{aligned}$$

So we have a repeated eigenvalue of $\lambda = 4$. Let's calculate it by solving $(A-4I)\eta = 0$.

$$\begin{pmatrix} 8 & 4 \\ -16 & -8 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6.291}$$

So the system of equations we want to solve is

$$8\eta_1 + 4\eta_2 = 0 \tag{6.292}$$
$$-16\eta_1 - 8\eta_2 = 0. \tag{6.293}$$

This is solved by anything of the form $\eta_2 = -2\eta_1$. So if we choose $\eta_1 = 1$, we get the eigenvector of

$$\eta = \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \tag{6.294}$$

We also need to find a generalized eigenvector ρ. So we need to solve $(A-4I)\rho = \eta$, or

$$\begin{pmatrix} 8 & 4 \\ -16 & -8 \end{pmatrix}\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \Rightarrow \quad \rho_2 = \frac{1}{4} - 2\rho_1 \tag{6.295}$$

So our generalize eigenvector has the form

$$\rho = \begin{pmatrix} \rho_1 \\ \frac{1}{4} - 2\rho_1 \end{pmatrix}. \tag{6.296}$$

Choose $\rho_1 = 0$ gives

$$\rho = \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix}. \tag{6.297}$$

Thus our general solution is

$$x(t) = c_1 e^{\lambda t}\eta + c_2(te^{\lambda t}\eta + e^{\lambda t}\rho) \tag{6.298}$$
$$= c_1 e^{-t}\rho = \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2(te^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + e^{-t}\begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix}. \tag{6.299}$$

■

■ **Example 6.22** Sketch the Phase Portrait of the system in Example 3.

Everything is completely analogous to the previous example's phase portrait. We sketch the eigensolution, and note that it will grow away from the origin as $t \to \infty$, so the origin will in this case be an unstable degenerate node.

Typical trajectories will once again come out of the origin parallel to the eigensolution and rotate around to be parallel to them again, and all we would need to do is to calculate the direction of rotation by computing the tangent vector at a point or two. At $(1,0)$, we would get

$$x' = \begin{pmatrix} 16 \\ 12 \end{pmatrix}. \tag{6.300}$$

which can only happen given that the solutions are growing if the direction of rotation is clockwise. Thus we get Figure 3.

■

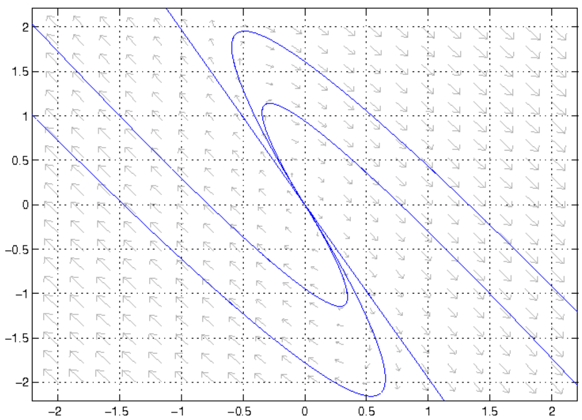
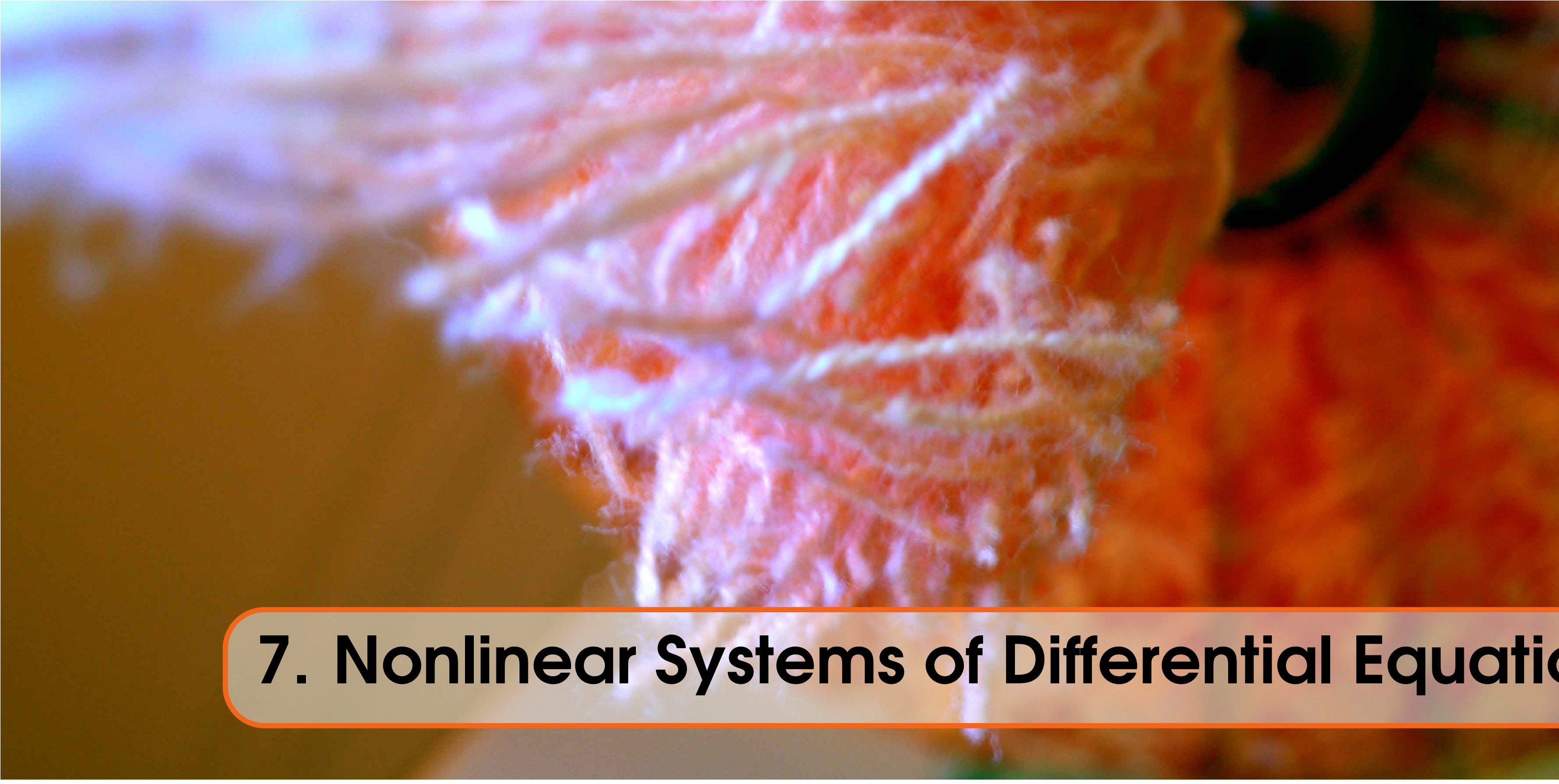


Figure 6.8: Phase Portrait of the unstable degenerate node in Example 3.



7. Nonlinear Systems of Differential Equatio

7.1 Phase Portrait Review

Last Time: We studied phase portraits and systems of differential equations with repeated eigenvalues.

Note in the last 3 sections 7.5, 7.6, 7.8 we have covered the information in Section 9.1, which is sketching phase portraits, and identifying the three distinct cases for 1. Real Distinct Eigenvalues, 2. Complex Eigenvalues, and 3. Repeated Eigenvalues. Think of this section as a good review.

In Chapter 1 and Section 2.5 we considered the autonomous equations

$$\frac{dy}{dt} = f(y) \tag{7.1}$$

Consider the simplest system, a second order linear homogeneous system with constant coefficients. Such a system has the form

$$\frac{dx}{dt} = Ax \tag{7.2}$$

where A is a 2×2 matrix. We spent three sections solving these types of systems. Recall we seek solutions of the form $x = \eta e^{rt}$, then if we substitute this into the equation we found

$$(A - rI)\eta = 0 \tag{7.3}$$

Thus r is an eigenvalue and η the corresponding eigenvector.

Definition 7.1.1 Points where $Ax = 0$ correspond to equilibrium or constant solutions, and are called **critical points**. Note if A is nonsingular, then the only critical point is $x = 0$.

We must consider the five possible situations we could be in.

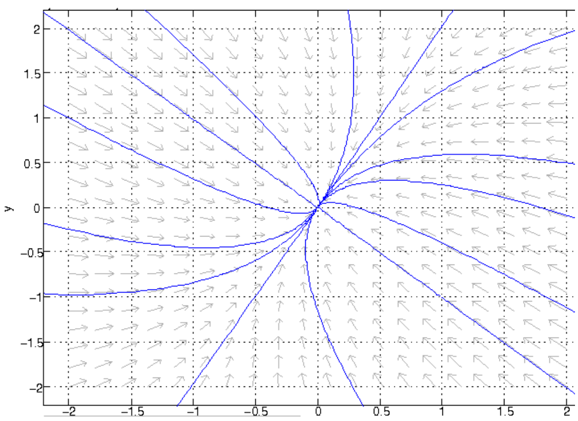


Figure 7.1: Phase Portrait of the Nodal Sink

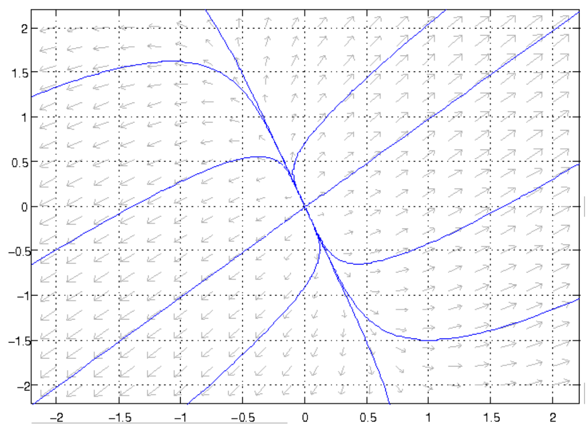


Figure 7.2: Phase Portrait of the Nodal Source

7.1.1 Case I: Real Unequal Eigenvalues of the Same Sign

The general solution of $x' = Ax$ is

$$x = c_1\eta^{(1)}e^{\lambda_1 t} + c_2\eta^{(2)}e^{\lambda_2 t} \tag{7.4}$$

where λ_1 and λ_2 are either both positive or both negative. Suppose first that $\lambda_1 < \lambda_2 < 0$. Both exponentials decay, so as $t \to \infty$, then $x(t) \to 0$ regardless of the values of c_1 and c_2. Note the eigenvalue with the bigger magnitude, $|\lambda_i|$, will determine where the trajectories are directed to. So the trajectories will tend towards $\eta^{(1)}$.

Definition 7.1.2 The type of critical point where all solutions decay to the origin is a **node** or **nodal sink**.

If λ_1 and λ_2 are both positive and $0 < \lambda_2 < \lambda_1$, then the trajectories have the same pattern as the previous case but as $t \to \infty$ the solutions blow up so all arrows change direction and move away from the origin. The critical point is still called a **node** or **nodal source**. Notice it is a source because trajectories come from it and leave, whereas the nodal sink before sucked all trajectories towards itself.

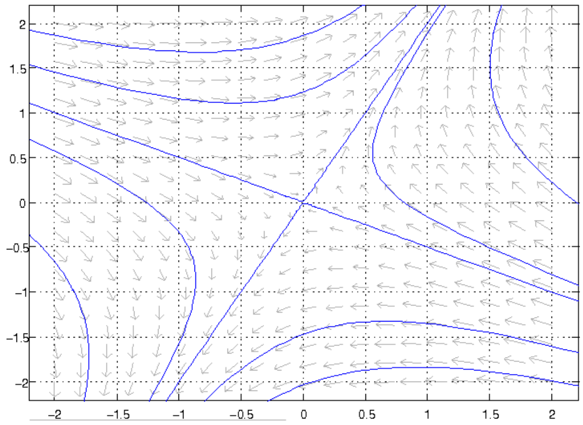


Figure 7.3: Phase Portrait of the saddle point

7.1.2 Case II: Real Eigenvalues of Opposite Signs

The general solution of $x' = Ax$ is

$$x = c_1\eta^{(1)}e^{\lambda_1 t} + c_2\eta^{(2)}e^{\lambda_2 t} \tag{7.5}$$

where $\lambda_1 > 0$ and $\lambda_2 < 0$. Notice as $t \to \infty$ the second term decays to zero and the first eigenvector becomes dominant. So as time goes to infinity all trajectories asymptotically approach $\eta^{(1)}$. The only solutions that approach 0 are the ones which start on $\eta^{(2)}$. This is because $c_1 = 0$ and all terms would decay as time increases.

Definition 7.1.3 The origin where some solutions tend towards it and some tend away is called a **saddle point**.

7.1.3 Case III: Repeated Eigenvalues

Here $\lambda_1 = \lambda_2 = \lambda$. We have two subcases.

Case IIIa: Two Independent Eigenvectors

The general solution of $x' = Ax$ is

$$x = c_1\eta^{(1)}e^{\lambda t} + c_2\eta^{(2)}e^{\lambda t} \tag{7.6}$$

where the eigenvectors are linearly independent. Every trajectory lies on a straight line through the origin. If $\lambda < 0$ all solutions decay to the origin, if $\lambda > 0$ then all solutions move away from the origin.

Definition 7.1.4 In either case, the critical point is called a **proper node** or a **star point**.

Case IIIb: One Independent Eigenvector

The general solution in this case is

$$x = c_1\eta e^{\lambda t} + c_2(\eta t e^{\lambda t} + \rho e^{\lambda t}). \tag{7.7}$$

where η is the eigenvector and ρ is the generalized eigenvector. For t large, $c_2\eta t e^{\lambda t}$ dominates. Thus as $t \to \infty$ every trajectory approaches the origin tangent to the line

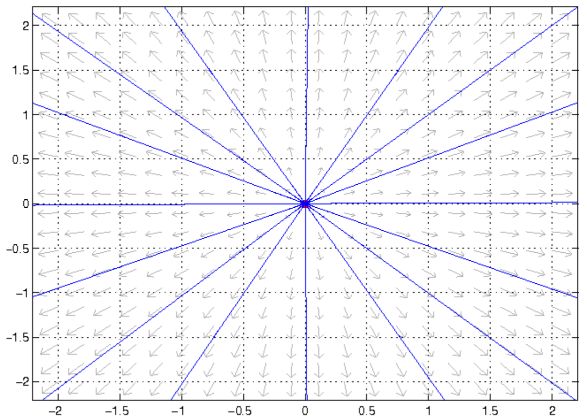


Figure 7.4: Phase Portrait of a star node

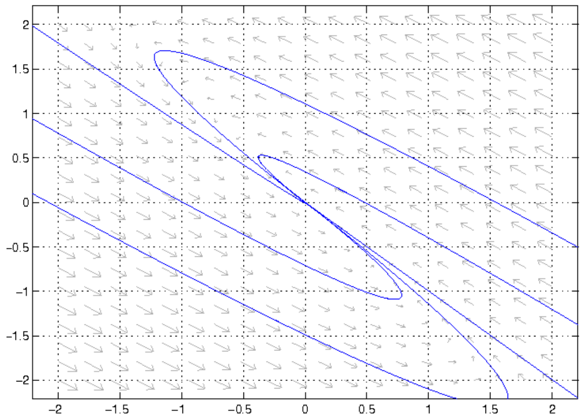


Figure 7.5: Phase Portrait for the asymptotically stable degenerate node

through the eigenvector. If the $\lambda > 0$ the trajectories move away from the origin, and if $\lambda < 0$ the trajectories moved towards the origin.

Definition 7.1.5 When a repeated eigenvalue has only a single independent eigenvector, the critical point is called an **improper** or **degenerate node**.

7.1.4 Case IV: Complex Eigenvalues

Suppose the eigenvalues are $\alpha \pm i\beta$, where α and β are real. In this case critical points are called **spiral point**. Depending on if the trajectories move toward or away from the origin it could be characterized as a **spiral sink** or **source**.

In the phase portrait we either spiral towards or away from the origin. If the real part $\alpha > 0$, then trajectories spiral away from the origin. If the real part $\alpha < 0$, then the trajectories spiral towards the origin.

7.1.5 Case V: Pure Imaginary Eigenvalues

Here $\alpha = 0$ and $\lambda = \pm\beta i$. In this case the critical point is called a **center**, because the trajectories are concentric circles around the origin. We can determine the direction of the circle by finding the tangent vector at a point like $(1,0)$.

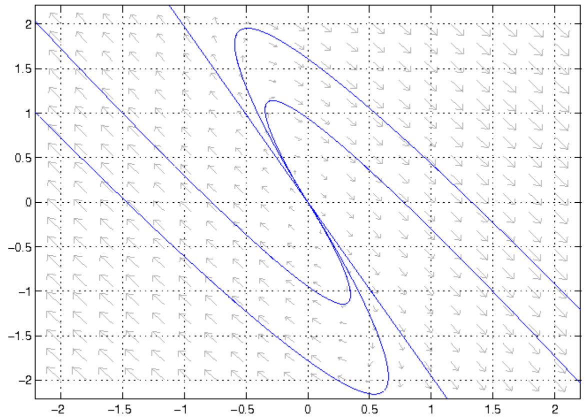


Figure 7.6: Phase Portrait of the unstable degenerate node

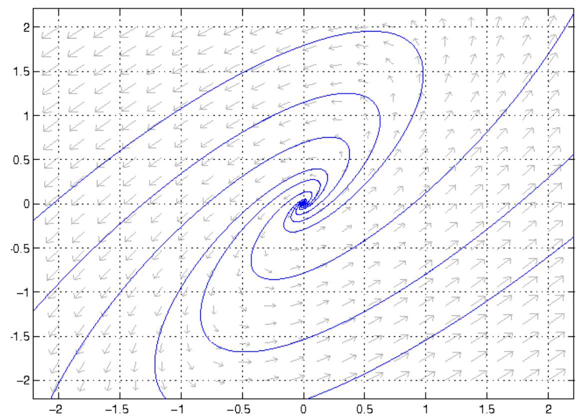


Figure 7.7: Phase Portrait of the unstable spiral

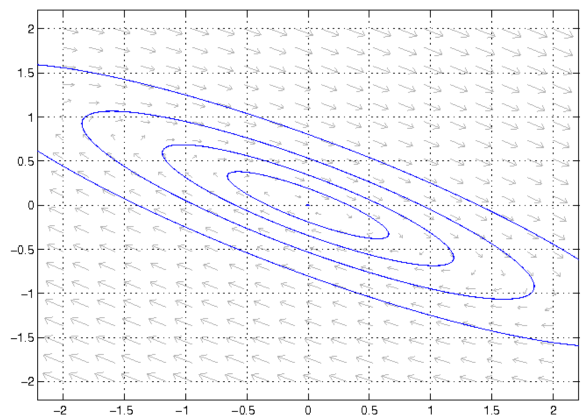


Figure 7.8: Phase Portrait of the center point

Eigenvalues	Type of Critical Point	Stability
$\lambda_1 > \lambda_2 > 0$	Node Source	Unstable
$\lambda_1 < \lambda_2 < 0$	Node Sink	Asymptotically Stable
$\lambda_2 < 0 < \lambda_1$	Saddle Point	Unstable
$\lambda_1 = \lambda_2 > 0$	Proper or Improper Node	Unstable
$\lambda_1 = \lambda_2 < 0$	Proper or Improper Node	Asymptotically Stable
$\lambda_1, \lambda_2 = \alpha + i\beta$		
$\alpha > 0$	Spiral Source	Unstable
$\alpha < 0$	Spiral Sink	Asymptotically Stable
$\lambda = \pm i\beta$	Center	Stable

Table 7.1: Stability Properties of Linear Systems $x' = Ax$ with $\det(A - rI) = 0$ and $\det(A) \neq 0$.

7.1.6 Summary and Observations

1. After a long time, each trajectory exhibits one of only three types of behavior. As $t \to \infty$, each trajectory approaches the critical point $x = 0$, repeatedly traverses a closed curve around the critical point, or becomes unbounded.

2. For each point there is only one trajectory. The trajectories do not cross each other. The only solutions passing through the critical point are $x = 0$, all other solutions only approach the origin as $t \to \infty$ or $-\infty$.

3. In each of the five cases we have one of three situations:

(1) All trajectories approach the critical point $x = 0$ as $t \to \infty$. This is the case if the eigenvalues are real and negative or complex with negative real part. The origin is nodal or a spiral sink.

(2) All trajectories remain bounded but do not approach the origin as $t \to \infty$. This is the case if the eigenvalues are pure imaginary. The origin is a center.

(3) Some trajectories, and possibly all trajectories except $x = 0$, become unbounded as $t \to \infty$. This is the case if at least one of the eigenvalues is positive or if the eigenvalues have positive real part. The origin is a nodal source, a spiral source, or a saddle point.

7.2 Autonomous Systems and Stability

What is an Autonomous System?

Definition 7.2.1 A system of two simultaneous differential equations of the form

$$\frac{dx}{dt} = F(x,y), \quad \frac{dy}{dt} = G(x,y) \tag{7.8}$$

where F and G are continuous and have continuous partial derivatives in some domain D. From Theorem 7.1 we know there exists a unique solution $x = \phi(t), y = \psi(t)$ of the system satisfying the initial conditions

$$x(t_0) = x_0, \quad y(t_0) = y_0 \tag{7.9}$$

The property that makes the system **autonomous** is that F and G only depend on x and y and not t.

7.2.1 Stability and Instability

Consider the autonomous system of the form

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}) \tag{7.10}$$

Definition 7.2.2 The points where $\mathbf{f}(\mathbf{x}) = 0$ are the critical points, which correspond to constant or equilibrium solutions of the autonomous system.

Definition 7.2.3 A critical point $\mathbf{x}^0$ is said to be **stable** if, given any $\varepsilon > 0$, there is a $\delta > 0$ such that every solution $x = \phi(t)$, which at $t = 0$ satisfies

$$||\phi(0) - x^0|| < \delta \tag{7.11}$$

exists for all positive t and satisfies

$$||\phi(t) - x^0|| < \varepsilon \tag{7.12}$$

for all $t \geq 0$. It's **asymptotically stable** if

$$||\phi(0) - x^0|| < \delta \tag{7.13}$$

then

$$\lim_{t \to \infty} \phi(t) = x^0 \tag{7.14}$$

Finally, it is **unstable** if a solution does not approach a critical point as $t \to \infty$.

■ **Example 7.1** Find the critical points of

$$\frac{dx}{dt} = -(x-y)(1-x-y), \quad \frac{dy}{dt} = x(2+y). \tag{7.15}$$

We find the critical points by solving the algebraic equations

$$(x-y)(1-x-y) = 0 \tag{7.16}$$
$$x(2+y) = 0 \tag{7.17}$$

One way to satisfy the second equation is to choose $x = 0$. Then the first equation becomes $y(1-y) = 0$, so $y = 0$ or $y = 1$. Now let's choose $y = -2$, then the first equation becomes $(x+2)(3-x) = 0$ so $x = -2$ or $x = 3$. So the four critical points are $(0,0), (0,1), (-2,-2)$, and $(3,-2)$. ■

7.3 Locally Linear Systems

We start with a few key theorem in this section. Consider the linear system

$$x' = Ax \tag{7.18}$$

Theorem 7.3.1 The critical point $x = 0$ of the linear system above is asymptotically stable if the eigenvalues r_1, r_2 are real and negative or have negative real part; stable, but not asymptotically stable if r_1 and r_2 are pure imaginary; unstable if r_1 and r_2 are real and either positive or if they have positive real part.

7.3.1 Introduction to Nonlinear Systems

The general form of the two dimensional system of differential equations is

$$x_1' = f_1(x_1, x_2) \tag{7.19}$$
$$x_2' = f_2(x_1, x_2) \tag{7.20}$$

For systems like this it is hard to find trajectories analytically, as we did for linear systems. Thus we need to discuss the behavior of these solutions.

There are some features of nonlinear phase portraits that we should be aware of:

(1) The **fixed** or **critical points** which are the equilibrium or steady-state solutions. These correspond to points x satisfying $\mathbf{f}(\mathbf{x}) = 0$. So x_1 and x_2 are zeroes for both f_1 and f_2.
(2) The **closed orbits**, which correspond to solutions that are periodic for both x_1 and x_2.
(3) How trajectories are arranged, new fixed points and closed orbits.
(4) The stability or instability of fixed points and closed orbits, which of these attract nearby trajectories and which repel them?

Theorem 7.3.2 *(Existence and Uniqueness)* Consider the initial value problem

$$\mathbf{x}' = \mathbf{f}(x) \quad x(0) = x_0 \tag{7.21}$$

If f is continuous and its partial derivatives on some region containing x_0, then the initial value problem has a unique solution $x(t)$ on some interval near $t = 0$.

Note: The theorem asserts that no two trajectories can intersect.

7.3.2 Linearization around Critical Points

To begin we always start by finding the critical points, which correspond to the equilibrium solutions of the system. If the system is linear the only critical point is the origin, $(0,0)$. Nonlinear systems can have many fixed points and we want to determine the behavior of the trajectories near these points. Consider,

$$x' = f(x,y) \tag{7.22}$$
$$y' = g(x,y) \tag{7.23}$$

The goal of linearization is to use our knowledge of linear systems to conclude what we can about the phase portrait near (x_0, y_0). We will try to approximate our nonlinear system by a linear system, which we can then classify. Since (x_0, y_0) is a fixed point, and the only fixed point of a linear system is the origin, we will want to change variables so that (x_0, y_0) becomes the origin of the new coordinate system. Thus, let

$$\begin{aligned} u &= x - x_0 && (7.24)\\ v &= y - y_0. && (7.25) \end{aligned}$$

We need to rewrite our differential equation in terms of u and v.

$$\begin{aligned} u' &= x' && (7.26)\\ &= f(x,y) && (7.27)\\ &= f(x_0+u, y_0+v) && (7.28) \end{aligned}$$

The natural thing to do is a Taylor Expansion of f near (x_0, y_0).

$$\begin{aligned} &= f(x_0,y_0) + u\frac{\partial f}{\partial x}(x_0,y_0) + v\frac{\partial f}{\partial y}(x_0,y_0) + \text{higher order terms} && (7.29)\\ &= u\frac{\partial f}{\partial x}(x_0,y_0) + v\frac{\partial f}{\partial y}(x_0,y_0) + H.O.T. && (7.30) \end{aligned}$$

recall that $f(x_0, y_0) = 0$ (since it is a fixed point). To simplify notation we will sometimes write $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ which are evaluated at (x_0, y_0), but it is important to keep this in mind. The partial derivatives are numbers not functions. Also, recall we are considering what happens very close to our fixed point, u and v are both small, and hence the higher order terms are smaller still and will be disregarded in computations. By a similar computation we have

$$v' = u\frac{\partial g}{\partial x} + v\frac{\partial g}{\partial y} + H.O.T. \tag{7.31}$$

Ignoring the small higher order terms, we can write this system of rewritten differential equations in matrix form. The **linearized system** near (x_0, y_0) is

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \\ \frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial y}(x_0,y_0) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{7.32}$$

We will use, from this point on, the notation $f_x = \frac{\partial f}{\partial x}$. The matrix

$$A = \begin{pmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{pmatrix} \tag{7.33}$$

is called the **Jacobian Matrix** at (x_0, y_0) of the vector-valued function $\mathbf{f}(x) = \begin{pmatrix} f(x_1,x_2) \\ g(x_1,x_2) \end{pmatrix}$. In multivariable calculus, the Jacobian matrix is appropriate analogue of the single variable calculus derivative. We then study this linear system with standard techniques.

7.4 Predator-Prey Equations

Recall from Chapter 1 we discussed equations representing populations of animals which grow without predators. Now we want to consider a system where one species (predator) preys on the other species (prey), while the prey lives on another food source.

An example is a closed forest where foxes prey on rabbits and rabbits eat vegetation. At a lake, ladybugs are predators and aphids are prey.

We denote by x and y the populations of prey and predators respectively, at time t. We need the following assumptions to construct our model.

(1) In the absence of a predator, the prey grows at a rate proportional to the current population. So $\frac{dx}{dt} = ax$, when $y = 0$.

(2) In the absence of the prey, the predator dies out. Thus $\frac{dy}{dt} = -cy$ where $c > 0$ when $x = 0$.

(3) Then number of encounters between predator and prey is proportional to the product of their populations. Each such encounter tends to promote the growth of the predator and to inhibit the growth of the prey. Thus the growth rate of the predator is increased by a term of the form γxy, while the growth rate of the prey is decreased by a term $-\alpha xy$, where γ and α are positive constants.

From these assumptions we can form the following equations

$$\frac{dx}{dt} = ax - \alpha xy = x(a - \alpha y), \tag{7.34}$$

$$\frac{dy}{dt} = -cy + \gamma xy = y(-c + \gamma y) \tag{7.35}$$

The constants a, c, α, and γ are all positive constants. a and c are the growth rate of the prey and the death rate of the predators respectively. α and γ are measures of the effect of interaction between the two species. Equations (7.34)-(7.35) are known as the **Lotka-Volterra** equations.

The goal is to determine the qualitative behavior of the solutions (trajectories) of the system (7.34)-(7.35) for arbitrary positive initial values of x and y.

■ **Example 7.2** Describe the solutions to the system

$$\frac{dx}{dt} = x - 0.5xy = F(x,y) \tag{7.36}$$

$$\frac{dy}{dt} = -0.75y + 0.25xy = G(x,y) \tag{7.37}$$

for x and y positive.

First we begin with techniques learned in 9.2 and find the critical points. The critical points of the system are the solutions of the equations

$$x - 0.5xy = 0, \quad -0.75y + 0.25xy = 0 \tag{7.38}$$

So the critical points are $(0,0)$ and $(3,2)$.

Next using 9.3, we want to consider the local behavior of the solutions near each critical point. At $(0,0)$ consider the Jacobian

$$J = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 1 - 0.5y & -0.5x \\ 0.25y & -0.75 + 0.25x \end{pmatrix} \tag{7.39}$$

For $(0,0)$ the Jacobian is

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -0.75 \end{pmatrix} \tag{7.40}$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -0.75$, since they are real of opposite signs Theorem 5 from Section 9.3 says we have a saddle point and thus $(0,0)$ is unstable.

At $(3,2)$ the Jacobian is

$$J = \begin{pmatrix} 0 & -1.5 \\ 0.5 & 0 \end{pmatrix} \tag{7.41}$$

The eigenvalues $\lambda_1 = \frac{\sqrt{3}i}{2}$ and $\lambda_2 = -\frac{\sqrt{3}i}{2}$. Since the eigenvalues are pure imaginary, $(3,2)$ is a center and thus a stable critical point. Besides solutions starting on coordinate axes, all solutions will circle around $(3,2)$. ■

Using this example consider the general system (7.34)-(7.35). First find the critical points of the system which are solutions to

$$x(a-\alpha y) = 0, \quad y(-c+\gamma x) = 0 \tag{7.42}$$

so the critical points are $(0,0)$ and $(\frac{c}{\gamma}, \frac{a}{\alpha})$. We first examine the solutions near each critical point. The Jacobian is

$$J = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} a-\alpha y & -\alpha x \\ \gamma y & -c+\gamma x \end{pmatrix} \tag{7.43}$$

Consider $(0,0)$, which has Jacobian

$$\begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix} \tag{7.44}$$

The eigenvalues are a and $-c$. Recall a and c are both positive real constants. So the eigenvalues are real values of opposite signs, thus $(0,0)$ is a saddle point and unstable. This makes sense since if we move slightly away from zero and introduce predators and prey, then the system would take off and the prey population would grow away from 0, while if we introduced a predator and no prey it would decay to zero.

Next consider $(\frac{c}{\gamma}, \frac{a}{\alpha})$. The Jacobian is

$$\begin{pmatrix} 0 & -\frac{\alpha c}{\gamma} \\ \frac{\gamma a}{\alpha} & 0 \end{pmatrix} \tag{7.45}$$

which has eigenvalues $\lambda_{1,2} = \pm i\sqrt{ac}$. So the eigenvalues are pure imaginary and the critical point is a stable center. The direction field for all predator prey equations will have those components and the direction fields will look like Figure 1 and Figure 2.

If we use the fact they are separable equations and solve for the trajectories we find that the solutions are ellipses, so $(\frac{c}{\gamma}, \frac{a}{\alpha})$ is a center.

This is what we expect, because a predator-prey system should be cyclic. When the predator population gets to high there is not enough prey and predators die out. When the

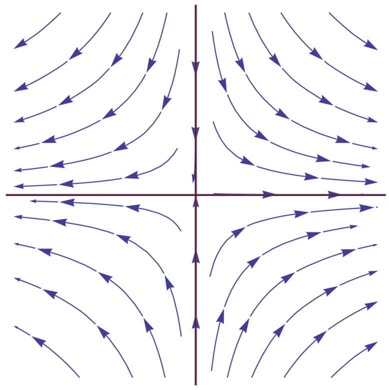


Figure 7.9: Direction Field for the critical point $(0,0)$

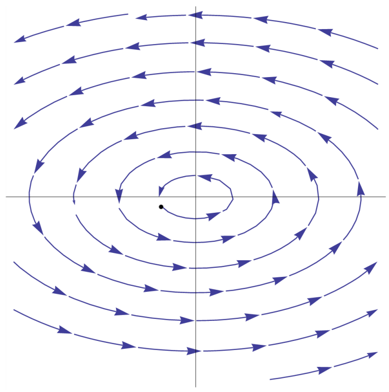


Figure 7.10: Direction Field for the critical point $(\frac{c}{\gamma}, \frac{a}{\alpha})$

prey population is high the system can support more predators. These systems are always cyclic. The solutions to this system are

$$x = \frac{c}{\gamma} + \frac{c}{\gamma} K \cos(\sqrt{ac}t + \Phi) \tag{7.46}$$

$$y = \frac{a}{\alpha} + \frac{a}{\alpha} K \sin(\sqrt{ac}t + \Phi) \tag{7.47}$$

where K and Φ are determined by the initial conditions.

From these solutions we get four main conclusions about predator prey models:

(1) The sizes of the predator and prey populations vary sinusoidally with period $\frac{2\pi}{\sqrt{ac}}$. This period of oscillation is independent of initial conditions.

(2) The predator and prey populations are out of phase by one quarter of a cycle. The prey leads and the predator lags as can be seen by plotting these due to sin and cos.

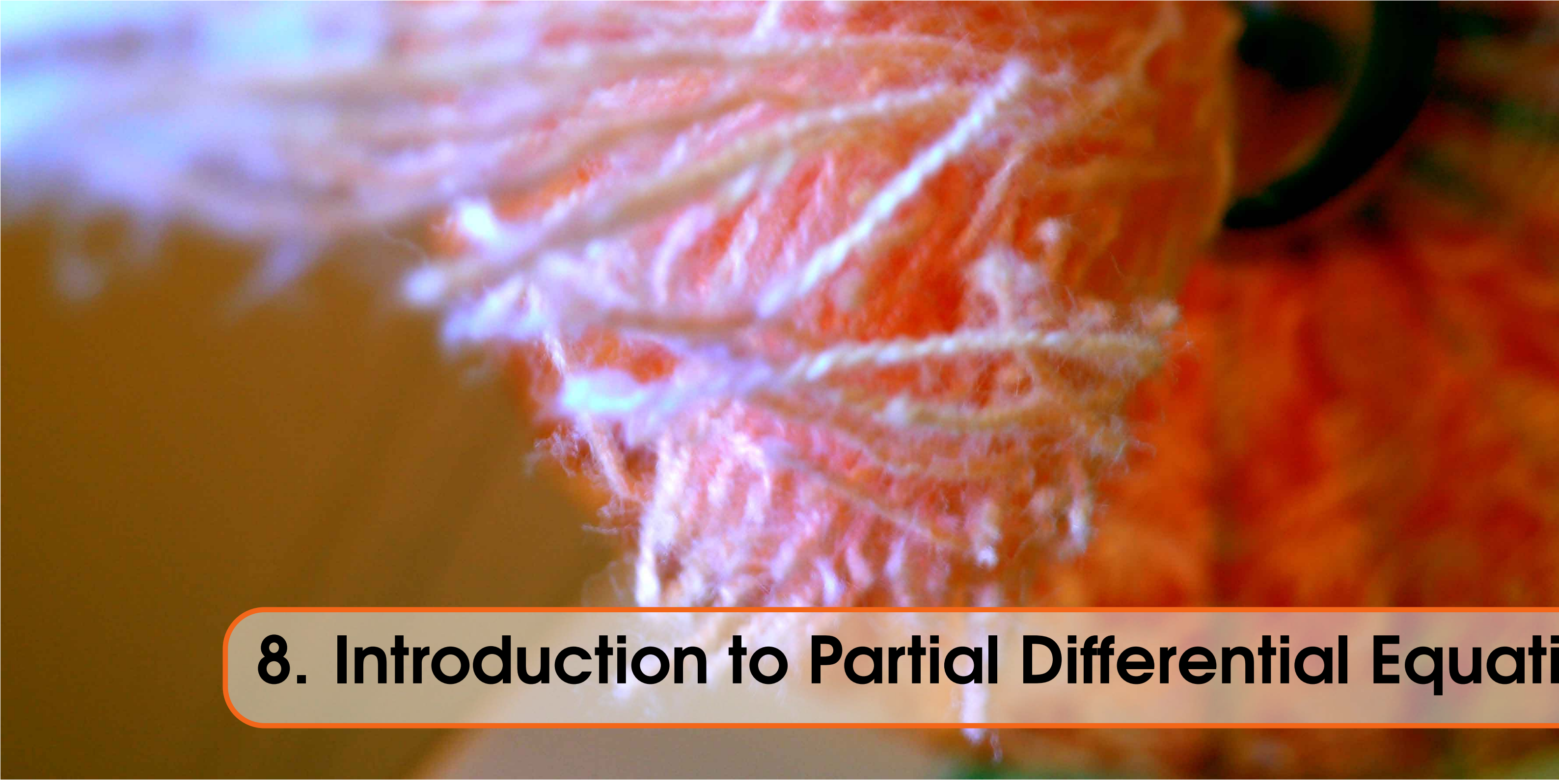
(3) The amplitudes of the oscillations are $\frac{Kc}{\gamma}$ for the prey and $\frac{a\sqrt{c}K}{\alpha\sqrt{a}}$ for the predator and hence depend on the initial conditions as well as the parameters of the problem.

(4) The average populations of predator and prey over one complete cycle are $\frac{c}{\gamma}$ and $\frac{a}{\alpha}$, respectively. These are the same as the equilibrium populations.

One criticism of this model is in the absence of a predator the prey grows without bound, which is not realistic due to a finite amount of food available for the prey. This is a basic model which is decently accurate and a building block for many advanced models.

VI

Partial Differential Equations



8. Introduction to Partial Differential Equati

8.1 Two-Point Boundary Value Problems and Eigenfunctions

8.1.1 Boundary Conditions

Up until now, we have studied ordinary differential equations and initial value problems. Now we shift to partial differential equations and boundary value problems. Partial differential equations are much more complicated, but are essential in modeling many complex systems found in nature. We need to specify how the solution should behave on the boundary of the region our equation is defined on. The data we prescribe are the **boundary values** or **boundary conditions**, and a combination of a differential equation and boundary conditions is called a **boundary value problem**.

Boundary Conditions depend on the domain of the problem. For an ordinary differential equation our domain was usually some interval on the real line. With a partial differential equation our domain might be an interval or it might be a square in the two-dimensional plane. To see how boundary conditions effect an equation let's examine how they affect the solution of an ordinary differential equation.

■ **Example 8.1** Let's consider the second order differential equation $y'' + y = 0$. Specifying boundary conditions for this equation involves specifying the values of the solution (or its derivatives) at two points, recall this is because the equation is second order. Consider the interval $(0, 2\pi)$ and specify the boundary conditions $y(0) = 0$ and $y(2\pi) = 0$. We know the solutions to the equation have the form

$$y(x) = A\cos(x) + B\sin(x). \tag{8.1}$$

by the method of characteristics. Applying the first boundary condition we see $0 = y(0) = A$. Applying the second condition gives $0 = y(2\pi) = B\sin(2\pi)$, but $\sin(2\pi)$ is already zero so B can be any number. So the solutions to this boundary value problem are any functions of the form

$$y(x) = B\sin(x). \tag{8.2}$$

■

■ **Example 8.2** Consider $y'' + y = 0$ with boundary conditions $y(0) = y(6) = 0$. this seems similar to the previous problem, the solutions still have the general form

$$y(x) = A\cos(x) + B\sin(x) \tag{8.3}$$

and the first condition still tells us $A = 0$. The second condition tells us that $0 = y(6) = B\sin(6)$. Now since $\sin(6) \neq 0$, so we must have $B = 0$ and the entire solution is $y(x) = 0$. ■

Boundary value problems occur in nature all the time. Examine the examples physically. We know from previous chapters $y'' + y = 0$ models an oscillator such as a rock hanging from a spring. The rock will oscillate with frequency $\frac{1}{2\pi}$. The condition $y(0) = 0$ just means that when we start observing, we want the rock to be at the equilibrium spot. If we specify $y(2\pi) = 0$, this will automatically happen, since the motion is 2π periodic. On the other hand, it is impossible for the rock to return to the equilibrium point after 6 seconds. It will come back in 2π seconds, which is more than 6. So the only possible way the rock can be at equilibrium after 6 seconds is if it does not leave, which is why the only solution is the zero solution.

The previous examples are **homogeneous boundary value problems**. We say that a boundary problem is homogeneous if the equation is homogeneous and the two boundary conditions involve zero. That is, homogeneous boundary conditions might be one of these types

$$\begin{aligned}
y(x_1) &= 0 \qquad y(x_2) = 0 && (8.4)\\
y'(x_1) &= 0 \qquad y(x_2) = 0 && (8.5)\\
y(x_1) &= 0 \qquad y'(x_2) = 0 && (8.6)\\
y'(x_1) &= 0 \qquad y'(x_2) = 0. && (8.7)
\end{aligned}$$

On the other hand, if the equation is nonhomogeneous or any of the boundary conditions do not equal zero, then the boundary value problem is **nonhomogenous** or **inhomogeneous**. Let's look at some examples of nonhomogeneous boundary value problems.

■ **Example 8.3** Take $y'' + 9y = 0$ with boundary conditions $y(0) = 2$ and $y(\frac{\pi}{6}) = 1$. The general solution to the differential equation is

$$y(x) = A\cos(3x) + B\sin(3x). \tag{8.8}$$

The two conditions give

$$\begin{aligned}
2 &= y(0) = A && (8.9)\\
1 &= y(\frac{\pi}{6}) = B && (8.10)
\end{aligned}$$

so that the solution is

$$y(x) = 2\cos(3x) + \sin(3x) \tag{8.11}$$

■

■ **Example 8.4** Take $y'' + 9y = 0$ with boundary conditions $y(0) = 2$ and $y(2\pi) = 2$. The general solution to the differential equation is

$$y(x) = A\cos(3x) + B\sin(3x). \tag{8.12}$$

The two conditions give

$$2 = y(0) = A \tag{8.13}$$
$$2 = y(2\pi) = A. \tag{8.14}$$

This time the second condition did not give and new information, like in Example 1 and B does not affect whether or not the solution satisfies the boundary conditions or not. We then have infinitely many solutions of the form

$$y(x) = 2\cos(3x) + B\sin(3x) \tag{8.15}$$

■

■ **Example 8.5** Take $y'' + 9y = 0$ with boundary conditions $y(0) = 2$ and $y(2\pi) = 4$. The general solution to the differential equation is

$$y(x) = A\cos(3x) + B\sin(3x). \tag{8.16}$$

The two conditions give

$$2 = y(0) = A \tag{8.17}$$
$$4 = y(2\pi) = A. \tag{8.18}$$

On one hand, $A = 2$ and by the second equation $A = 4$. This is impossible and this boundary value problem has no solutions. ■

These examples illustrate that a small change to the boundary conditions can dramatically change the problem, unlike small changes in the initial data for initial value problems.

8.1.2 Eigenvalue Problems

Recall the system studied extensively in previous chapters

$$Ax = \lambda x \tag{8.19}$$

where for certain values of λ, called eigenvalues, there are nonzero solutions called eigenvectors. We have a similar situation with boundary value problems.

Consider the problem

$$y'' + \lambda y = 0 \tag{8.20}$$

with boundary conditions $y(0) = 0$ and $y(\pi) = 0$. The values of λ where we get nontrivial (nonzero) solutions will be **eigenvalues**. The nontrivial solutions themselves are called **eigenfunctions**.

We need to consider three cases separately.

(1) If $\lambda > 0$, then it is convenient to let $\lambda = \mu^2$ and rewrite the equation as

$$y'' + \mu^2 y = 0 \tag{8.21}$$

The characteristic polynomial is $r^2 + \mu^2 = 0$ with roots $r = \pm i\mu$. So the general solution is

$$y(x) = A\cos(\mu x) + B\sin(\mu x) \tag{8.22}$$

Note that $\mu \neq 0$ since $\lambda > 0$. Recall the boundary conditions are $y(0) = 0$ and $y(\pi) = 0$. So the first boundary condition gives $A = 0$. The second boundary condition reduces to

$$B\sin(\mu\pi) = 0 \tag{8.23}$$

For nontrivial solutions $B \neq 0$. So $\sin(\mu\pi) = 0$. Thus $\mu = 1, 2, 3, \ldots$ and thus the eigenvalues λ_n are $1, 4, 9, \ldots, n^2$. The eigenfunctions are only determined up to arbitrary constant, so convention is to choose the arbitrary constant to be 1. Thus the eigenfunctions are

$$y_1(x) = \sin(x) \quad y_2(x) = \sin(2x), \ldots, y_n(x) = \sin(nx) \tag{8.24}$$

(2) If $\lambda < 0$, let $\lambda = -\mu^2$. So the above equation becomes

$$y'' - \mu^2 y = 0 \tag{8.25}$$

The characteristic equation is $r^2 - \mu^2 = 0$ with roots $r = \pm\mu$, so its general solution can be written as

$$y(x) = A\cosh(\mu x) + B\sinh(\mu x) = Ce^{\mu x} + De^{-\mu x} \tag{8.26}$$

The first boundary condition, if considering the first form, gives $A = 0$. The second boundary condition gives $B\sinh(\mu\pi) = 0$. Since $\mu \neq 0$, then $\sinh(\mu\pi) \neq 0$, and therefore $B = 0$. So for $\lambda < 0$ the only solution is $y = 0$, there are no nontrivial solutions and thus no eigenvalues.

(3) If $\lambda = 0$, then the equation above becomes

$$y'' = 0 \tag{8.27}$$

and the general solution if we integrate twice is

$$y(x) = Ax + B \tag{8.28}$$

The boundary conditions are only satisfied when $A = 0$ and $B = 0$. So there is only the trivial solution $y = 0$ and $\lambda = 0$ is not an eigenvalue.

To summarize we only get **real** eigenvalues and eigenvectors when $\lambda > 0$. There may be complex eigenvalues. A basic problem studied later in the chapter is

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0 \tag{8.29}$$

Hence the eigenvalues and eigenvectors are

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad y_n(x) = \sin(\frac{n\pi x}{L}) \quad \text{for } n = 1, 2, 3, \ldots \tag{8.30}$$

This is the classical **Euler Buckling Problem**.

Review Euler's Equations:

■ **Example 8.6** Consider equation of the form

$$t^2y'' + ty' + y = 0 \tag{8.31}$$

and let $x = \ln(t)$. Then

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{1}{t}\frac{dy}{dx} \tag{8.32}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}(\frac{dy}{dx})\frac{1}{t} + \frac{dy}{dx}(\frac{1}{t})\frac{dy}{dx} \tag{8.33}$$

$$= \frac{d^2y}{dx^2}\frac{1}{t^2} + \frac{dy}{dx}(-\frac{1}{t^2}) \tag{8.34}$$

$$\tag{8.35}$$

Plug these back into the original equation

$$t^2y'' + ty + y = \frac{d^2y}{dx^2} - \frac{dy}{dx} + \frac{dy}{dx} + y = 0 \tag{8.36}$$

$$= y'' + y = 0 \tag{8.37}$$

Thus the characteristic equation is $r^2 + 1 = 0$, which has roots $r = \pm i$. So the general solution is

$$\hat{y}(x) = c_1 \cos(x) + c_2 \sin(x) \tag{8.38}$$

Recalling that $x = \ln(t)$ our final solution is

$$y(x) = c_1 \cos(\ln(t)) + c_2 \sin(\ln(t)) \tag{8.39}$$

■

8.2 Fourier Series

Last lecture, we identified solutions of the heat equation having the form

$$u_t = u_{xx} \tag{8.40}$$

$0 < x < l$, $t > 0$, with homogeneous Dirichlet conditions at $u(0,t) = u(l,t) = 0$, had the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{l})^2 kt} \sin(\frac{n\pi x}{l}). \tag{8.41}$$

while the heat equation with homogeneous Neumann conditions $u_x(0,t) = u_x(l,t) = 0$ had solutions of the form

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{l})^2 kt} \cos(\frac{n\pi x}{l}). \tag{8.42}$$

For this to make sense given an initial condition $u(x,0) = f(x)$, for the Dirichlet case we need to be able to write $f(x)$ as

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l}) \tag{8.43}$$

for some coefficients A_n, while in the Neumann case it must have the form

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{l}) \tag{8.44}$$

for appropriate coefficients. Equation (8.348) is called a **Fourier Sine Series** of $f(x)$ and an expression like Equation (8.363) is called a **Fourier Cosine Series** of $f(x)$.

There are two key things to keep in mind:

(1) Is it possible to find appropriate coefficients for the Fourier Sine and Cosine series for a given $f(x)$?

(2) For which $f(x)$ will the Fourier series converge, if any? What will the Fourier Series converge to?

8.2.1 The Euler-Fourier Formula

We have a famous formula for the Fourier Coefficients, called the **Euler-Fourier Formula**.

Fourier Sine Series

Start by considering the Fourier Sine Series

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l}). \tag{8.45}$$

How can we find the coefficients A_n? Observe that sine functions have the following property

$$\int_0^l \sin(\frac{n\pi x}{l}) \sin(\frac{m\pi x}{l}) dx = 0 \tag{8.46}$$

if $m \neq n$ are both positive integers. This can be seen by direct integration. Recall the trig identity

$$\sin(a)\sin(b) = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b). \tag{8.47}$$

Then the integral in Equation (8.253) equals

$$\frac{l}{2(m-n)\pi} \sin(\frac{(m-n)\pi x}{l})|_0^l - \frac{l}{2(m+n)\pi} \sin(\frac{(m+n)\pi x}{l})|_0^l \tag{8.48}$$

so long as $m \neq n$. But these terms are just linear combinations of $\sin((m \pm n)\pi)$ and $\sin(0)$, and thus everything is zero.

Now, fix m and multiply Equation (8.348) (Fourier Sine Series) by $\sin(\frac{m\pi x}{l})$. Integrating term by term we get

$$\int_0^l f(x)\sin(\frac{m\pi x}{l})dx = \int_0^l \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l})\sin(\frac{m\pi x}{l})dx \quad (8.49)$$

$$= \sum_{n=1}^{\infty} \int_0^l \sin(\frac{n\pi x}{l})\sin(\frac{m\pi x}{l})dx. \quad (8.50)$$

Due to the above work the only term that remains is when $m = n$. So all we have left is

$$\int_0^l f(x)\sin(\frac{m\pi x}{l})dx = A_m \int_0^l \sin^2(\frac{m\pi x}{l}) = \frac{1}{2}lA_m \quad (8.51)$$

and so

$$A_m = \frac{2}{l}\int_0^l f(x)\sin(\frac{m\pi x}{l})dx. \quad (8.52)$$

In summary. If $f(x)$ has a Fourier sine expansion, the coefficients must be given by Equation (8.313). These are the only possible coefficients for such a series, but we have not shown that the Fourier Sine Series is a valid expression for $f(x)$.

■ **Example 8.7** Compute a Fourier Sine Series for $f(x) = 1$ on $0 \le x \le l$.

By Equation (8.313), the coefficients must be given by

$$A_m = \frac{2}{l}\int_0^l \sin(\frac{m\pi x}{l})dx. \quad (8.53)$$

$$= -\frac{2}{m\pi}\cos(\frac{m\pi x}{l})|_0^l \quad (8.54)$$

$$= \frac{2}{m\pi}(1-\cos(m\pi)) = \frac{2}{m\pi}(1-(-1)^m). \quad (8.55)$$

So we have $A_m = \frac{4}{m\pi}$ if m is odd and $A_m = 0$ if m is even. Thus, on $(0,l)$

$$1 = \frac{4}{\pi}(\sin(\frac{\pi x}{l}) + \frac{1}{3}\sin(\frac{3\pi x}{l}) + \frac{1}{5}\sin(\frac{5\pi x}{l}) + \ldots) \quad (8.56)$$

$$= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(\frac{(2n-1)\pi x}{l}). \quad (8.57)$$

■

■ **Example 8.8** Compute the Fourier Sine Series for $f(x) = x$ on $0 \le x \le l$.

In this case Equation (8.313) yields a formula for the coefficients

$$A_m = \frac{2}{l}\int_0^l x\sin(\frac{m\pi x}{l})dx \quad (8.58)$$

$$= -\frac{2x}{m\pi}\cos(\frac{m\pi x}{l})|_0^l + \frac{2l}{m^2\pi^2}\sin(\frac{m\pi x}{l})|_0^l \quad (8.59)$$

$$= -\frac{2l}{m\pi}\cos(m\pi) + \frac{2l}{m^2\pi^2}\sin(m\pi) \quad (8.60)$$

$$= (-1)^{m+1}\frac{2l}{m\pi}. \quad (8.61)$$

So on $(0,l)$, we have

$$x = \frac{2l}{\pi}\left(\sin(\frac{\pi x}{l}) - \frac{1}{2}\sin(\frac{2\pi x}{l}) + \frac{1}{3}\sin(\frac{3\pi x}{l}) - ...\right) \tag{8.62}$$

$$= \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left(\frac{(2n-1)\pi x}{l}\right) - \frac{1}{2n}\sin\left(\frac{2n\pi x}{l}\right). \tag{8.63}$$

■

Fourier Cosine Series

Now let's consider the Fourier Cosine Series

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{l}). \tag{8.64}$$

We can use the following property of cosine

$$\int_0^l \cos(\frac{n\pi x}{l})\cos(\frac{m\pi x}{l})dx = 0. \tag{8.65}$$

Verify this for an exercise.

By the exact same computation as before for sines, we replace sines with cosines, if $m \neq 0$ we get

$$\int_0^l f(x)\cos(\frac{m\pi x}{l})dx = A_m \int_0^l \cos^2(\frac{m\pi x}{l})dx = \frac{1}{2}lA_m. \tag{8.66}$$

If $m = 0$, we have

$$\int_0^l f(x)\cdot 1 dx = \frac{1}{2}A_0 \int_0^l 1^2 = \frac{1}{2}lA_0. \tag{8.67}$$

Thus, for all $m > 0$, we have

$$A_m = \frac{2}{l}\int_0^l f(x)\cos(\frac{m\pi x}{l})dx. \tag{8.68}$$

This is why we have the $\frac{1}{2}$ in front of A_0 (so it has the same form as A_m for $m \neq 0$).

■ **Example 8.9** Compute the Fourier Cosine Series for $f(x) = 1$ on $0 \leq x \leq l$.

By Equation (8.103), the coefficients when $m \neq 0$ are

$$A_m = \frac{2}{l}\int_0^l \cos(\frac{m\pi x}{l})dx \tag{8.69}$$

$$= \frac{2}{m\pi}\sin(\frac{m\pi x}{l})|_0^l \tag{8.70}$$

$$= \frac{2}{m\pi}\sin(m\pi) = 0. \tag{8.71}$$

So the only coefficient we have occurs are A_0, and this Fourier Cosine Series is then trivial

$$1 = 1 + 0\cos(\frac{\pi x}{l}) + 0\cos(\frac{2\pi x}{l}) + ... \tag{8.72}$$

■

■ **Example 8.10** Compute the Fourier Cosine Series for $f(x) = x$.
For $m \neq 0$,

$$A_m = \frac{2}{l}\int_0^l x\cos(\frac{m\pi x}{l})dx \tag{8.73}$$
$$= \frac{2x}{m\pi}\sin(\frac{m\pi x}{l}) + \frac{2l}{m^2\pi^2}\cos(\frac{m\pi x}{l})|_0^l \tag{8.74}$$
$$= \frac{2l}{m\pi}\sin(m\pi) + \frac{2l}{m^2\pi^2}(\cos(m\pi) - 1) \tag{8.75}$$
$$= \frac{2l}{m^2\pi^2}((-1)^m - 1) \tag{8.76}$$
$$= \begin{cases} -\frac{4l}{m^2\pi^2} & \text{m odd} \\ 0 & \text{m even} \end{cases}. \tag{8.77}$$

If $m = 0$, we have

$$A_0 = \frac{2}{l}\int_0^l xdx = l. \tag{8.78}$$

So on $(0,l)$, we have the Fourier Cosine Series

$$x = \frac{l}{2} - \frac{4l}{\pi^2}\left(\cos(\frac{\pi x}{l}) + \frac{1}{9}\cos(\frac{3\pi x}{l}) + \frac{1}{25}\cos(\frac{5\pi x}{l}) + ...\right) \tag{8.79}$$
$$= \frac{l}{2} + \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left(\frac{(2n-1)\pi x}{l}\right). \tag{8.80}$$

■

Full Fourier Series

The full Fourier Series of $f(x)$ on the interval $-l < x < l$, is defined as

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n\cos(\frac{n\pi x}{l}) + B_n\sin(\frac{n\pi x}{l}). \tag{8.81}$$

REMARK: Be careful, now the interval we are working with is twice as long $-l < x < l$.

The computation of the coefficients for the formulas is analogous to the Fourier Sine and Cosine series. We need the following set of identities:

$$\int_{-l}^{l}\cos(\frac{n\pi x}{l})\sin(\frac{m\pi x}{l})dx = 0 \quad \text{for } n,m \tag{8.82}$$
$$\int_{-l}^{l}\cos(\frac{n\pi x}{l})\cos(\frac{m\pi x}{l})dx = 0 \quad \text{for } n \neq m \tag{8.83}$$
$$\int_{-l}^{l}\sin(\frac{n\pi x}{l})\sin(\frac{m\pi x}{l})dx = 0 \quad \text{for } n \neq m \tag{8.84}$$
$$\int_{-l}^{l}1\cdot\cos(\frac{n\pi x}{l})dx = 0 = \int_{-l}^{l}1\cdot\sin(\frac{n\pi x}{l})dx. \tag{8.85}$$

Thus, using the same procedure as above for sine and cosine, we can get the coefficients. We fix m and multiply by $\cos(\frac{m\pi x}{l})$, and do the same for $\sin(\frac{m\pi x}{l})$. So we need to calculate the integrals of the squares

$$\int_{-l}^{l}\cos^2(\frac{n\pi x}{l})dx = 1 = \int_{-l}^{l}\sin^2(\frac{n\pi x}{l})dx \quad \text{and} \quad \int_{-l}^{l}1^2dx = 2l \tag{8.86}$$

EXERCISE: Verify the above integrals.

So we get the following formulas

$$A_m = \frac{1}{l}\int_{-l}^{l} f(x)\cos(\frac{m\pi x}{l})dx \quad (n = 1,2,3,...) \tag{8.87}$$

$$B_m = \frac{1}{l}\int_{-l}^{l} f(x)\sin(\frac{m\pi x}{l})dx \quad (n = 1,2,3,...) \tag{8.88}$$

for the coefficients of the full Fourier Series. Notice that the first equation is exactly the same as we got when considering the Fourier Cosine Series and the second equation is the same as the solution for the Fourier Sine Series. **NOTE:** The intervals of integration are different!

■ **Example 8.11** Compute the Fourier Series of $f(x) = 1+x$.

Using the above formulas we have

$$A_0 = \frac{1}{l}\int_{-l}^{l}(1+x)dx = 2 \tag{8.89}$$

$$A_m = \frac{1}{l}\int_{-l}^{l}(1+x)\cos(\frac{m\pi x}{l})dx \tag{8.90}$$

$$= \frac{1+x}{m\pi}\sin(\frac{m\pi x}{l}) + \frac{1}{m^2\pi^2}\cos(\frac{m\pi x}{l})|_{-l}^{l} \tag{8.91}$$

$$= \frac{1}{m^2\pi^2}(\cos(m\pi) - \cos(-m\pi)) = 0 \quad m \neq 0 \tag{8.92}$$

$$B_m = \frac{1}{l}\int_{-l}^{l}(1+x)\sin(\frac{m\pi x}{l})dx \tag{8.93}$$

$$= -\frac{1+x}{m\pi}\cos(\frac{m\pi x}{l}) + \frac{1}{m^2\pi^2}\sin(\frac{m\pi x}{l})|_{-l}^{l} \tag{8.94}$$

$$= -\frac{2l}{m\pi}\cos(m\pi) = (-1)^{m+1}\frac{2l}{m\pi}. \tag{8.95}$$

So the full Fourier series of $f(x)$ is

$$1+x = 1+\frac{2l}{\pi}(\sin(\frac{\pi x}{l}) - \frac{1}{2}\sin(\frac{2\pi x}{l}) + \frac{1}{3}\sin(\frac{3\pi x}{l}) - ...) \tag{8.96}$$

$$= 1+\frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(\frac{(2n-1)\pi x}{l}) - \frac{1}{2n}\sin(\frac{2n\pi x}{l}). \tag{8.97}$$

■

8.3 Convergence of Fourier Series

Last class we derived the Euler-Fourier formulas for the coefficients of the Fourier Series of a given function $f(x)$. For the **Fourier Sine Series** of $f(x)$ on the interval $(0,l)$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l}), \tag{8.98}$$

we have

$$A_n = \frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi x}{l})dx, \tag{8.99}$$

where $n = 1,2,3,...$ For the **Fourier Cosine Series** of $f(x)$ on $(0,l)$,

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{l}), \tag{8.100}$$

with the coefficients given by

$$A_n = \frac{2}{l}\int_0^l f(x)\cos(\frac{n\pi x}{l})dx, \tag{8.101}$$

where $n = 1,2,3,...$ Finally, for the full **Fourier Series** of $f(x)$, which is valid on the interval $(-l,l)$ (Note: **Not** the same interval as the previous two cases!),

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{l}) + B_n \sin(\frac{n\pi x}{l}), \tag{8.102}$$

the coefficients are given by

$$A_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos(\frac{n\pi x}{l})dx \quad n = 1,2,3,... \tag{8.103}$$

$$B_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin(\frac{n\pi x}{l})dx \quad n = 1,2,3,... \tag{8.104}$$

■ **Example 8.12** Compute the Fourier Series for $f(x) = \begin{cases} 2 & -2 \le x < -1 \\ 1-x & -1 \le x < 2 \end{cases}$ on the interval $(-2,2)$.

We start by using the Euler-Fourier Formulas. For the Cosine terms we find

$$\begin{aligned} A_0 &= \frac{1}{2}\int_{-2}^{2} f(x)dx \\ &= \frac{1}{2}\left(\int_{-2}^{-1} 2dx + \int_{-1}^{2} 1 - xdx\right) \\ &= \frac{1}{2}(2+\frac{3}{2}) = \frac{7}{4} \end{aligned}$$

and

$$\begin{aligned} A_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos(\frac{n\pi x}{2})dx \\ &= \frac{1}{2}\left(\int_{-2}^{-1} 2\cos(\frac{n\pi x}{2})dx + \int_{-1}^{2}(1-x)\cos(\frac{n\pi x}{2})dx\right) \\ &= \frac{1}{2}\left(\frac{4}{n\pi}\sin(\frac{n\pi x}{2})|_{-2}^{-1} + \frac{2(1-x)}{n\pi}\sin(\frac{n\pi x}{2})|_{-1}^{2} - \frac{4}{n^2\pi^2}\left(\cos(\frac{n\pi x}{2})|_{-1}^{2}\right)\right) \\ &= \frac{1}{2}\left(-\frac{4}{n\pi}\sin(\frac{n\pi}{2}) + \frac{4}{n\pi}\sin(\frac{n\pi}{2}) - \frac{4}{n^2\pi^2}(\cos(n\pi) - \cos(\frac{n\pi}{2}))\right) \\ &= \begin{cases} \frac{2}{n^2\pi^2} & n \text{ odd} \\ 0 & n = 4m \\ -\frac{4}{n^2\pi^2} & n = 4m+2 \end{cases} . \end{aligned}$$

Also, for the sine terms

$$
\begin{aligned}
B_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin(\frac{n\pi x}{2})dx \\
&= \frac{1}{2}\left(\int_{-2}^{-1} 2\sin(\frac{n\pi x}{2})dx + \int_{-1}^{2}(1-x)\sin(\frac{n\pi x}{2})dx\right) \\
&= \frac{1}{2}\left(-\frac{4}{n\pi}\cos(\frac{n\pi x}{2})|_{-2}^{-1} - \frac{2(1-x)}{n\pi}\cos(\frac{n\pi x}{2})|_{-1}^{2} - \frac{4}{n^2\pi^2}(\sin(\frac{n\pi x}{2})|_{-1}^{2}\right) \\
&= \frac{1}{2}\left(\frac{6}{n\pi}\cos(n\pi) - \frac{4}{n^2\pi^2}\sin(\frac{n\pi}{2})\right) \\
&= \begin{cases} \frac{3}{n\pi} & n \text{even} \\ -\frac{3}{n\pi} - \frac{2}{n^2\pi^2} & n = 4m+1 \\ -\frac{3}{n\pi} + \frac{2}{n^2\pi^2} & n = 4m+3 \end{cases}.
\end{aligned}
$$

So we have

$$
\begin{aligned}
f(x) = \frac{7}{8} &+ \sum_{m=1}^{\infty} \frac{2}{(4m+1)^2\pi^2}\cos\left(\frac{(4m+1)\pi x}{2}\right) + \left(-\frac{3}{(4m+1)\pi} - \frac{2}{(4m+1)^2\pi^2}\right)\sin\left(\frac{(4m+1)\pi x}{2}\right) \\
&- \frac{4}{(4m+2)^2\pi^2}\cos\left(\frac{(4m+2)\pi x}{2}\right) + \frac{3}{(4m+2)\pi}\sin\left(\frac{(4m+2)\pi x}{2}\right) \\
&+ \frac{2}{(4m+3)^2\pi^2}\cos\left(\frac{(4m+3)\pi x}{2}\right) + \left(-\frac{3}{(4m+3)\pi} + \frac{2}{(4m+3)^2\pi^2}\right)\sin\left(\frac{(4m+3)\pi x}{2}\right) \\
&+ \frac{3}{4m\pi}\sin\left(\frac{4m\pi x}{2}\right).
\end{aligned}
$$

■

This example represents a worst case scenario. There are a lot of Fourier coefficients to keep track of. Notice that for each value of m, the summand specifies four different Fourier terms (for $4m, 4m+1, 4m+2, 4m+3$). This can often happen and depending on l, even more terms maybe required.

8.3.1 Convergence of Fourier Series

So we know that if a function $f(x)$ is to have a Fourier Series on an appropriate interval, the coefficients have to be in the form of a Fourier Sine series (8.348) on $(0,l)$, a Fourier Cosine Series (8.363) on $(0,l)$, or the full Fourier Series (8.253) on $(-l,l)$. What do these series converge to? First consider the full Fourier Series.

We require that $f(x)$ is **piecewise smooth**. This is even stronger than the piecewise continuity we saw with Laplace Transforms. We want to divide $(-l,l)$ into a finite number of subintervals so that both $f(x)$ and its derivative $f'(x)$ are continuous on each interval. We also require that the only discontinuities at the boundary points of the subintervals are jump discontinuities (not asymptotically approaching infinity).

■ **Example 8.13** Any continuous function with continuous derivative on the desired interval is automatically piecewise smooth. This is proven later in an advanced analysis class. ■

■ **Example 8.14** Consider the function from Example 1

$$f(x) = \begin{cases} 2 & -2 \le x < -1 \\ 1-x & -1 \le x \le 2 \end{cases} . \tag{8.105}$$

$f(x)$ is continuous for all x in $(-2,2)$, but the derivative $f'(x)$ has a discontinuity at $x=-1$. This is a jump discontinuity, with $\lim_{x\to -1^-} f'(x) = 0$ and $\lim_{x\to -1^+} f'(x) = -1$. Thus, $f(x)$ is piecewise smooth. ■

The next thing to note is that even though we only need $f(x)$ to be defined on $(-l,l)$ to compute the Fourier Series, the Fourier Series itself is defined for all x. Also, all of the terms in a Fourier Series are $2l$-periodic. They are either constants of have the form $\sin(\frac{n\pi x}{l})$ or $\cos(\frac{n\pi x}{l})$. So we can regard the Fourier Series either as the expansion of a function on $(-l,l)$ or as the expansion of a $2l$-periodic function on $-\infty < x < \infty$.

What will this $2l$-periodic function be? It will have to coincide on $(-l,l)$ with $f(x)$ (since it is also the expansion of $f(x)$ on that interval) and still be $2l$-periodic. Define the **periodic extension** of $f(x)$ to be

$$f_{per}(x) = f(x-2lm) \quad \text{for} \quad -l+2lm < x < l+2lm \tag{8.106}$$

for all integers m.

REMARK: The definition (8.106) does not specify what the periodic extension is at the endpoints $x = l+2lm$. This is because the extension will, in general, have jumps at these points. This happens when $f(-l^+) \neq f(l^-)$.

No what does the Fourier Series of $f(x)$ converge to?

Theorem 8.3.1 *(Fourier Convergence Theorem)* Suppose $f(x)$ is piecewise smooth on $(-l,l)$. Then at $x = x_0$, the Fourier Series of $f(x)$ will converge to
(1) $f_{per}(x_0)$ if f_{per} is continuous at x_0 or
(2) The average of the one sided limits $\frac{1}{2}[f_{per}(x_0^+) + f_{per}(x_0^-)]$ is f_{per} has a jump discontinuity at $x = x_0$.

Theorem 1 tells us that on the interval $(-l,l)$ the Fourier Series will **almost** converge to the original function $f(x)$, with the only problems occurring at the discontinuities.

■ **Example 8.15** What does the Fourier Series of $f(x) = \begin{cases} 1 & -3 \le x \le 0 \\ 2x & 0 < x \le 3 \end{cases}$ will converge to at $x = -2, 0, 3, 5, 6$?

The first two points are inside the original interval of definition of $f(x)$, so we can just directly consider $f_{per}(x)$. The only discontinuity of $f(x)$ occurs at $x=0$. So at $x=-2$, $f(x)$ is nice and continuous. The Fourier Series will converge to $f(-2) = 1$. On the other hand, at $x = 0$ we have a jump discontinuity, so the Fourier Series will converge to the average of the one-sided limits. $f(0^+) = \lim_{x\to 0^+} f(x) = 0$ and $f(0^-) = \lim_{x\to 0^-} f(x) = 1$, so the Fourier Series will converge to $\frac{1}{2}[f(0^+) + f(0^-)] = \frac{1}{2}$.

What happens at the other points? Here we consider $f_{per}(x)$ and where it has jump discontinuities. These can only occur either at $x = x_0 + 2lm$ where $-l < x_0 < l$ is a jump discontinuity of $f(x)$ or at endpoints $x = \pm l + 2lm$, since the periodic extension might not "sync up" at these points, producing a jump discontinuity.

At $x = 3$, we are at one of these "boundary points" and the left-sided limit is 6 while the right-sided limit is 1. Thus the Fourier Series will converge here to $\frac{6+1}{2} = \frac{7}{2}$. $x = 5$ is a point of continuity for $f_{per}(x)$ and so the Fourier Series will converge to $f_{per}(5) = f(-1) = 1$. $x = 6$, is a jump discontinuity (corresponding to $x = 0$), so the Fourier Series will converge to $\frac{1}{2}$. ■

■ **Example 8.16** Where does the Fourier Series for $f(x) = \begin{cases} 2 & -2 \le x < -1 \\ 1-x & -1 \le x \le 2 \end{cases}$ converge at $x = -7, -1, 6$?

None of the points are inside $(-2,2)$ where $f(x)$ is discontinuous. The only points where the periodic extension might be discontinuous are the "boundary points" $x = \pm 2 + 4k$. In fact, since $f(-2) \neq f(2)$, these will be points of discontinuity. So $f_{per}(x)$ is continuous at $x = -7$, since it is not a boundary point and we have $f_{per}(-7) = f(1) = 0$, which is what the Fourier Series will converge to. The same for $x = -1$, the Fourier Series will converge to $f(-1) = \frac{2+2}{2} = 2$.

For $x = 6$ we are at an endpoint. The left-sided limit is -1, while the right-sided limit is 2, so the Fourier Series will converge to their average $\frac{1}{2}$. ■

8.4 Even and Odd Functions

Before we can apply the discussion from Section 10.3 to the Fourier Sine and Cosine Series, we need to review some facts about Even and Odd Functions.

Recall that an **even** function is a function satisfying

$$g(-x) = g(x). \tag{8.107}$$

This means that the graph $y = g(x)$ is symmetric with respect to the y-axis. An **odd** function satisfies

$$g(-x) = -g(x) \tag{8.108}$$

meaning that its graph $y = g(x)$ is symmetric with respect to the origin.

■ **Example 8.17** A monomial x^n is even if n is even and odd if n is odd. $\cos(x)$ is even and $\sin(x)$ is odd. Note $\tan(x)$ is odd. ■

There are some general rules for how products and sums behave:
(1) If $g(x)$ is odd and $h(x)$ is even, their product $g(x)h(x)$ is odd.
(2) If $g(x)$ and $h(x)$ are either both even or both odd, $g(x)h(x)$ is even.
(3) The sum of two even functions or two odd functions is even or odd, respectively.
To remember the rules consider how many negative signs come out of the arguments.
EXERCISE: Verify these rules.
(4) The sum of an even and an odd function can be anything. In fact, any function on $(-l,l)$ can be written as a sum of an even function, called the **even part**, and an odd function, called the **odd part**.
(5) Differentiation and Integration can change the parity of a function. If $f(x)$ is even, $\frac{df}{dx}$ and $\int_0^x f(s)ds$ are both odd, and vice versa.

The graph of an odd function $g(x)$ must pass through the origin by definition. This also tells us that if $g(x)$ is even, as long as $g'(0)$ exists, then $g'(0) = 0$.

Definite Integrals on symmetric intervals of odd and even functions have useful properties

$$\int_{-l}^{l}(odd)dx = 0 \quad \text{and} \quad \int_{-l}^{l}(even)dx = 2\int_{0}^{l}(even)dx \tag{8.109}$$

Given a function $f(x)$ defined on $(0,l)$, there is only one way to extend it to $(-l,l)$ to an even or odd function. The **even extension** of $f(x)$ is

$$f_{even}(x) = \begin{cases} f(x) & \text{for } 0 < x < l \\ f(-x) & \text{for } -l < x < 0. \end{cases} \tag{8.110}$$

This is just its reflection across the y-axis. Notice that the even extension is not necessarily defined at the origin.

The **odd extension** of $f(x)$ is

$$f_{odd}(x) = \begin{cases} f(x) & \text{for } 0 < x < l \\ -f(-x) & \text{for } -l < x < 0 \\ 0 & \text{for } x = 0 \end{cases} \quad . \tag{8.111}$$

This is just its reflection through the origin.

8.4.1 Fourier Sine Series

Each of terms in the Fourier Sine Series for $f(x)$, $\sin(\frac{n\pi x}{l})$, is odd. As with the full Fourier Series, each of these terms also has period $2l$. So we can think of the Fourier Sine Series as the expansion of an odd function with period $2l$ defined on the entire line which coincides with $f(x)$ on $(0,l)$.

One can show that the full Fourier Series of f_{odd} is the same as the Fourier Sine Series of $f(x)$. Let

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n\cos(\frac{n\pi x}{l}) + B_n\sin(\frac{n\pi x}{l}) \tag{8.112}$$

be the Fourier Series for $f_{odd}(x)$, with coefficients given in Section 10.3

$$A_n = \frac{1}{l}\int_{-l}^{l} f_{odd}(x)\cos(\frac{n\pi x}{l})dx = 0 \tag{8.113}$$

But f_{odd} is odd and cos is even, so their product is again odd.

$$B_n = \frac{1}{l}\int_{-l}^{l} f_{odd}(x)\sin(\frac{n\pi x}{l})dx \tag{8.114}$$

But both f_{odd} and sin are odd, so their product is even.

$$B_n = \frac{2}{l}\int_{0}^{l} f_{odd}(x)\sin(\frac{n\pi x}{l})dx \tag{8.115}$$

$$= \frac{2}{l}\int_{0}^{l} f(x)\sin(\frac{n\pi x}{l})dx, \tag{8.116}$$

which are just the Fourier Sine coefficients of $f(x)$. Thus, as the Fourier Sine Series of $f(x)$ is the full Fourier Series of $f_{odd}(x)$, the $2l$-periodic odd function that the Fourier Sine Series expands is just the periodic extension f_{odd}.

This goes both ways. If we want to compute a Fourier Series for an odd function on $(-l,l)$ we can just compute the Fourier Sine Series of the function restricted to $(0,l)$. It will **almost** converge to the original function on $(-l,l)$ with the only issues occurring at any jump discontinuities. The **only works for odd functions**. Do not use the formula for the coefficients of the Sine Series, unless you are working with an odd function.

■ **Example 8.18** Write down the odd extension of $f(x) = l - x$ on $(0,l)$ and compute its Fourier Series.

To get the odd extension of $f(x)$ we will need to see how to reflect f across the origin. What we end up with is the function

$$f_{odd}(x) = \begin{cases} l-x & 0<x<l \\ -l-x & -l<x<0 \end{cases} . \tag{8.117}$$

Now. what is the Fourier Series of $f_{odd}(x)$? By the previous discussion, we know that is will be identical to the Fourier Sine Series of $f(x)$, as this will converge on $(-l,0)$ to f_{odd}. So we have

$$f_{odd}(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l}), \tag{8.118}$$

where

$$\begin{aligned} A_n &= \frac{2}{l}\int_0^l (l-x)\sin(\frac{n\pi x}{l})dx && (8.119) \\ &= \frac{2}{l}[-\frac{l(l-x)}{n\pi}\cos(\frac{n\pi x}{l}) - \frac{l^2}{n^2\pi^2}\sin(\frac{n\pi x}{l})]_0^l && (8.120) \\ &= \frac{2l}{n\pi}. && (8.121) \end{aligned}$$

Thus the desired Fourier Series is

$$f_{odd}(x) = \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(\frac{n\pi x}{l}). \tag{8.122}$$

■

You might wonder how we were able a few lectures ago to compute the Fourier Sine Series of a constant function like $f(x) = 1$ which is even. It is important to remember that if we are computing the Fourier Sine Series for $f(x)$, it only needs to converge to $f(x)$ on $(0,l)$, where issues of evenness and oddness do not occur. The Fourier Sine Series will converge to the odd extension of $f(x)$ on $(-l,l)$.

■ **Example 8.19** Find the Fourier Series for the odd extension of

$$f(x) = \begin{cases} \frac{3}{2} & 0<x<\frac{3}{2} \\ x-\frac{3}{2} & \frac{3}{2}<x<3. \end{cases} \tag{8.123}$$

on $(-3,3)$.

The Fourier Series for $f_{odd}(x)$ on $(-3,3)$ will just be the Fourier Sine Series for $f(x)$ on $(0,3)$. The Fourier Sine coefficients for $f(x)$ are

$$\begin{aligned}
A_n &= \frac{2}{3}\int_0^3 f(x)\sin(\frac{n\pi x}{l})dx & (8.124)\\
&= \frac{2}{3}\left(\int_0^{\frac{3}{2}} \frac{3}{2}\sin(\frac{n\pi x}{3})dx + \int_{\frac{3}{2}}^3 (x-\frac{3}{2})\sin(\frac{n\pi x}{3})\right) & (8.125)\\
&= \frac{2}{3}\left(-\frac{9}{2n\pi}\cos(\frac{n\pi x}{3})|_0^{\frac{3}{2}} + \frac{3(x-\frac{3}{2})}{n\pi}\cos(\frac{n\pi x}{3})|_{\frac{3}{2}}^3 + \frac{9}{n^2\pi^2}\sin(\frac{n\pi x}{3})|_{\frac{3}{2}}^3\right) & (8.126)\\
&= \frac{2}{3}\left(-\frac{9}{2n\pi}(\cos(\frac{n\pi}{2})-1) - \frac{9}{2n\pi}\cos(n\pi) - \frac{9}{n^2\pi^2}\sin(\frac{n\pi x}{2})\right) & (8.127)\\
&= \frac{2}{3}(\frac{9}{2n\pi}(1-\cos(\frac{n\pi}{2})+(-1)^{n+1}) - \frac{9}{n^2\pi^2}\sin(\frac{n\pi}{2})\Big) & (8.128)\\
&= \frac{3}{n\pi}\left(1-\cos(\frac{n\pi}{2})+(-1)^{n+1} - \frac{2}{n\pi}\sin(\frac{n\pi}{2})\right) & (8.129)
\end{aligned}$$

and the Fourier Series is

$$f_{odd}(x) = \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}[1-\cos(\frac{n\pi}{2})+(-1)^{n+1} - \frac{2}{n\pi}\sin(\frac{n\pi}{2})]\sin(\frac{n\pi x}{3}). \quad (8.130)$$

■

EXERCISE: Sketch the Odd Extension of $f(x)$ given in the previous Example and write down the formula for it.

8.4.2 Fourier Cosine Series

Now consider what happens for the Fourier Cosine Series of $f(x)$ on $(0,l)$. This is analogous to the Sine Series case. Every term in the Cosine Series has the form

$$A_n\cos(\frac{n\pi x}{l}) \quad (8.131)$$

and hence is even, so the entire Cosine Series is even. So the Cosine Series must converge on $(-l,l)$ to an even function which coincides on $(0,l)$ with $f(x)$. this must be the even extension

$$f_{even}(x) = \begin{cases} f(x) & 0<x<l \\ f(-x) & -l<x<0 \end{cases}. \quad (8.132)$$

Notice that this definition does not specify the value of the function at zero, the only restriction on an even function at zero is that, if it exists, the derivative should be zero.

It is straight forward enough to show that the Fourier coefficients of $f_{even}(x)$ coincide with the Fourier Cosine coefficients of $f(x)$. The Euler-Fourier formulas give

$$\begin{aligned}
A_n &= \frac{1}{l}\int_{-l}^{l} f_{even}(x)\cos(\frac{n\pi x}{l})dx & (8.133)\\
&= \frac{2}{l}\int_0^l f_{even}(x)\cos(\frac{n\pi x}{l})dx \quad \text{since } f_{even}(x)\cos(\frac{n\pi x}{l}) \text{ is even} & (8.134)\\
&= \frac{2}{l}\int_0^l f_{even}(x)\cos(\frac{n\pi x}{l})dx & (8.135)
\end{aligned}$$

which are the Fourier Cosine coefficients of $f(x)$ on $(0,l)$

$$B_n = \frac{1}{l}\int_{-l}^{l} f_{even}(x)\sin(\frac{n\pi x}{l})dx = 0 \tag{8.136}$$

since $f_{even}(x)\sin(\frac{n\pi x}{l})$ is odd. Thus the Fourier Cosine Series of $f(x)$ on $(0,l)$ can be considered as the Fourier expansion of $f_{even}(x)$ on $(-l,l)$, and therefore also as expansion of the periodic extension of $f_{even}(x)$. It will converge as in the Fourier Convergence Theorem to this periodic extension.

This also means that if we want to compute the Fourier Series of an even function, we can just compute the Fourier Cosine Series of its restriction to $(0,l)$. It is very important that this only be attempted if the function we are starting with is even.

■ **Example 8.20** Write down the even extension of $f(x) = l - x$ on $(0,l)$ and compute its Fourier Series.

The even extension will be

$$f_{even}(x) = \begin{cases} l-x & 0 < x < l \\ l+x & -l < x < 0 \end{cases} . \tag{8.137}$$

Its Fourier Series is the same as the Fourier Cosine Series of $f(x)$, by the previous discussion. So we can just compute the coefficients. Thus we have

$$f_{even}(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{l}), \tag{8.138}$$

where

$$A_0 = \frac{2}{l}\int_0^l f(x)dx = \frac{2}{l}\int_0^l (l-x)dx = l \tag{8.139}$$

$$A_n = \frac{2}{l}\int_0^l f(x)\cos(\frac{n\pi x}{l})dx \tag{8.140}$$

$$= \frac{2}{l}\int_0^l (l-x)\cos(\frac{n\pi x}{l})dx \tag{8.141}$$

$$= \frac{2}{l}[\frac{l(l-x)}{n\pi}\sin(\frac{n\pi x}{l}) - \frac{l^2}{n^2\pi^2}\cos(\frac{n\pi x}{l})]_0^l \tag{8.142}$$

$$= \frac{2}{l}\left(\frac{l^2}{n^2\pi^2}(-\cos(n\pi)+\cos(0))\right) \tag{8.143}$$

$$= \frac{2l}{n^2\pi^2}((-1)^{n+1}+1). \tag{8.144}$$

So we have

$$f_{even}(x) = \frac{l}{2} + \sum_{n=1}^{\infty} \frac{2l}{n^2\pi^2}((-1)^{n+1}+1). \tag{8.145}$$

■

■ **Example 8.21** Write down the even extension of

$$f(x) = \begin{cases} \frac{3}{2} & 0 \le x < \frac{3}{2} \\ x - \frac{3}{2} & \frac{3}{2} \le x \le 3 \end{cases} \tag{8.146}$$

and compute its Fourier Series.

Using Equation (8.132) we see that the even extension is

$$f_{even}(x) = \begin{cases} x-\frac{3}{2} & \frac{3}{2}<x<3 \\ \frac{3}{2} & 0\leq x<\frac{3}{2} \\ \frac{3}{2} & -\frac{3}{2}<x<0 \\ -x-\frac{3}{2} & -3\leq x\leq -\frac{3}{2} \end{cases}. \tag{8.147}$$

We just need to compute the Fourier Cosine coefficients of the original $f(x)$ on $(0,3)$.

$$\begin{aligned}
A_0 &= \frac{2}{3}\int_0^3 f(x)dx && (8.148)\\
&= \frac{2}{3}\left(\int_0^{3/2}\frac{3}{2}dx+\int_{3/2}^3 x-\frac{3}{2}dx\right) && (8.149)\\
&= \frac{2}{3}(\frac{9}{4}+\frac{9}{8}) = \frac{9}{4} && (8.150)\\
A_n &= \frac{2}{3}\int_0^3 f(x)\cos(\frac{n\pi x}{3})dx && (8.151)\\
&= \frac{2}{3}\left(\int_0^{3/2}\frac{3}{2}\cos(\frac{n\pi x}{3})dx+\int_{3/2}^3(x-\frac{3}{2})\cos(\frac{n\pi x}{3})dx\right) && (8.152)\\
&= \frac{2}{3}\left(\frac{9}{2n\pi}\sin(\frac{n\pi x}{3})|_0^{3/2}+\frac{3(x-\frac{3}{2})}{n\pi}\sin(\frac{n\pi x}{3})|_{3/2}^3+\frac{9}{n^2\pi^2}\cos(\frac{n\pi x}{3})|_{3/2}^3\right) && (8.153)\\
&= \frac{2}{3}\left(\frac{9}{2n\pi}\sin(\frac{n\pi}{2})+\frac{9}{n^2\pi^2}\left(cos(n\pi)-\cos(\frac{n\pi}{2})\right)\right) && (8.154)\\
&= \frac{6}{n\pi}\left(\frac{1}{2}\sin(\frac{n\pi}{2})+\frac{1}{n\pi}\left((-1)^n-\cos(\frac{n\pi}{2})\right)\right) && (8.155)\\
&= \frac{6}{n\pi}\left(\frac{1}{n\pi}((-1)^n-\cos(\frac{n\pi}{2}))+\frac{1}{2}\sin(\frac{n\pi}{2})\right). && (8.156)
\end{aligned}$$

So the Fourier Series is

$$f_{even} = \frac{9}{8}+\frac{6}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{n\pi}((-1)^n-\cos(\frac{n\pi}{2}))+\frac{1}{2}\sin(\frac{n\pi}{2})\right)\cos(\frac{n\pi x}{3}). \tag{8.157}$$

■

8.5 The Heat Equation

We will soon see that partial differential equations can be far more complicated than ordinary differential equations. For PDEs, there is no general theory, the methods need to be adapted for smaller groups of equations. This course will only do an introduction, you can find out much more in advanced courses. We will be focusing on a single solution method called **Separation of Variables**, which is pervasive in engineering and mathematics.

The first partial differential equation to consider is the famous **heat equation** which models the temperature distribution in some object. We will focus on the one-dimensional

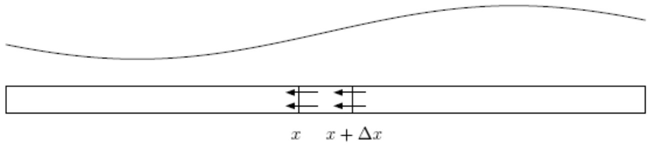


Figure 8.1: Heat Flux across the boundary of a small slab with length Δx. The graph is the graph of temperature at a given time t. In accordance with Fourier's Law, the heat leaves or enters the boundary by flowing from hot to cold; hence at x the flux is opposing the sign of u_x, while at $x+\Delta x$ it is agreeing.

heat equation, where we want to find the temperature distributions in a one-dimensional bar of length l. In particular we will assume that our bar corresponds to the interval $(0,l)$ on the real line.

The assumption is made purely for simplicity. If we assume we have a real bar, the one-dimensional assumption is equivalent to assuming at every lateral cross-section and every instant of time, the temperature is constant. While this is unrealistic it is not a terrible assumption. Also, if the length is much larger than the width in advanced mathematics one can assume the width is 0 since it is such a small fraction of the length. We are also assuming the bar is perfectly insulated, so the only way heat can enter or leave the bar is through the ends $x=0$ and $x=l$. So any heat transfer will be one-dimensional.

8.5.1 Derivation of the Heat Equation

Many PDEs come from basic physical laws. Let $u(x,t)$ denote the temperature at a point x at time t. c will be the specific heat of the material the bar is made from (which is the amount of heat needed to raise one unit of mass of this material by one temperature unit) and ρ is the density of the rod. Note that in general, the specific heat and density of the rods do not have to be constants, they may vary with x. We greatly simplify the problem by allowing them to be constant.

Let's consider a small slab of length Δx. We will let $H(t)$ be the amount of heat contained in this slab. The mass of the slab is $\rho\Delta x$ and the heat energy contained in this small region is given by

$$H(t) = cu\rho\Delta x \tag{8.158}$$

On the other hand, within the slab, heat will flow from hot to cold (this is **Fourier's Law**). The only way heat can leave is by leaving through the boundaries, which are at x and $x+\Delta x$ (This is the **Law of Conservation of Energy**). So the change of heat energy of the slab is equal to the heat flux across the boundary. If κ is the conductivity of the bar's material

$$\frac{dH}{dt} = \kappa u_x(x+\Delta x,t) - \kappa u_x(x,t) \tag{8.159}$$

This is illustrated in Figure 8.5.1. Setting the derivative of $H(t)$ from above equal to the previous equations we find

$$(cu(x,t)\rho\Delta x)_t = \kappa u_x(x+\Delta x,t) - \kappa u_x(x,t) \tag{8.160}$$

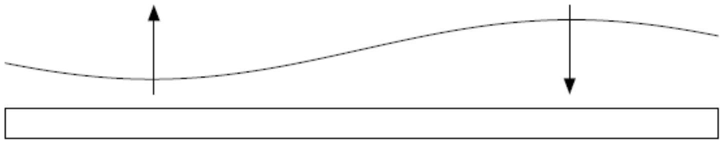


Figure 8.2: Temperature versus position on a bar. The arrows show time dependence in accordance with the heat equation. The temperature graph is concave up, so the left side of the bar is warming up. While on the right the temperature is concave down and so th right side is cooling down..

or

$$c\rho u_t(x,t) = \frac{\kappa u_x(x+\Delta x,t) - \kappa u_x(x,t)}{\Delta x}. \tag{8.161}$$

If we take the limit as $\Delta x \to 0$, the right hand side is just the x-derivative of $\kappa u_x(x,t)$ or

$$c\rho u_t(x,t) = \kappa u_{xx}(x,t). \tag{8.162}$$

Setting $k = \frac{\kappa}{c\rho} > 0$, we have the heat equation

$$u_t = k u_{xx}. \tag{8.163}$$

Notice that the heat equation is a linear PDE, since all of the derivatives of u are only multiplied by constants. What is the constant k? It is called the **Thermal Diffusivity** of the bar and is a measure of how quickly heat spreads through a given material.

How do we interpret the heat equation? Graph the temperature of the bar at a fixed time. Suppose it looks like Figure 2. On the left side the bar is concave up. If the graph is concave up, that means that the second derivative of the temperature (with respect to position x) is positive. The heat equation tells us that the time derivative of the temperature at any of the points on the left side of the bar will be increasing. The left side of the bar will be warming up. Similarly, on the right side of the bar, the graph is concave down. Thus the second x-derivative of the temperature is negative, and so will be the first t-derivative, and we can conclude that the right side of the bar is cooling down.

8.6 Separation of Variables and Heat Equation IVPs

8.6.1 Initial Value Problems

Partial Differential Equations generally have a lot of solutions. To specify a unique one, we will need additional conditions. These conditions are motivated by physics and are initial or boundary conditions. An IVP for a PDE consists for the heat equation, initial conditions, and boundary conditions.

An initial condition specifies the physical state at a given time t_0. For example, and initial condition for the heat equation would be the starting temperature distribution

$$u(x,0) = f(x) \tag{8.164}$$

This is the only condition required because the heat equation is first order with respect to time. The wave equation, considered in a future section is second order in time and needs two initial conditions.

PDEs are only valid on a given domain. Boundary conditions specify how the solution behaves on the boundaries of the given domain. These need to be specified, because the solution does not exist on one side of the boundary, we might have problems with differentiability there.

Our heat equation was derived for a one-dimensional bar of length l, so the relevant domain in question can be taken to be the interval $0 < x < l$ and the boundary consists of the two points $x = 0$ and $x = l$. We could have derived a two-dimensional heat equation, for example, in which case the domain would be some region in the xy-plane with the boundary being some closed curve.

It will be clear from the physical description of the problem what the appropriate boundary conditions are. We might know at the endpoints $x = 0$ and $x = l$, the temperature $u(0,t)$ and $u(l,t)$ are fixed. Boundary conditions that give the value of the solution are called **Dirichlet Boundary Conditions**. Or we might insulate the ends of the bar, meaning there should be no heat flow out of the boundary. This would yield the boundary conditions $u_x(0,t) = u_x(l,t) = 0$. If the boundary conditions specify the derivative at the boundary, they are called **Neumann Conditions**. If the boundary conditions specify that we have one insulated end and at the other we control the temperature. This is an example of a **Mixed Boundary Condition**.

As we have seen, changing boundary conditions can significantly change the solution. Initially, we will work with homogeneous Dirichlet conditions $u(0,t) = u(l,t) = 0$, giving us the following initial value problem

$$
\begin{aligned}
(DE): \quad u_t &= k u_x x && (8.165)\\
(BC): \quad u(0,t) &= u(l,t) = 0 && (8.166)\\
(IC): u(x,0) &= f(x) && (8.167)
\end{aligned}
$$

After we have seen the general method, we will see what happens with homogeneous Neumann conditions. We will discuss nonhomogeneous equations later.

8.6.2 Separation of Variables

Above we have derived the heat equation for the bar of length L. Suppose we have an initial value problem such as Equation (8.165)-(8.167). How should we proceed? We want to try to build a general solution out of smaller solutions which are easier to find.

We start by assuming we have a **separated solution**, where

$$u(x,t) = X(x)T(t). \tag{8.168}$$

Our solution is the product of a function that depends only on x and a function that depends only on t. We can then try to write down an equation depending only on x and another solution depending only on t before using our knowledge of ODEs to try and solve them.

It should be noted that this is a very special situation and will not occur in general. Even when we can use it sometimes it is hard to move beyond the first step. However, it works for all equations we will be considering in this class, and is a good starting point.

How does this method work? Plug the separated solution into the heat equation.

$$\frac{\partial}{\partial t}[X(x)T(t)] = k\frac{\partial^2}{\partial x^2}[X(x)T(t)] \tag{8.169}$$
$$X(x)T'(t) = kX''(x)T(t) \tag{8.170}$$

Now notice that we can move everything depending on x to one side and everything depending on t to the other.

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} \tag{8.171}$$

This equation should says that both sides are equal for any x or t we choose. Thus they both must be equal to a constant. Since if what they equal depended on x or t both sides would not be equal for all x and t. So

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda \tag{8.172}$$

We have written the minus sign for convenience. It will turn out that $\lambda > 0$.

The equation above contains a pair of separate ordinary differential equations

$$X'' + \lambda X = 0 \tag{8.173}$$
$$T' + \lambda kT = 0. \tag{8.174}$$

Notice that our boundary conditions becomes $X(0) = 0$ and $X(l) = 0$. Now the second equation can easily be solved, since we have $T' = -\lambda kT$, so that

$$T(t) = Ae^{-\lambda kt}. \tag{8.175}$$

The first equation gives a boundary value problem

$$X'' + \lambda X = 0 \quad X(0) = 0 \quad X(l) = 0 \tag{8.176}$$

This should look familiar. The is the basic eigenfunction problem studied in section 10.1. As in that example, it turns out our eigenvalues have to be positive. Let $\lambda = \mu^2$ for $\mu > 0$, our general solution is

$$X(x) = B\cos(\mu x) + C\sin(\mu x). \tag{8.177}$$

The first boundary condition says $B = 0$. The second condition says that $X(l) = C\sin(\mu l) = 0$. To avoid only having the trivial solution, we must have $\mu l = n\pi$. In other words,

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad \text{and} \quad X_n(x) = \sin\left(\frac{n\pi x}{l}\right) \tag{8.178}$$

for $n = 1, 2, 3, \ldots$

So we end up having found infinitely many solutions to our boundary value problem, one for each positive integer n. They are

$$u_n(x,t) = A_n e^{-(\frac{n\pi}{l})^2 kt}\sin\left(\frac{n\pi x}{l}\right). \tag{8.179}$$

The heat equation is linear and homogeneous. As such, the Principle of Superposition still holds. So a linear combination of solutions is again a solution. So any function of the form

$$u(x,t) = \sum_{n=0}^{N} A_n e^{-(\frac{n\pi}{l})^2 kt} \sin(\frac{n\pi x}{l}) \tag{8.180}$$

is also a solution to our problem.

Notice we have not used our initial condition yet. We have

$$f(x) = u(x,0) = \sum_{n=0}^{N} A_n \sin(\frac{n\pi x}{l}). \tag{8.181}$$

So if our initial condition has this form, the result of superposition Equation (8.180) is in a good form to use the IC. The coefficients A_n just being the associated coefficients from $f(x)$.

■ **Example 8.22** Find the solutions to the following heat equation problem on a rod of length 2.

$$u_t = u_{xx} \tag{8.182}$$

$$u(0,t) = u(2,t) = 0 \tag{8.183}$$

$$u(x,0) = \sin(\frac{3\pi x}{2}) - 5\sin(3\pi x) \tag{8.184}$$

In this problem, we have $k = 1$. Now we know that our solution will have the form like Equation (8.180), since our initial condition is just the difference of two sine functions. We just need to figure out which terms are represented and what the coefficients A_n are.

Our initial condition is

$$f(x) = \sin(\frac{3\pi x}{2}) - 5\sin(3\pi x) \tag{8.185}$$

Looking at (8.180) with $l = 2$, we can see that the first term corresponds to $n = 3$ and the second $n = 6$, and there are no other terms. Thus we have $A_3 = 1$ and $A_6 = -5$, and all other $A_n = 0$. Our solution is then

$$u(x,t) = e^{-(\frac{9\pi^2}{4})t} \sin(\frac{3\pi x}{2}) - 5e^{(-9\pi^2)t} \sin(3\pi x). \tag{8.186}$$

■

There is no reason to suppose that our initial distribution is a finite sum of sine functions. Physically, such situations are special. What do we do if we have a more general initial temperature distribution?

Let's consider what happens if we take an **infinite** sum of our separated solutions. Then our solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-(\frac{n\pi}{l})^2 kt} \sin(\frac{n\pi x}{l}). \tag{8.187}$$

Now the initial condition gives

$$f(x) = \sum_{n=0}^{\infty} A_n \sin(\frac{n\pi x}{l}). \tag{8.188}$$

This idea is due to the French Mathematician Joseph Fourier and is called the **Fourier Sine Series** for $f(x)$.

There are several important questions that arise. Why should we believe that our initial condition $f(x)$ ought to be able to be written as an infinite sum of sines? why should we believe that such a sum would converge to anything?

8.6.3 Neumann Boundary Conditions

Now let's consider a heat equation problem with homogeneous Neumann conditions

$$(DE): \quad u_t = u_{xx} \tag{8.189}$$
$$(BC): \quad u_x(0,t) = u_x(l,t) = 0 \tag{8.190}$$
$$(IC): \quad u(x,0) = f(x) \tag{8.191}$$

We will start by again supposing that our solution to Equation (8.189) is separable, so we have $u(x,t) = X(x)T(t)$ and we obtain a pair of ODEs, which are the same as before

$$X'' + \lambda X = 0 \tag{8.192}$$
$$T' + \lambda kT = 0. \tag{8.193}$$

The solution to the first equation is still

$$T(t) = Ae^{-\lambda kt} \tag{8.194}$$

Now we need to determine the boundary conditions for the second equation. Our boundary conditions are $u_x(0,t)$ and $u_x(l,t)$. Thus they are conditions for $X'(0)$ and $X'(l)$, since the t-derivative is not controlled at all. So we have the boundary value problem

$$X'' + \lambda X = 0 \quad X'(0) = 0 \quad X'(l) = 0. \tag{8.195}$$

Along the lines of the analogous computation last lecture, this has eigenvalues and eigenfunctions

$$\lambda_n = (\frac{n\pi}{l})^2 \tag{8.196}$$
$$y_n(x) = \cos(\frac{n\pi x}{l}) \tag{8.197}$$

for $n = 0,1,2,...$ So the individual solutions to Equation (8.189) have the form

$$u_(x,t) = A_n e^{(\frac{n\pi}{l})^2 kt} \cos(\frac{n\pi x}{l}). \tag{8.198}$$

Taking finite linear combinations of these work similarly to the Dirichlet case (and is the solution to Equation (8.189) when $f(x)$ is a finite linear combination of constants and cosines, but in general we are interested in knowing when we can take infinite sums, i.e.

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{l})^2 kt} \cos(\frac{n\pi x}{l}) \tag{8.199}$$

Notice how we wrote the $n = 0$ case, as $\frac{1}{2}A_0$. The reason will be clear when talking about Fourier Series. The initial conditions means we need

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{l}). \tag{8.200}$$

An expression of the form above is called the **Fourier Cosine Series** of $f(x)$.

8.6.4 Other Boundary Conditions

It is also possible for certain boundary conditions to require the "full" Fourier Series of the initial data, this is an expression of the form

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \left(A_n \cos(\frac{n\pi x}{l}) + B_n \sin(\frac{n\pi x}{l}) \right). \tag{8.201}$$

but in most cases we will work with Dirichlet or Neumann conditions. However, in the process of learning about Fourier sine and cosine series, we will also learn how to compute the full Fourier series of a function.

8.7 Heat Equation Problems

In the previous lecture on the Heat Equation we saw that the product solutions to the heat equation with homogeneous Dirichlet boundary conditions problem

$$u_t = ku_{xx} \tag{8.202}$$
$$u(0,t) = u(l,t) = 0 \tag{8.203}$$
$$u(x,0) = f(x) \tag{8.204}$$

had the form

$$u_n(x,t) = B_n e^{-(\frac{n\pi}{l})kt} \sin(\frac{n\pi x}{l}) \quad n = 1,2,3,... \tag{8.205}$$

Taking linear combinations of these (over each n) gives a general solution to the above problem.

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-(\frac{n\pi}{l})kt} \sin(\frac{n\pi x}{l}) \tag{8.206}$$

Setting $t = 0$, this implies that we must have

$$f(x) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{l}) \tag{8.207}$$

In other words, the coefficients in the general solution for the given initial condition are the **Fourier Sine** coefficients of $f(x)$ on $(0,l)$, which are given by

$$B_n = \frac{2}{l}\int_0^l f(x) \sin(\frac{n\pi x}{l}) dx. \tag{8.208}$$

We also, saw that if we instead have a problem with homogeneous Neumann boundary conditions

$$\begin{aligned} u_t &= ku_{xx} \quad 0 < x < l, \quad t > 0 & (8.209) \\ u_x(0,t) &= u_x(l,t) = 0 & (8.210) \\ u(0,t) &= f(x) & (8.211) \end{aligned}$$

the product solutions had the form

$$u_n(x,t) = A_n e^{-(\frac{n\pi}{l})^2 kt} \cos\left(\frac{n\pi x}{l}\right) \quad n = 1,2,3,... \tag{8.212}$$

and the general solution has the form

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{l})^2 kt} \cos\left(\frac{n\pi x}{l}\right). \tag{8.213}$$

With $t = 0$ this means that the initial condition must satisfy

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right). \tag{8.214}$$

and so the coefficients for a particular initial condition are the **Fourier Cosine** coefficients of $f(x)$, given by

$$A_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx. \tag{8.215}$$

One way to think about this difference is that given the initial data $u(x,0) = f(x)$, the Dirichlet conditions specify the **odd extension** of $f(x)$ as the desired periodic solution, while the Neumann conditions specify the **even extension**. This should make sense since odd functions must have $f(0) = 0$, while even functions must have $f'(0) = 0$.

So to solve a homogeneous heat equation problem, we begin by identifying the type of boundary conditions we have. If we have Dirichlet conditions, we know our solution will have the form of Equation (8.348). All we then have to do is compute the Fourier Sine coefficients of $f(x)$. Similarly, if we have Neumann conditions, we know the solution has the form of Equation (8.363) and we have to compute the Fourier Cosine coefficients of $f(x)$.

REMARK: Observe that for any homogeneous Dirichlet problem, the temperature distribution (8.348) will go to 0 as $t \to \infty$. This should make sense because these boundary conditions have a physical interpretation where we keep the ends of our rod at freezing temperature without regulating the heat flow in and out of the endpoints. As a result, if the interior of the rod is initially above freezing, that heat will radiate towards the endpoints and into our reservoirs at the endpoints. On the other hand, if the interior of the rod is below freezing, heat will come from the reservoirs at the endpoints and warm it up until the temperature is uniform.

For the Neumann problem, the temperature distribution (8.363) will converge to $\frac{1}{2}A_0$. Again, this should make sense because these boundary conditions correspond to a situation where we have insulated ends, since we are preventing any heat from escaping the bar. Thus all heat energy will move around inside the rod until the temperature is uniform.

8.7.1 Examples

■ **Example 8.23** Solve the initial value problem

$$\begin{aligned} u_t &= 3u_{xx} \quad 0 < x < 2, \quad t > 0 && (8.216)\\ u(0,t) &= u(2,t) = 0 && (8.217)\\ u(x,0) &= 20. && (8.218) \end{aligned}$$

This problem has homogeneous Dirichlet conditions, so by (8.348) our general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-3(\frac{n\pi}{2})^2 t} \sin(\frac{n\pi x}{2}). \tag{8.219}$$

The coefficients for the particular solution are the Fourier Sine coefficients of $u(x,0) = 20$, so we have

$$\begin{aligned} B_n &= \frac{2}{2}\int_0^2 20\sin(\frac{n\pi x}{2})dx && (8.220)\\ &= [-\frac{40}{n\pi}\cos(\frac{n\pi x}{2})]_0^2 && (8.221)\\ &= -\frac{40}{n\pi}(\cos(n\pi) - \cos(0)) && (8.222)\\ &= \frac{40}{n\pi}(1 + (-1)^{n+1}) && (8.223) \end{aligned}$$

and the solution to the problem is

$$u(x,t) = \frac{40}{\pi}\sum_{n=1}^{\infty} \frac{1+(-1)^{n+1}}{n} e^{-\frac{3n^2\pi^2}{4}t} \sin(\frac{n\pi x}{2}). \tag{8.224}$$

■

■ **Example 8.24** Solve the initial value problem

$$\begin{aligned} u_t &= 3u_{xx} \quad 0 < x < 2, \quad t > 0 && (8.225)\\ u_x(0,t) &= u_x(2,t) = 0 && (8.226)\\ u(x,0) &= 3x. && (8.227) \end{aligned}$$

This problem has homogeneous Neumann conditions, so by (8.363) our general solution is

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-3(\frac{n\pi}{2})^2 t} \cos(\frac{n\pi x}{2}). \tag{8.228}$$

The coefficients for the particular solution are the Fourier Cosine coefficients of $u(x,0) =$

$3x$, so we have

$$A_0 = \frac{2}{2}\int_0^2 3x dx = 6 \tag{8.229}$$

$$A_n = \frac{2}{2}\int_0^2 3x\cos(\frac{n\pi x}{2})dx \tag{8.230}$$

$$= [-\frac{6x}{n\pi}\cos(\frac{n\pi x}{2}) + \frac{12}{n^2\pi^2}\sin(\frac{n\pi x}{2})]_0^2 \tag{8.231}$$

$$= -\frac{12}{n\pi}\cos(n\pi) \tag{8.232}$$

$$= \frac{12}{n\pi}(-1)^{n+1} \tag{8.233}$$

and the solution to the problem is

$$u(x,t) = \frac{3}{2} + \frac{12}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-\frac{3n^2\pi^2}{4}t}\cos(\frac{n\pi x}{2}). \tag{8.234}$$

■

■ **Example 8.25** Solve the initial value problem

$$u_t = 4u_{xx} \quad 0 < x < 2\pi, \quad t > 0 \tag{8.235}$$

$$u(0,t) = u(2\pi,t) = 0 \tag{8.236}$$

$$u(x,0) = \begin{cases} 1 & 0 < x < \pi \\ x & \pi < x < 2\pi \end{cases}. \tag{8.237}$$

This problem has homogeneous Dirichlet conditions, so our general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2 t}\sin(\frac{nx}{2}). \tag{8.238}$$

The coefficients for the particular solution are the Fourier Sine coefficients of $u(x,0)$, so we have

$$B_n = \frac{2}{2\pi}\left(\int_0^{\pi}\sin(\frac{nx}{2})dx + \int_{\pi}^{2\pi} x\sin(\frac{nx}{2})dx\right) \tag{8.239}$$

$$= -\frac{2}{n\pi}\cos(\frac{nx}{2})|_0^{\pi} - \frac{2x}{n\pi}\cos(\frac{nx}{2})|_{\pi}^{2\pi} + \frac{4}{n^2\pi}\sin(\frac{nx}{2})|_{\pi}^{2\pi} \tag{8.240}$$

$$= -\frac{2}{n\pi}(\cos(\frac{nx}{2}) - \cos(0)) - \frac{4}{n}\cos(n\pi) + \frac{2}{n}\cos(\frac{n\pi}{2}) - \frac{4}{n^2\pi}\sin(\frac{n\pi}{2}) \tag{8.241}$$

$$= -\frac{2}{n\pi}(\cos(\frac{n\pi}{2}) - 1) + \frac{4}{n}(-1)^{n+1} + \frac{2}{n}\cos(\frac{n\pi}{2}) - \frac{4}{n^2\pi}\sin(\frac{n\pi}{2}) \tag{8.242}$$

$$= \frac{2}{n}\left(-\frac{1}{\pi}(\cos(\frac{n\pi}{2}) - 1) + 2(-1)^{n+1}\cos(\frac{n\pi}{2}) - \frac{2}{n\pi}\sin(\frac{n\pi}{2})\right) \tag{8.243}$$

and the solution to the problem is

$$u(x,t) = 2\sum_{n=1}^{\infty}\frac{1}{n}\left(-\frac{1}{\pi}(\cos(\frac{n\pi}{2}) - 1) + 2(-1)^{n+1}\cos(\frac{n\pi}{2}) - \frac{2}{n\pi}\sin(\frac{n\pi}{2})\right)e^{-n^2 t}\sin(\frac{nx}{2}). \tag{8.244}$$

■

8.8 Other Boundary Conditions

So far, we have used the technique of separation of variables to produce solutions to the heat equation

$$u_t = ku_{xx} \tag{8.245}$$

on $0 < x < l$ with either homogeneous Dirichlet boundary conditions $[u(0,t) = u(l,t) = 0]$ or homogeneous Neumann boundary conditions $[u_x(0,t) = u_x(l,t) = 0]$. What about for some other physically relevant boundary conditions?

8.8.1 Mixed Homogeneous Boundary Conditions

We could have the following boundary conditions

$$u(0,t) = u_x(l,t) = 0 \tag{8.246}$$

Physically, this might correspond to keeping the end of the rod where $x = 0$ in a bowl of ice water, while the other end is insulated.

Use Separation of Variables. Let $u(x,t) = X(x)T(t)$, and we get the pair of ODEs

$$T' = -k\lambda T \tag{8.247}$$
$$X'' = -\lambda X. \tag{8.248}$$

Thus

$$T(t) = Be^{-k\lambda t}. \tag{8.249}$$

We now have a boundary value problem for X to deal with, where the boundary conditions are $X(0) = X'(l) = 0$. There are only positive eigenvalues, which are given by

$$\lambda_n = \left(\frac{(2n-1)\pi}{2l}\right)^2 \tag{8.250}$$

and their associated eigenfunctions are

$$X_n(x) = \sin\left(\frac{(2n-1)\pi x}{2l}\right). \tag{8.251}$$

The separated solutions are then given by

$$u_n(x,t) = B_n e^{-(\frac{(2n-1)\pi}{2l})^2 kt} \sin\left(\frac{(2n-1)\pi x}{2l}\right) \tag{8.252}$$

and the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-(\frac{(2n-1)\pi}{2l})^2 kt} \sin\left(\frac{(2n-1)\pi x}{2l}\right). \tag{8.253}$$

with an initial condition $u(x,0) = f(x)$, we have that

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{(2n-1)\pi x}{2l}\right). \tag{8.254}$$

This is an example of a specialized sort of Fourier Series, the coefficients are given by

$$B_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{(2n-1)\pi x}{2l}\right)dx. \tag{8.255}$$

REMARK: The convergence for a series like the one above is different than that of our standard Fourier Sine or Cosine series, which converge to the periodic extension of the odd or even extensions of the original function, respectively. Notice that the terms in the sum above are periodic with period $4l$ (as opposed to the $2l$-periodic series we have seen before). In this case, we need to first extend our function $f(x)$, given on $(0,l)$, to a function on $(0,2l)$ symmetric around $x=l$. Then, as our terms are all sines, the convergence on $(-2l,2l)$ will be to the odd extension of this extended function, and the periodic extension of this will be what the series converges to on the entire real line.

■ **Example 8.26** Solve the following heat equation problem

$$u_t = 25u_{xx} \tag{8.256}$$
$$u(0,t) = 0 \quad u_x(10,t) = 0 \tag{8.257}$$
$$u(x,0) = 5. \tag{8.258}$$

By (8.253) our general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-25(\frac{(2n-1)\pi}{20})^2 t}\sin\left(\frac{(2n-1)\pi x}{20}\right). \tag{8.259}$$

The coefficients for the particular solution are given by

$$\begin{aligned} B_n &= \frac{2}{10}\int_0^{10} 5\sin\left(\frac{(2n-1)\pi x}{20}\right)dx && (8.260)\\ &= -\frac{10}{(2n-1)\pi}\cos\left(\frac{(2n-1)\pi x}{20}\right)\Big|_0^{10} && (8.261)\\ &= -\frac{10}{(2n-1)\pi}\left(\cos\left(\frac{(2n-1)\pi}{2}\right)-\cos(0)\right) && (8.262)\\ &= \frac{10}{(2n-1)\pi}. && (8.263)\end{aligned}$$

and the solution to the problem is

$$u(x,t) = \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}e^{-\frac{(2n-1)^2\pi^2}{16}t}\sin\left(\frac{(2n-1)\pi x}{20}\right). \tag{8.264}$$

■

8.8.2 Nonhomogeneous Dirichlet Conditions

The next type of boundary conditions we will look at are Dirichlet conditions, which fix the value of u at the endpoints $x=0$ and $x=l$. For the heat equation, this corresponds to fixing the temperature at the ends of the rod. We have already looked at homogeneous conditions where the ends of the rod had fixed temperature 0. Now consider the nonhomogeneous Dirichlet conditions

$$u(0,t) = T_1, \quad u(l,t) = T_2 \tag{8.265}$$

This problem is slightly more difficult than the homogeneous Dirichlet condition problem we have studied. Recall that for separation of variables to work, the differential equations and the boundary conditions must be homogeneous. When we have nonhomogeneous conditions we need to try to split the problem into one involving homogeneous conditions, which we know how to solve, and another dealing with the nonhomogeneity.

REMARK: We used a similar approach when we applied the method of Undetermined Coefficients to nonhomogeneous linear ordinary differential equations.

How can we separate the "core" homogeneous problem from what is causing the nonhomogeneity? Consider what happens as $t \to \infty$. We should expect that, since we fix the temperatures at the endpoints and allow free heat flux at the boundary, at some point the temperature will stabilize and we will be at equilibrium. Such a temperature distribution would clearly not depend on time, and we can write

$$\lim_{t\to\infty} u(x,t) = v(x) \tag{8.266}$$

Notice that $v(x)$ must still satisfy the boundary conditions and the heat equation, but we should not expect it to satisfy the initial conditions (since for large t we are far from where we initially started). A solution such as $v(x)$ which does not depend on t is called a **steady-state** or **equilibrium solution**.

For a steady-state solution the boundary value problem becomes

$$0 = kv'' \quad v(0) = T_1 \quad v(l) = T_2. \tag{8.267}$$

It is easy to see that solutions to this second order differential equation are

$$v(x) = c_1 x + c_2 \tag{8.268}$$

and applying the boundary conditions, we have

$$v(x) = T_1 + \frac{T_2 - T_1}{l}x. \tag{8.269}$$

Now, let

$$w(x,t) = u(x,t) - v(x) \tag{8.270}$$

so that

$$u(x,t) = w(x,t) + v(x). \tag{8.271}$$

This function $w(x,t)$ represents the **transient** part of $u(x,t)$ (since $v(x)$ is the equilibrium part). Taking derivatives we have

$$u_t = w_t + v_t = w_t \quad \text{and} \quad u_{xx} = w_{xx} + v_{xx} = w_{xx}. \tag{8.272}$$

Here we use the fact that $v(x)$ is independent of t and must satisfy the differential equation. Also, using the equilibrium equation $v'' = v_{xx} = 0$.

Thus $w(x,t)$ must satisfy the heat equation, as the relevant derivatives of it are identical to those of $u(x,t)$, which is known to satisfy the equation. What are the boundary and initial conditions?

$$w(0,t) = u(0,t) - v(0) = T_1 - T_1 = 0 \tag{8.273}$$
$$w(l,t) = u(l,t) - v(l) = T_2 - T_2 = 0 \tag{8.274}$$
$$w(x,0) = u(x,0) - v(x) = f(x) - v(x) \tag{8.275}$$

where $f(x) = u(x,0)$ is the given initial condition for the nonhomogeneous problem. Now, even though our initial condition is slightly messier, we now have homogeneous boundary conditions, since $w(x,t)$ must solve the problem

$$w_t = kw_{xx} \tag{8.276}$$
$$w(0,t) = w(l,t) = 0 \tag{8.277}$$
$$w(x,0) = f(x) - v(x) \tag{8.278}$$

This is just a homogeneous Dirichlet problem. We know the general solution is

$$w(x,t) = \sum_{n=1}^{\infty} B_n e^{-(\frac{n\pi}{l})^2 kt} \sin(\frac{n\pi x}{l}). \tag{8.279}$$

where the coefficients are given by

$$B_n = \frac{2}{l}\int_0^l (f(x) - v(x)) \sin(\frac{n\pi x}{l}) dx. \tag{8.280}$$

Notice that $\lim_{t\to\infty} w(x,t) = 0$, so that $w(x,t)$ is transient.

Thus, the solution to the nonhomogeneous Dirichlet problem

$$u_t = ku_{xx} \tag{8.281}$$
$$u(0,t) = T_1, \quad u(l,t) = T_2 \tag{8.282}$$
$$u(x,0) = f(x) \tag{8.283}$$

is $u(x,t) = w(x,t) + v(x)$, or

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-(\frac{n\pi}{l})^2 lt} \sin\Big(\frac{n\pi x}{l}\Big) + T_1 + \frac{T_2 - T_1}{l} x \tag{8.284}$$

with coefficients

$$B_n = \frac{2}{l}\int_0^l (f(x) - T_1 - \frac{T_2 - T_1}{l} x) \sin\Big(\frac{n\pi x}{l}\Big) dx. \tag{8.285}$$

REMARK: Do not memorize the formulas but remember what problem $w(x,t)$ has to solve and that the final solution is $u(x,t) = v(x) + w(x,t)$. For $v(x)$, it is not a hard formula, but if one is not sure of it, remember $v_{xx} = 0$ and it has the same boundary conditions as $u(x,t)$. This will recover it.

■ **Example 8.27** Solve the following heat equation problem

$$u_t = 3u_{xx} \tag{8.286}$$
$$u(0,t) = 20, \quad u(40,t) = 100 \tag{8.287}$$
$$u(x,0) = 40 - 3x \tag{8.288}$$

We start by writing

$$u(x,t) = v(x) + w(x,t) \tag{8.289}$$

where $v(x) = 20+2x$. Then $w(x,t)$ must satisfy the problem

$$\begin{aligned} w_t &= 3w_{xx} && (8.290) \\ w(0,t) &= w(40,t) = 0 && (8.291) \\ w(x,0) &= 40-3x-(20+2x) = 20-x && (8.292) \end{aligned}$$

This is a homogeneous Dirichlet problem, so the general solution for $w(x,t)$ will be

$$w(x,t) = \sum_{n=1}^{\infty} e^{-3(\frac{n\pi}{40})^2 t} \sin(\frac{n\pi x}{40}). \tag{8.293}$$

The coefficients are given by

$$\begin{aligned} B_n &= \frac{2}{40}\int_0^{40}(20-x)\sin(\frac{n\pi x}{40})dx && (8.294) \\ &= \frac{1}{20}[-\frac{40(20-x)}{n\pi}\cos(\frac{n\pi x}{40}) - \frac{1600}{n^2\pi^2}\sin(\frac{n\pi x}{40})]_0^{40} && (8.295) \\ &= \frac{1}{20}\left(\frac{800}{n\pi}\cos(n\pi) + \frac{800}{n\pi}\cos(0)\right) && (8.296) \\ &= \frac{40}{n\pi}((-1)^n+1). && (8.297) \end{aligned}$$

So the solution is

$$u(x,t) = 20+2x+\frac{40}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n+1}{n}e^{-\frac{3n^2\pi^2}{1600}t}\sin(\frac{n\pi x}{40}). \tag{8.298}$$

■

8.8.3 Other Boundary Conditions

There are many other boundary conditions one could use, most of which have a physical interpretation. For example the boundary conditions

$$u(0,t)+u_x(0,t) = 0 \quad u(l,t)+u_x(l,t) = 0 \tag{8.299}$$

say that the heat flux at the end points should be proportional to the temperature. We could also have had nonhomogeneous Neumann conditions

$$u_x(0,t) = F_1 \quad u_x(l,t) = F_2 \tag{8.300}$$

which would specify allowing a certain heat flux at the boundaries. These conditions are not necessarily well suited for the method of separation of variables though and are left for future classes.

8.9 The Wave Equation

8.9.1 Derivation of the Wave Equation

Consider a completely flexible string of length l and constant density ρ. We will assume that the string will only undergo relatively small vertical vibrations, so that points do not move from side to side. An example might be a plucked guitar string. Thus we can let $u(x,t)$ be its displacement from equilibrium at time t. The assumption of complete flexibility means that the tension force is tangent to the string, and the string itself provides no resistance to bending. This means the tension force only depends on the slope of the string.

Take a small piece of string going from x to $x+\Delta x$. Let $\Theta(x,t)$ be the angle from the horizontal of the string. Our goal is to use Newton's Second Law $F = ma$ to describe the motion. What forces are acting on this piece of string?
(a) Tension pulling to the right, which has magnitude $T(x+\Delta x,t)$ and acts at an angle of $\Theta(x+\Delta x,t)$ from the horizontal.
(b) Tension pulling to the left, which has magnitude $T(x,t)$ and acts at an angle of $\Theta(x,t)$ from the horizontal.
(c) Any external forces, which we denote by $F(x,t)$.
Initially, we will assume that $F(x,t) = 0$. The length of the string is essentially $\sqrt{(\Delta x)^2+(\Delta u)^2}$, so the vertical component of Newton's Law says that

$$\rho\sqrt{(\Delta x)^2+(\Delta u)^2}u_{tt}(x,t) = T(x+\Delta x,t)\sin(\Theta(x+\Delta x,t)) - T(x,t)\sin(\Theta(x,t)). \quad (8.301)$$

Dividing by Δx and taking the limit as $\Delta x \to 0$, we get

$$\rho\sqrt{1+(u_x)^2}u_{tt}(x,t) = \frac{\partial}{\partial x}[T(x,t)\sin(\Theta(x,t))]. \quad (8.302)$$

We assumed our vibrations were relatively small. This means that $\Theta(x,t)$ is very close to zero. As a result, $\sin(\Theta(x,t)) \equiv \tan(\Theta(x,t))$. Moreover, $\tan(\Theta(x,t))$ is just the slope of the string $u_x(x,t)$. We conclude, since $\Theta(x,t)$ is small, that $u_x(x,t)$ is also very small. The above equation becomes

$$\rho u_{tt}(x,t) = (T(x,t)u_x(x,t))_x. \quad (8.303)$$

We have not used the horizontal component of Newton's Law yet. Since we assume there are only vertical vibrations, our tiny piece of string can only move vertically. Thus the net horizontal force is zero.

$$T(x+\Delta x,t)\cos(\Theta(x+\Delta x,t)) - T(x,t)\cos(\Theta(x,t)) = 0. \quad (8.304)$$

Dividing by Δx and taking the limit as $\Delta x \to \infty$ yields

$$\frac{\partial}{\partial x}[T(x,t)\cos(\Theta(x,t))] = 0. \quad (8.305)$$

Since $\Theta(x,t)$ is very close to zero, $\cos(\Theta(x,t))$ is close to one. thus we have that $\frac{\partial T}{\partial x}(x,t)$ is close to zero. So $T(x,t)$ is constant along the string, and independent of x. We will also assume that T is independent of t. Then Equation (8.363) becomes the one-dimensional wave equation

$$u_{tt} = c^2 u_{xx} \quad (8.306)$$

where $c^2 = \frac{T}{\rho}$.

8.9.2 The Homogeneous Dirichlet Problem

Now that we have derived the wave equation, we can use Separation of Variables to obtain basic solutions. We will consider homogeneous Dirichlet conditions, but if we had homogeneous Neumann conditions the same techniques would give us a solution. The wave equation is second order in t, unlike the heat equation which was first order in t. We will need to initial conditions in order to obtain a solution, one for the initial displacement and the other for the initial speed.

The relevant wave equation problem we will study is

$$\begin{aligned} u_{tt} &= c^2 u_{xx} && (8.307)\\ u(0,t) &= u(l,t) = 0 && (8.308)\\ u(x,0) &= f(x), \quad u_t(x,0) = g(x) && (8.309) \end{aligned}$$

The physical interpretation of the boundary conditions is that the ends of the string are fixed in place. They might be attached to guitar pegs.

We start by assuming our solution has the form

$$u(x,t) = X(x)T(t). \tag{8.310}$$

Plugging this into the equation gives

$$T''(t)X(x) = c^2 T(t)X''(x). \tag{8.311}$$

Separating variables, we have

$$\frac{X''}{X} = \frac{T''}{c^2 T} = -\lambda \tag{8.312}$$

where λ is a constant. This gives a pair of ODEs

$$\begin{aligned} T'' + c^2\lambda T &= 0 && (8.313)\\ X'' + \lambda X &= 0. && (8.314) \end{aligned}$$

The boundary conditions transform into

$$\begin{aligned} u(0,t) &= X(0)T(t) = 0 \quad \Rightarrow \quad X(0) = 0 && (8.315)\\ u(l,t) &= X(l)T(t) = 0 \quad \Rightarrow \quad X(l) = 0. && (8.316) \end{aligned}$$

This is the same boundary value problem that we saw for the heat equation and thus the eigenvalues and eigenfunctions are

$$\begin{aligned} \lambda_n &= \left(\frac{n\pi}{l}\right)^2 && (8.317)\\ X_n(x) &= \sin\left(\frac{n\pi x}{l}\right) && (8.318) \end{aligned}$$

for $n = 1, 2, \ldots$ The first ODE (8.313) is then

$$T'' + \left(\frac{cn\pi}{l}\right)^2 T = 0, \tag{8.319}$$

and since the coefficient of T is clearly positive this has a general solution

$$T_n(t) = A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right). \tag{8.320}$$

There is no reason to think either of these are zero, so we end up with separated solutions

$$u_n(x,t) = \left[A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right)\right] \sin\left(\frac{n\pi x}{l}\right) \tag{8.321}$$

and the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right)\right] \sin\left(\frac{n\pi x}{l}\right). \tag{8.322}$$

We can directly apply our first initial condition, but to apply the second we will need to differentiate with respect to t. This gives us

$$u_t(x,t) = \sum_{n=1}^{\infty} \left[-\frac{n\pi c}{l} A_n \sin\left(\frac{n\pi ct}{l}\right) + \frac{n\pi c}{l} B_n \cos\left(\frac{n\pi ct}{l}\right)\right] \sin\left(\frac{n\pi x}{l}\right) \tag{8.323}$$

Plugging in the initial condition then yields the pair of equations

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) \tag{8.324}$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \sin\left(\frac{n\pi x}{l}\right). \tag{8.325}$$

These are both Fourier Sine series. The first is directly the Fourier Since series for $f(x)$ on $(0,l)$. The second equation is the Fourier Sine series for $g(x)$ on $(0,l)$ with a slightly messy coefficient. The Euler-Fourier formulas then tell us that

$$A_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx \tag{8.326}$$

$$\frac{n\pi c}{l} B_n = \frac{2}{l}\int_0^l g(x)\sin\left(\frac{n\pi x}{l}\right)dx \tag{8.327}$$

$$A_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx \tag{8.328}$$

$$B_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\left(\frac{n\pi x}{l}\right)dx. \tag{8.329}$$

8.9.3 Examples

■ **Example 8.28** Find the solution (displacement $u(x,t)$) for the problem of an elastic string of length L whose ends are held fixed. The string has no initial velocity ($u_t(x,0) = 0$) from an initial position

$$u(x,0) = f(x) = \begin{cases} \frac{4x}{L} & 0 \le x \le \frac{L}{4} \\ 1 & \frac{L}{4} < x < \frac{3L}{4} \\ \frac{4(L-x)}{L} & \frac{3L}{4} \le x \le L \end{cases} \tag{8.330}$$

By the formulas above we see if we separate variables we have the following equation for T

$$T'' + (\frac{cn\pi}{L})^2 T = 0 \tag{8.331}$$

with the general solution

$$T_n(t) = A_n \cos(\frac{n\pi ct}{L}) + B_n \sin(\frac{n\pi ct}{L}). \tag{8.332}$$

since the initial speed is zero, we find $T'(0) = 0$ and thus $B_n = 0$. Therefore the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi ct}{L}) \sin(\frac{n\pi x}{L}). \tag{8.333}$$

where the coefficients are the Fourier Sine coefficients of $f(x)$. So

$$\begin{aligned}
A_n &= \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx & (8.334)\\
&= \frac{2}{L}\left[\int_0^{L/4} \frac{4x}{L}\sin(\frac{n\pi x}{L})dx + \int_{L/4}^{3L/4}\sin(\frac{n\pi x}{L})dx + \int_{3L/4}^{L}\frac{4L-4x}{L}\sin(\frac{n\pi x}{L})dx\right] & (8.335)\\
&= 8\frac{\sin(\frac{n\pi}{4}) + \sin(\frac{3n\pi}{4})}{n^2\pi^2} & (8.336)
\end{aligned}$$

Thus the displacement of the string will be

$$u(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{\sin(\frac{n\pi}{4}) + \sin(\frac{3n\pi}{4})}{\pi^2} \cos(\frac{n\pi ct}{L}) \sin(\frac{n\pi x}{L}). \tag{8.337}$$

■

■ **Example 8.29** Find the solution (displacement $u(x,t)$) for the problem of an elastic string of length L whose ends are held fixed. The string has no initial velocity ($u_t(x,0) = 0$) from an initial position

$$u(x,0) = f(x) = \frac{8x(L-x)^2}{L^3} \tag{8.338}$$

By the formulas above we see if we separate variables we have the following equation for T

$$T'' + (\frac{cn\pi}{L})^2 T = 0 \tag{8.339}$$

with the general solution

$$T_n(t) = A_n \cos(\frac{n\pi ct}{L}) + B_n \sin(\frac{n\pi ct}{L}). \tag{8.340}$$

since the initial speed is zero, we find $T'(0) = 0$ and thus $B_n = 0$. Therefore the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi ct}{L}) \sin(\frac{n\pi x}{L}). \tag{8.341}$$

where the coefficients are the Fourier Sine coefficients of $f(x)$. So

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx && (8.342)\\ &= \frac{2}{L}\int_0^L \frac{8x(L-x)^2}{L^3}\sin(\frac{n\pi x}{L})dx && (8.343)\\ &= 32\frac{2+\cos(n\pi)}{n^3\pi^3} \quad \text{Integrate By Parts} && (8.344)\end{aligned}$$

Thus the displacement of the string will be

$$u(x,t) = \frac{32}{\pi^3}\sum_{n=1}^{\infty}\frac{2+\cos(n\pi)}{n^3}\cos(\frac{n\pi ct}{L})\sin(\frac{n\pi x}{L}). \tag{8.345}$$

■

■ **Example 8.30** Problem 12 is a great exercise. ■

8.10 Laplace's Equation

We will consider the two-dimensional and three-dimensional Laplace Equations

$$\begin{aligned} (2D):\quad u_{xx}+u_{yy} &= 0, && (8.346)\\ (3D):\quad u_{xx}+u_{yy}+u_{zz} &= 0. && (8.347)\end{aligned}$$

8.10.1 Dirichlet Problem for a Rectangle

We want to find the function u satisfying Laplace's Equation

$$u_{xx}+u_{yy} = 0 \tag{8.348}$$

in the rectangle $0<x<a, 0<y<b$, and satisfying the boundary conditions

$$\begin{aligned} u(x,0) &= 0, \quad u(x,b)=0, \quad 0<x<a, && (8.349)\\ u(0,y) &= 0, \quad u(a,y)=f(y), \quad 0\le y\le b. && (8.350)\end{aligned}$$

We need four boundary conditions for the four spatial derivatives.

Start by using Separation of Variables and assume $u(x,y) = X(x)Y(y)$. Substitute u into Equation (8.348). This yields

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda, \tag{8.351}$$

where λ is a constant. We obtain the following system of ODEs

$$\begin{aligned} X''-\lambda X &= 0 && (8.352)\\ Y''+\lambda Y &= 0. && (8.353)\end{aligned}$$

From the boundary conditions we find

$$\begin{aligned} X(0) &= 0 && (8.354)\\ Y(0) &= 0, Y(b) = 0. && (8.355)\end{aligned}$$

We first solve the ODE for Y, which we have seen numerous times before. Using the BCs we find there are nontrivial solutions if and only if λ is an eigenvalue

$$\lambda = \left(\frac{n\pi}{b}\right)^2, \quad n = 1,2,3,... \tag{8.356}$$

and $Y_n(y) = \sin(\frac{n\pi y}{b})$, the corresponding eigenfunction. Now substituting in for λ we want to solve the ODE for X. This is another problem we have seen regularly and the solution is

$$X_n(x) = c_1 \cosh\left(\frac{n\pi x}{b}\right) + c_2 \sinh\left(\frac{n\pi x}{b}\right) \tag{8.357}$$

The BC implies that $c_1 = 0$. So the fundamental solution to the problem is

$$u_n(x,y) = \sinh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right). \tag{8.358}$$

By linear superposition the general solution is

$$u(x,y) = \sum_{n=1}^{\infty} c_n u_n(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right). \tag{8.359}$$

Using the last boundary condition $u(a,y) = f(y)$ solve for the coefficients c_n.

$$u(a,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi a}{b}\right)\sin\left(\frac{n\pi y}{b}\right) = f(y) \tag{8.360}$$

Using the Fourier Since Series coefficients we find

$$c_n = \frac{2}{b\sinh(\frac{n\pi a}{b})}\int_0^b f(y)\sin\left(\frac{n\pi y}{b}\right)dy. \tag{8.361}$$

8.10.2 Dirichlet Problem For A Circle

Consider solving Laplace's Equation in a circular region $r < a$ subject to the boundary condition

$$u(a,\theta) = f(\theta) \tag{8.362}$$

where f is a given function on $0 \leq \theta \leq 2\pi$. In polar coordinates Laplace's Equation becomes

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0. \tag{8.363}$$

Try Separation of Variables in Polar Coordinates

$$u(r,\theta) = R(r)\Theta(\theta), \tag{8.364}$$

plug into the differential equation, Equation (8.363). This yields

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \tag{8.365}$$

or

$$r^2\frac{R''}{R}+r\frac{R'}{R}=-\frac{\Theta''}{\Theta}=\lambda \tag{8.366}$$

where λ is a constant. We obtain the following system of ODEs

$$r^2R''+rR'-\lambda R = 0, \tag{8.367}$$
$$\Theta''+\lambda\theta = 0. \tag{8.368}$$

Since we have no homogeneous boundary conditions we must use instead the fact that the solutions must be bounded and also periodic in Θ with period 2π. It can be shown that we need λ to be real. Consider the three cases when $\lambda<0, \lambda=0, \lambda>0$.

If $\lambda<0$, let $\lambda=-\mu^2$, where $\mu>0$. So we find the equation for Θ becomes $\Theta''-\mu^2\Theta=0$. So

$$\Theta(\theta)=c_1e^{\mu\theta}+c_2e^{-\mu\theta} \tag{8.369}$$

Θ can only be periodic if $c_1=c_2=0$, so λ cannot be negative (Since we do not get any nontrivial solutions.

If $\lambda=0$, then the equation for Θ becomes $\Theta''=0$ and thus

$$\Theta(\theta)=c_1+c_2\theta \tag{8.370}$$

For Θ to be periodic $c_2=0$. Then the equation for R becomes

$$r^2R''+rR'=0. \tag{8.371}$$

This equation is an Euler equation and has solution

$$R(r)=k_1+k_2\ln(r) \tag{8.372}$$

Since we also need the solution bounded as $r\to\infty$, then $k_2=0$. So $u(r,\theta)$ is a constant, and thus proportional to the solution $u_0(r,\theta)=1$.

If $\lambda>0$, we let $\lambda=\mu^2$, where $\mu>0$. Then the system of equations becomes

$$r^2R''+rR'-\mu^2R = 0 \tag{8.373}$$
$$\Theta''+\mu^2\Theta=0 \tag{8.374}$$

The equation for R is an Euler equation and has the solution

$$R(r)=k_1r^{\mu}+k_2r^{-\mu} \tag{8.375}$$

and the equation for Θ has the solution

$$\Theta(\theta)=c_1\sin(\mu\theta)+c_2\cos(\mu\theta). \tag{8.376}$$

For Θ to be periodic we need μ to be a positive integer n, so $\mu=n$. Thus the solution $r^{-\mu}$is unbounded as $r\to 0$. So $k_2=0$. So the solutions to the original problem are

$$u_n(r,\theta)=r^n\cos(n\theta), \quad v_n(r,\theta)=r^n\sin(n\theta), \quad n=1,2,3,... \tag{8.377}$$

Together with $u_0(r,\theta) = 1$, by linear superposition we find

$$u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n (c_n \cos(n\theta) + k_n \sin(n\theta)). \tag{8.378}$$

Using the boundary condition from the beginning

$$u(a,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} a^n (c_n \cos(n\theta) + k_n \sin(n\theta)) = f(\theta) \tag{8.379}$$

for $0 \le \theta \le 2\pi$. We compute to coefficients by using our previous Fourier Series equations

$$c_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta)\cos(n\theta)d\theta, \quad n = 1,2,3,\ldots \tag{8.380}$$

$$k_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta)\sin(n\theta)d\theta, \quad n = 1,2,3,\ldots \tag{8.381}$$

Note we need both terms since sine and cosine terms remain throughout the general solution.

8.10.3 Example 1

Find the solution $u(x,y)$ of Laplace's Equation in the rectangle $0 < x < a$, $0 < y < b$, that satisfies the boundary conditions

$$u(0,y) = 0, \quad u(a,y) = 0, \quad 0 < y < b \tag{8.382}$$

$$u(x,0) = h(x), \quad u(x,b) = 0, \quad 0 \le x \le a \tag{8.383}$$

Answer: Using the method of Separation of Variables, write $u(x,y) = X(x)Y(y)$. We get the following system of ODEs

$$X'' + \lambda X = 0, \quad X(0) = X(a) = 0 \tag{8.384}$$

$$Y'' - \lambda Y = 0, \quad Y(b) = 0 \tag{8.385}$$

It follows that $\lambda_n = (\frac{n\pi}{a})^2$ and $X_n(x) = \sin(\frac{n\pi x}{a})$. The solution of the second ODE gives

$$Y(y) = d_1 \cosh(\lambda(b-y)) + d_2 \sinh(\lambda(b-y)). \tag{8.386}$$

Using $y(b) = 0$, we find that $d_1 = 0$. Therefore the fundamental solutions are

$$u_n(x,y) = \sin(\frac{n\pi x}{a})\sinh(\lambda_n(b-y)), \tag{8.387}$$

and the general solution is

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{a})\sinh(\frac{n\pi(b-y)}{a}). \tag{8.388}$$

Using another boundary condition

$$h(x) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{a})\sinh(\frac{n\pi b}{a}). \tag{8.389}$$

The coefficients are calculated using the equation from the Fourier Sine Series

$$c_n = \frac{2}{a\sinh(\frac{n\pi b}{a})} \int_0^a h(x)\sin(\frac{n\pi x}{a})dx. \tag{8.390}$$

8.10.4 Example 2

Consider the problem of finding a solution $u(x,y)$ of Laplace's Equation in the rectangle $0 < x < a$, $0 < y < b$, that satisfies the boundary conditions

$$u_x(0,y) = 0, \quad u_x(a,y) = f(y), \quad 0 < y < b, \tag{8.391}$$
$$u_y(x,0) = 0, \quad u_y(x,b) = 0, \quad 0 \le x \le a \tag{8.392}$$

This is an example of a Neumann Problem. We want to find the fundamental set of solutions.

$$X'' - \lambda X = 0, \quad X'(0) = 0 \tag{8.393}$$
$$Y'' + \lambda Y = 0, \quad Y'(0) = Y'(b) = 0. \tag{8.394}$$

The solution to the equation for Y is

$$Y(y) = c_1 \cos(\lambda^{1/2} y) + c_2 \sin(\lambda^{1/2} y), \tag{8.395}$$

with $Y'(y) = -c_1\lambda^{1/2}\sin(\lambda^{1/2}y) + c_2\lambda^{1/2}\cos(\lambda^{1/2}y)$. Using the boundary conditions we find $c_2 = 0$ and the eigenvalues are $\lambda_n = \frac{n^2\pi^2}{b^2}$, for $n = 1,2,3,\ldots$. The corresponding Eigenfunctions are $Y(y) = \cos(\frac{n\pi y}{b})$ for $n = 1,2,3,\ldots$ The solution of the equation for X becomes $X(x) = d_1 \cosh(\frac{n\pi x}{b}) + d_2 \sinh(\frac{n\pi x}{b})$, with

$$X'(x) = d_1 \frac{n\pi}{b} \sinh(\frac{n\pi x}{b}) + d_2 \frac{n\pi}{b} \cosh(\frac{n\pi x}{b}). \tag{8.396}$$

Using the boundary conditions, $X(x) = d_1 \cosh(\frac{n\pi x}{b})$.So the fundamental set of solutions is

$$u_n(x,y) = \cosh(\frac{n\pi x}{b}) \cos(\frac{n\pi y}{b}), \quad n = 1,2,3,\ldots \tag{8.397}$$

The general solution is given by

$$u(x,y) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cosh(\frac{n\pi x}{b}) \cos(\frac{n\pi y}{b}) \tag{8.398}$$