Lecture Notes for Math 251: ODE and PDE. Lecture 32: 10.2 Fourier Series

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Last Time: We studied the heat equation and the method of Separation of Variables. We then used Separation of Variables to solve the heat equation and looked at the form of the typical solution.

1 Fourier Series

Last lecture, we identified solutions of the heat equation having the form

$$u_t = u_{xx} \tag{1}$$

$0 < x < l$, $t > 0$, with homogeneous Dirichlet conditions at $u(0,t) = u(l,t) = 0$, had the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{l})^2 kt} \sin(\frac{n\pi x}{l}). \tag{2}$$

while the heat equation with homogeneous Neumann conditions $u_x(0,t) = u_x(l,t) = 0$ had solutions of the form

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{l})^2 kt} \cos(\frac{n\pi x}{l}). \tag{3}$$

For this to make sense given an initial condition $u(x,0) = f(x)$, for the Dirichlet case we need to be able to write $f(x)$ as

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l}) \tag{4}$$

for some coefficients A_n, while in the Neumann case it must have the form

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{l}) \tag{5}$$

for appropriate coefficients. Equation (**??**) is called a **Fourier Sine Series** of $f(x)$ and an expression like Equation (**??**) is called a **Fourier Cosine Series** of $f(x)$.

There are two key things to keep in mind:

(1) Is it possible to find appropriate coefficients for the Fourier Sine and Cosine series for a given $f(x)$?

(2) For which $f(x)$ will the Fourier series converge, if any? What will the Fourier Series converge to?

Section 10.2 will focus on finding the coefficients of the Fourier Series, while Section 10.3 will address convergence.

1.1 The Euler-Fourier Formula

We have a famous formula for the Fourier Coefficients, called the **Euler-Fourier Formula**.

1.1.1 Fourier Sine Series

Start by considering the Fourier Sine Series

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l}). \tag{6}$$

How can we find the coefficients A_n? Observe that sine functions have the following property

$$\int_0^l \sin(\frac{n\pi x}{l}) \sin(\frac{m\pi x}{l}) dx = 0 \tag{7}$$

if $m \neq n$ are both positive integers. This can be seen by direct integration. Recall the trig identity

$$\sin(a)\sin(b) = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b). \tag{8}$$

Then the integral in Equation (**??**) equals

$$\frac{l}{2(m-n)\pi} \sin(\frac{(m-n)\pi x}{l})|_0^l - \frac{l}{2(m+n)\pi} \sin(\frac{(m+n)\pi x}{l})|_0^l \tag{9}$$

so long as $m \neq n$. But these terms are just linear combinations of $\sin((m \pm n)\pi)$ and $\sin(0)$, and thus everything is zero.

Now, fix m and multiply Equation (**??**) (Fourier Sine Series) by $\sin(\frac{m\pi x}{l})$. Integrating term by term we get

$$\int_0^l f(x) \sin(\frac{m\pi x}{l}) dx = \int_0^l \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l}) \sin(\frac{m\pi x}{l}) dx \tag{10}$$

$$= \sum_{n=1}^{\infty} \int_0^l \sin(\frac{n\pi x}{l}) \sin(\frac{m\pi x}{l}) dx. \tag{11}$$

Due to the above work the only term that remains is when $m = n$. So all we have left is

$$\int_0^l f(x)\sin(\frac{m\pi x}{l})dx = A_m \int_0^l \sin^2(\frac{m\pi x}{l}) = \frac{1}{2} l A_m \tag{12}$$

and so

$$A_m = \frac{2}{l}\int_0^l f(x)\sin(\frac{m\pi x}{l})dx. \tag{13}$$

In summary. If $f(x)$ has a Fourier sine expansion, the coefficients must be given by Equation (**??**). These are the only possible coefficients for such a series, but we have not shown that the Fourier Sine Series is a valid expression for $f(x)$.

Example 1. Compute a Fourier Sine Series for $f(x) = 1$ on $0 \leq x \leq l$.

By Equation (**??**), the coefficients must be given by

$$\begin{aligned} A_m &= \frac{2}{l}\int_0^l \sin(\frac{m\pi x}{l})dx. && (14)\\ &= -\frac{2}{m\pi}\cos(\frac{m\pi x}{l})|_0^l && (15)\\ &= \frac{2}{m\pi}(1-\cos(m\pi)) = \frac{2}{m\pi}(1-(-1)^m). && (16) \end{aligned}$$

So we have $A_m = \frac{4}{m\pi}$ if m is odd and $A_m = 0$ if m is even. Thus, on $(0, l)$

$$\begin{aligned} 1 &= \frac{4}{\pi}(\sin(\frac{\pi x}{l}) + \frac{1}{3}\sin(\frac{3\pi x}{l}) + \frac{1}{5}\sin(\frac{5\pi x}{l}) + ...) && (17)\\ &= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(\frac{(2n-1)\pi x}{l}). && (18) \end{aligned}$$

Example 2. Compute the Fourier Sine Series for $f(x) = x$ on $0 \leq x \leq l$.

In this case Equation (**??**) yields a formula for the coefficients

$$\begin{aligned} A_m &= \frac{2}{l}\int_0^l x\sin(\frac{m\pi x}{l})dx && (19)\\ &= -\frac{2x}{m\pi}\cos(\frac{m\pi x}{l})|_0^l + \frac{2l}{m^2\pi^2}\sin(\frac{m\pi x}{l})|_0^l && (20)\\ &= -\frac{2l}{m\pi}\cos(m\pi) + \frac{2l}{m^2\pi^2}\sin(m\pi) && (21)\\ &= (-1)^{m+1}\frac{2l}{m\pi}. && (22) \end{aligned}$$

So on $(0, l)$, we have

$$\begin{aligned} x &= \frac{2l}{\pi}\left(\sin(\frac{\pi x}{l}) - \frac{1}{2}\sin(\frac{2\pi x}{l}) + \frac{1}{3}\sin(\frac{3\pi x}{l}) - ...\right) && (23)\\ &= \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left(\frac{(2n-1)\pi x}{l}\right) - \frac{1}{2n}\sin\left(\frac{2n\pi x}{l}\right). && (24) \end{aligned}$$

1.1.2 Fourier Cosine Series

Now let's consider the Fourier Cosine Series

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{l}). \tag{25}$$

We can use the following property of cosine

$$\int_0^l \cos(\frac{n\pi x}{l}) \cos(\frac{m\pi x}{l})dx = 0. \tag{26}$$

Verify this for an exercise.

By the exact same computation as before for sines, we replace sines with cosines, if $m \neq 0$ we get

$$\int_0^l f(x) \cos(\frac{m\pi x}{l})dx = A_m \int_0^l \cos^2(\frac{m\pi x}{l})dx = \frac{1}{2}lA_m. \tag{27}$$

If $m = 0$, we have

$$\int_0^l f(x) \cdot 1dx = \frac{1}{2}A_0 \int_0^l 1^2 = \frac{1}{2}lA_0. \tag{28}$$

Thus, for all $m > 0$, we have

$$A_m = \frac{2}{l} \int_0^l f(x) \cos(\frac{m\pi x}{l})dx. \tag{29}$$

This is why we have the $\frac{1}{2}$ in front of A_0 (so it has the same form as A_m for $m \neq 0$).

Example 3. Compute the Fourier Cosine Series for $f(x) = 1$ on $0 \leq x \leq l$.

By Equation (**??**), the coefficients when $m \neq 0$ are

$$\begin{aligned} A_m &= \frac{2}{l} \int_0^l \cos(\frac{m\pi x}{l})dx & (30) \\ &= \frac{2}{m\pi} \sin(\frac{m\pi x}{l})\Big|_0^l & (31) \\ &= \frac{2}{m\pi} \sin(m\pi) = 0. & (32) \end{aligned}$$

So the only coefficient we have occurs are A_0, and this Fourier Cosine Series is then trivial

$$1 = 1 + 0\cos(\frac{\pi x}{l}) + 0\cos(\frac{2\pi x}{l}) + ... \tag{33}$$

Example 4. Compute the Fourier Cosine Series for $f(x) = x$.

For $m \neq 0$,

$$\begin{aligned}
A_m &= \frac{2}{l}\int_0^l x\cos(\frac{m\pi x}{l})dx && (34)\\
&= \frac{2x}{m\pi}\sin(\frac{m\pi x}{l}) + \frac{2l}{m^2\pi^2}\cos(\frac{m\pi x}{l})|_0^l && (35)\\
&= \frac{2l}{m\pi}\sin(m\pi) + \frac{2l}{m^2\pi^2}(\cos(m\pi)-1) && (36)\\
&= \frac{2l}{m^2\pi^2}((-1)^m - 1) && (37)\\
&= \begin{cases} -\frac{4l}{m^2\pi^2} & \text{m odd} \\ 0 & \text{m even} \end{cases}. && (38)
\end{aligned}$$

If $m = 0$, we have

$$A_0 = \frac{2}{l}\int_0^l x dx = l. \quad (39)$$

So on $(0, l)$, we have the Fourier Cosine Series

$$\begin{aligned}
x &= \frac{l}{2} - \frac{4l}{\pi^2}\left(\cos(\frac{\pi x}{l}) + \frac{1}{9}\cos(\frac{3\pi x}{l}) + \frac{1}{25}\cos(\frac{5\pi x}{l}) + ...\right) && (40)\\
&= \frac{l}{2} + \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left(\frac{(2n-1)\pi x}{l}\right). && (41)
\end{aligned}$$

1.1.3 Full Fourier Series

The full Fourier Series of $f(x)$ on the interval $-l < x < l$, is defined as

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n\cos(\frac{n\pi x}{l}) + B_n\sin(\frac{n\pi x}{l}). \quad (42)$$

REMARK: Be careful, now the interval we are working with is twice as long $-l < x < l$.

The computation of the coefficients for the formulas is analogous to the Fourier Sine and Cosine series. We need the following set of identities:

$$\begin{aligned}
\int_{-l}^{l}\cos(\frac{n\pi x}{l})\sin(\frac{m\pi x}{l})dx &= 0 \quad \text{for } n, m && (43)\\
\int_{-l}^{l}\cos(\frac{n\pi x}{l})\cos(\frac{m\pi x}{l})dx &= 0 \quad \text{for } n \neq m && (44)\\
\int_{-l}^{l}\sin(\frac{n\pi x}{l})\sin(\frac{m\pi x}{l})dx &= 0 \quad \text{for } n \neq m && (45)\\
\int_{-l}^{l}1\cdot\cos(\frac{n\pi x}{l})dx &= 0 = \int_{-l}^{l}1\cdot\sin(\frac{n\pi x}{l})dx. && (46)
\end{aligned}$$

Thus, using the same procedure as above for sine and cosine, we can get the coefficients. We fix m and multiply by $\cos(\frac{m\pi x}{l})$, and do the same for $\sin(\frac{m\pi x}{l})$. So we need to calculate the integrals of the squares

$$\int_{-l}^{l} \cos^2(\frac{n\pi x}{l})dx = 1 = \int_{-l}^{l} \sin^2(\frac{n\pi x}{l})dx \quad \text{and} \quad \int_{-l}^{l} 1^2 dx = 2l \tag{47}$$

EXERCISE: Verify the above integrals.

So we get the following formulas

$$A_m = \frac{1}{l}\int_{-l}^{l} f(x)\cos(\frac{m\pi x}{l})dx \quad (n = 1, 2, 3, ...) \tag{48}$$

$$B_m = \frac{1}{l}\int_{-l}^{l} f(x)\sin(\frac{m\pi x}{l})dx \quad (n = 1, 2, 3, ...) \tag{49}$$

for the coefficients of the full Fourier Series. Notice that the first equation is exactly the same as we got when considering the Fourier Cosine Series and the second equation is the same as the solution for the Fourier Sine Series. **NOTE:** The intervals of integration are different!

Example 5. Compute the Fourier Series of $f(x) = 1 + x$.

Using the above formulas we have

$$A_0 = \frac{1}{l}\int_{-l}^{l} (1+x)dx = 2 \tag{50}$$

$$A_m = \frac{1}{l}\int_{-l}^{l} (1+x)\cos(\frac{m\pi x}{l})dx \tag{51}$$

$$= \frac{1+x}{m\pi}\sin(\frac{m\pi x}{l}) + \frac{1}{m^2\pi^2}\cos(\frac{m\pi x}{l})|_{-l}^{l} \tag{52}$$

$$= \frac{1}{m^2\pi^2}(\cos(m\pi) - \cos(-m\pi)) = 0 \quad m \neq 0 \tag{53}$$

$$B_m = \frac{1}{l}\int_{-l}^{l} (1+x)\sin(\frac{m\pi x}{l})dx \tag{54}$$

$$= -\frac{1+x}{m\pi}\cos(\frac{m\pi x}{l}) + \frac{1}{m^2\pi^2}\sin(\frac{m\pi x}{l})|_{-l}^{l} \tag{55}$$

$$= -\frac{2l}{m\pi}\cos(m\pi) = (-1)^{m+1}\frac{2l}{m\pi}. \tag{56}$$

So the full Fourier series of $f(x)$ is

$$1 + x = 1 + \frac{2l}{\pi}(\sin(\frac{\pi x}{l}) - \frac{1}{2}\sin(\frac{2\pi x}{l}) + \frac{1}{3}\sin(\frac{3\pi x}{l}) - ...) \tag{57}$$

$$= 1 + \frac{2l}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1}\sin(\frac{(2n-1)\pi x}{l}) - \frac{1}{2n}\sin(\frac{2n\pi x}{l}). \tag{58}$$

HW 10.2 # 14,16,18