Lecture Notes for Math 251: ODE and PDE. Lecture 2: 1.2 Solutions to Some DEs

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1 Solutions of Some Differential Equations

Last Time: We derived two formulas:

$$m\frac{dv}{dt} = mg - \gamma v \quad \text{(Falling Bodies)} \tag{1}$$

$$\frac{dp}{dt} = rp - k \quad \text{(Population Growth)} \tag{2}$$

Both equations have the form:

$$\frac{dy}{dt} = ay - b \tag{3}$$

Example 1. *(Field Mice / Predator-Prey Model)*
Consider

$$\frac{dp}{dt} = 0.5p - 450 \tag{4}$$

we want to now solve this equation. Rewrite equation (4) as

$$\frac{dp}{dt} = \frac{p - 900}{2}. \tag{5}$$

Note $p = 900$ is an equilbrium solution and the system does not change. If $p \neq 900$

$$\frac{dp/dt}{p - 900} = \frac{1}{2} \tag{6}$$

By Chain Rule we can rewrite as

$$\frac{d}{dt}\Big[\ln|p - 900|\Big] = \frac{1}{2} \tag{7}$$

So by integrating both sides we find

$$\ln|p - 900| = \frac{t}{2} + C \tag{8}$$

Therefore,

$$p = 900 + Ce^{t/2} \tag{9}$$

Thus we have infinitely many solutions where a different arbitrary constant C produces a different solution. What if the initial population of mice was 850. How do we account for this?

Definition 2. The additional condition, $p(0) = 850$, that is used to determine C is an example of an **initial condition**.

Definition 3. The differential equation together with the initial condition form the **initial value problem**

Consider the general problem

$$\frac{dy}{dt} = ay - b \tag{10}$$

$$y(0) = y_0 \tag{11}$$

The solution has the form

$$y = (b/a) + [y_0 - (b/a)]e^{at} \tag{12}$$

when $a \neq 0$ this contains all possible solutions to the general equation and is thus called **the general solution** The geometric representation of the general solution is an infinite family of curves called **integral curves**.

Example 4. *(Dropping a ball)* System under consideration:

$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \tag{13}$$

$$v(0) = 0 \tag{14}$$

From the formula above we have

$$v = \left(\frac{-9.8}{-1/5}\right) + \left[0 - \frac{-9.8}{-1/5}\right]e^{-\frac{t}{5}} \tag{15}$$

and the general solution is

$$v = 49 + Ce^{-t/5} \tag{16}$$

with the I.C. $C = -49$.

HW 1.2 # 7, 8, 13