Lecture Notes for Math 251: ODE and PDE. Lecture 1: 1.1 Direction Fields

Shawn D. Ryan

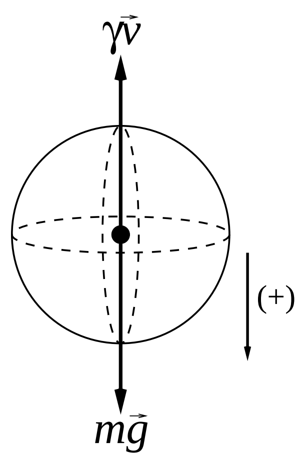
Spring 2012

1 Some Basic Mathematical Models; Direction Fields

Definition 1. A **differential equation** is an equation containing derivatives.

Definition 2. A differential equation that describes some physical process is often called a **mathematical model**

Example 3. *(Falling Object)*



Consider an object falling from the sky. From Newton's Second Law we have

$$F = ma = m\frac{dv}{dt} \tag{1}$$

When we consider the forces from the free body diagram we also have

$$F = mg - \gamma v \tag{2}$$

where γ is the **drag coefficient**. Combining the two

$$m\frac{dv}{dt} = mg - \gamma v \tag{3}$$

Suppose $m = 10kg$ and $\gamma = 2kg/s$. Then we have

$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \tag{4}$$

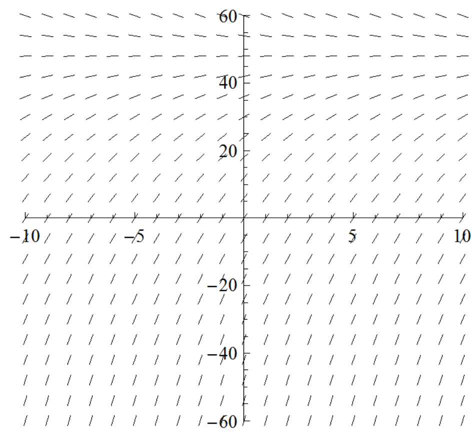


Figure 1: Direction field for above example

It looks like the direction field tends towards $v = 49m/s$. We plot the direction field by plugging in the values for v and t and letting dv/dt be the slope of a line at that point.

Direction Fields are valuable tools in studying the solutions of differential equations of the form

$$\frac{dy}{dt} = f(t, y) \tag{5}$$

where f is a given function of the two variables t and y, sometimes referred to as a **rate function**. At each point on the grid, a short line is drawn whose slope is the value of f at the point. This technique provides a good picture of the overall behavior of a solution.

Two Things to keep in mind:
1. In constructing a direction field we never have to solve the differential equation only evaluate it at points.

2. This method is useful if one has access to a computer because a computer can generate the plots well.

Example 4. *(Population Growth)* Consider a population of field mice, assuming there is nothing to eat the field mice, the population will grow at a constant rate. Denote time by t (in months) and the mouse population by $p(t)$, then we can express the model as

$$\frac{dp}{dt} = rp \tag{6}$$

where the proportionality factor r is called the **rate constant** or **growth constant**. Now suppose owls are killing mice (15 per day), the model becomes

$$\frac{dp}{dt} = 0.5p - 450 \tag{7}$$

note that we subtract 450 rather than 15 because time was measured in months. In general

$$\frac{dp}{dt} = rp - k \tag{8}$$

where the growth rate is r and the predation rate k is unspecified. Note the equilibrium solution would be k/r.

Definition 5. The **equilibrium solution** is the value of $p(t)$ where the system no longer changes, $\frac{dp}{dt} = 0$.

In this example solutions above equilibrium will increase, while solutions below will decrease.

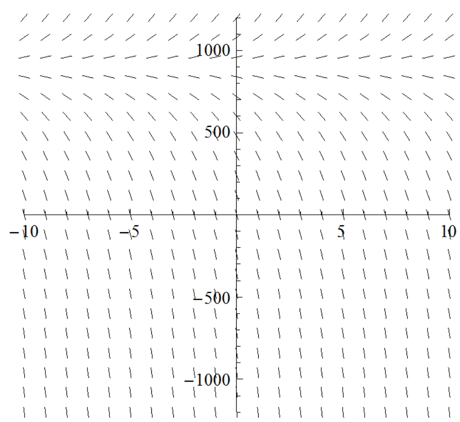


Figure 2: Direction field for above example

Steps to Constructing Mathematical Models:

1. Identify the independent and dependent variables and assign letters to represent them. Often the independent variable is time.
2. Choose the units of measurement for each variable.
3. Articulate the basic principle involved in the problem.
4. Express the principle in the variables chosen above.
5. Make sure each term has the same physical units.
6. We will be dealing with models in this chapter which are single differential equations.

Example 6. Draw the direction field for the following, describe the behavior of y as $t \to \infty$. Describe the dependence on the initial value:

$$y' = 2y + 3 \tag{9}$$

Ans: For $y > -1.5$ the slopes are positive, and hence the solutions increase. For $y < -1.5$ the slopes are negative, and hence the solutions decrease. All solutions appear to diverge away from the equilibrium solution $y(t) = -1.5$.

Example 7. Write down a DE of the form $dy/dt = ay + b$ whose solutions have the required behavior as $t \to \infty$. It must approach $\frac{2}{3}$.
Answer: For solutions to approach the equilibrium solution $y(t) = 2/3$, we must have $y' < 0$ for $y > 2/3$, and $y' > 0$ for $y < 2/3$. The required rates are satisfied by the DE $y' = 2 - 3y$.

Example 8. Find the direction field for $y' = y(y - 3)$

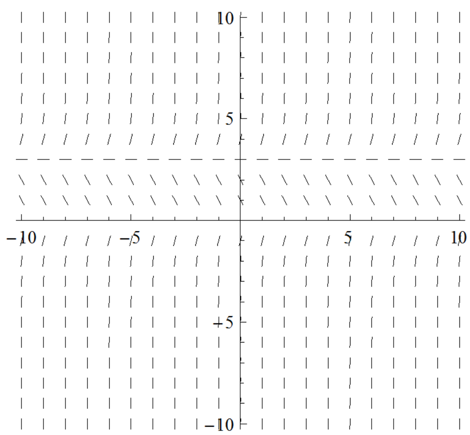


Figure 3: Direction field for above example

HW 1.1 # 6, 9, 16, 17, 20