Lecture Notes for Math 251: ODE and PDE. Lecture 30: 10.1 Two-Point Boundary Value Problems

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Last Time: We finished Chapter 9: Nonlinear Differential Equations and Stability. Now we start Chapter 10: Partial Differential Equations and Fourier Series

1 Two-Point Boundary Value Problems and Eigenfunctions

1.1 Boundary Conditions

Up until now, we have studied ordinary differential equations and initial value problems. Now we shift to partial differential equations and boundary value problems. Partial differential equations are much more complicated, but are essential in modeling many complex systems found in nature. We need to specify how the solution should behave on the boundary of the region our equation is defined on. The data we prescribe are the **boundary values** or **boundary conditions**, and a combination of a differential equation and boundary conditions is called a **boundary value problem**.

Boundary Conditions depend on the domain of the problem. For an ordinary differential equation our domain was usually some interval on the real line. With a partial differential equation our domain might be an interval or it might be a square in the two-dimensional plane. To see how boundary conditions effect an equation let's examine how they affect the solution of an ordinary differential equation.

Example 1. Let's consider the second order differential equation $y'' + y = 0$. Specifying boundary conditions for this equation involves specifying the values of the solution (or its derivatives) at two points, recall this is because the equation is second order. Consider the interval $(0, 2\pi)$ and specify the boundary conditions $y(0) = 0$ and $y(2\pi) = 0$. We know the solutions to the equation have the form

$$y(x) = A\cos(x) + B\sin(x). \tag{1}$$

by the method of characteristics. Applying the first boundary condition we see $0 = y(0) = A$. Applying the second condition gives $0 = y(2\pi) = B\sin(2\pi)$, but $\sin(2\pi)$ is already zero so B can be any number. So the solutions to this boundary value problem are any functions of the form

$$y(x) = B\sin(x). \tag{2}$$

Example 2. Consider $y'' + y = 0$ with boundary conditions $y(0) = y(6) = 0$. this seems similar to the previous problem, the solutions still have the general form

$$y(x) = A\cos(x) + B\sin(x) \tag{3}$$

and the first condition still tells us $A = 0$. The second condition tells us that $0 = y(6) = B\sin(6)$. Now since $\sin(6) \neq 0$, so we must have $B = 0$ and the entire solution is $y(x) = 0$.

Boundary value problems occur in nature all the time. Examine the examples physically. We know from previous chapters $y'' + y = 0$ models an oscillator such as a rock hanging from a spring. The rock will oscillate with frequency $\frac{1}{2\pi}$. The condition $y(0) = 0$ just means that when we start observing, we want the rock to be at the equilibrium spot. If we specify $y(2\pi) = 0$, this will automatically happen, since the motion is 2π periodic. On the other hand, it is impossible for the rock to return to the equilibrium point after 6 seconds. It will come back in 2π seconds, which is more than 6. So the only possible way the rock can be at equilibrium after 6 seconds is if it does not leave, which is why the only solution is the zero solution.

The previous examples are **homogeneous boundary value problems**. We say that a boundary problem is homogeneous if the equation is homogeneous and the two boundary conditions involve zero. That is, homogeneous boundary conditions might be one of these types

$$\begin{aligned}
y(x_1) &= 0 \qquad y(x_2) = 0 && (4)\\
y'(x_1) &= 0 \qquad y(x_2) = 0 && (5)\\
y(x_1) &= 0 \qquad y'(x_2) = 0 && (6)\\
y'(x_1) &= 0 \qquad y'(x_2) = 0. && (7)
\end{aligned}$$

On the other hand, if the equation is nonhomogeneous or any of the boundary conditions do not equal zero, then the boundary value problem is **nonhomogenous** or **inhomogeneous**. Let's look at some examples of nonhomogeneous boundary value problems.

Example 3. Take $y'' + 9y = 0$ with boundary conditions $y(0) = 2$ and $y(\frac{\pi}{6}) = 1$. The general solution to the differential equation is

$$y(x) = A\cos(3x) + B\sin(3x). \tag{8}$$

The two conditions give

$$\begin{aligned}
2 &= y(0) = A && (9)\\
1 &= y(\frac{\pi}{6}) = B && (10)
\end{aligned}$$

so that the solution is

$$y(x) = 2\cos(3x) + \sin(3x) \tag{11}$$

Example 4. Take $y'' + 9y = 0$ with boundary conditions $y(0) = 2$ and $y(2\pi) = 2$. The general solution to the differential equation is

$$y(x) = A\cos(3x) + B\sin(3x). \tag{12}$$

The two conditions give

$$\begin{aligned} 2 &= y(0) = A && (13) \\ 2 &= y(2\pi) = A. && (14) \end{aligned}$$

This time the second condition did not give and new information, like in Example 1 and B does not affect whether or not the solution satisfies the boundary conditions or not. We then have infinitely many solutions of the form

$$y(x) = 2\cos(3x) + B\sin(3x) \tag{15}$$

Example 5. Take $y'' + 9y = 0$ with boundary conditions $y(0) = 2$ and $y(2\pi) = 4$. The general solution to the differential equation is

$$y(x) = A\cos(3x) + B\sin(3x). \tag{16}$$

The two conditions give

$$\begin{aligned} 2 &= y(0) = A && (17) \\ 4 &= y(2\pi) = A. && (18) \end{aligned}$$

On one hand, $A = 2$ and by the second equation $A = 4$. This is impossible and this boundary value problem has no solutions.

These examples illustrate that a small change to the boundary conditions can dramatically change the problem, unlike small changes in the initial data for initial value problems.

1.2 Eigenvalue Problems

Recall the system studied extensively in previous chapters

$$Ax = \lambda x \tag{19}$$

where for certain values of λ, called eigenvalues, there are nonzero solutions called eigenvectors. We have a similar situation with boundary value problems.

Consider the problem

$$y'' + \lambda y = 0 \tag{20}$$

with boundary conditions $y(0) = 0$ and $y(\pi) = 0$. The values of λ where we get nontrivial (nonzero) solutions will be **eigenvalues**. The nontrivial solutions themselves are called **eigenfunctions**.

We need to consider three cases separately.

(1) If $\lambda > 0$, then it is convenient to let $\lambda = \mu^2$ and rewrite the equation as

$$y'' + \mu^2 y = 0 \tag{21}$$

The characteristic polynomial is $r^2 + \mu^2 = 0$ with roots $r = \pm i\mu$. So the general solution is

$$y(x) = A\cos(\mu x) + B\sin(\mu x) \tag{22}$$

Note that $\mu \neq 0$ since $\lambda > 0$. Recall the boundary conditions are $y(0) = 0$ and $y(\pi) = 0$. So the first boundary condition gives $A = 0$. The second boundary condition reduces to

$$B\sin(\mu\pi) = 0 \tag{23}$$

For nontrivial solutions $B \neq 0$. So $\sin(\mu\pi) = 0$. Thus $\mu = 1, 2, 3, ...$ and thus the eigenvalues λ_n are $1, 4, 9, ..., n^2$. The eigenfunctions are only determined up to arbitrary constant, so convention is to choose the arbitrary constant to be 1. Thus the eigenfunctions are

$$y_1(x) = \sin(x) \quad y_2(x) = \sin(2x), ..., y_n(x) = \sin(nx) \tag{24}$$

(2) If $\lambda < 0$, let $\lambda = -\mu^2$. So the above equation becomes

$$y'' - \mu^2 y = 0 \tag{25}$$

The characteristic equation is $r^2 - \mu^2 = 0$ with roots $r = \pm\mu$, so its general solution can be written as

$$y(x) = A\cosh(\mu x) + B\sinh(\mu x) = Ce^{\mu x} + De^{-\mu x} \tag{26}$$

The first boundary condition, if considering the first form, gives $A = 0$. The second boundary condition gives $B\sinh(\mu\pi) = 0$. Since $\mu \neq 0$, then $\sinh(\mu\pi) \neq 0$, and therefore $B = 0$. So for $\lambda < 0$ the only solution is $y = 0$, there are no nontrivial solutions and thus no eigenvalues.

(3) If $\lambda = 0$, then the equation above becomes

$$y'' = 0 \tag{27}$$

and the general solution if we integrate twice is

$$y(x) = Ax + B \tag{28}$$

The boundary conditions are only satisfied when $A = 0$ and $B = 0$. So there is only the trivial solution $y = 0$ and $\lambda = 0$ is not an eigenvalue.

To summarize we only get **real** eigenvalues and eigenvectors when $\lambda > 0$. There may be complex eigenvalues. A basic problem studied later in the chapter is

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0 \tag{29}$$

Hence the eigenvalues and eigenvectors are

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad y_n(x) = \sin(\frac{n\pi x}{L}) \quad \text{for } n = 1, 2, 3, ... \tag{30}$$

This is the classical **Euler Buckling Problem**.

Review Euler's Equations:

Example 6. Consider equation of the form

$$t^2y'' + ty' + y = 0 \tag{31}$$

and let $x = \ln(t)$. Then

$$\begin{aligned}
\frac{dy}{dt} &= \frac{dy}{dx}\frac{dx}{dt} = \frac{1}{t}\frac{dy}{dx} && (32)\\
\frac{d^2y}{dx^2} &= \frac{d}{dt}(\frac{dy}{dx})\frac{1}{t} + \frac{dy}{dx}(\frac{1}{t})\frac{dy}{dx} && (33)\\
&= \frac{d^2y}{dx^2}\frac{1}{t^2} + \frac{dy}{dx}(-\frac{1}{t^2}) && (34)\\
& && (35)
\end{aligned}$$

Plug these back into the original equation

$$\begin{aligned}
t^2y'' + ty + y &= \frac{d^2y}{dx^2} - \frac{dy}{dx} + \frac{dy}{dx} + y = 0 && (36)\\
&= y'' + y = 0 && (37)
\end{aligned}$$

Thus the characteristic equation is $r^2 + 1 = 0$, which has roots $r = \pm i$. So the general solution is

$$\hat{y}(x) = c_1\cos(x) + c_2\sin(x) \tag{38}$$

Recalling that $x = \ln(t)$ our final solution is

$$y(x) = c_1\cos(\ln(t)) + c_2\sin(\ln(t)) \tag{39}$$

HW 10.1 # 1,4,8,10,11,15,17

Hint: You may have to use the method of undetermined coefficients for nonhomogeneous problems.