Lecture Notes for Math 251: ODE and PDE. Lecture 32: 10.3 The Fourier Convergence Theorem

Shawn D. Ryan

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Last Time: We found Fourier Series for given functions and now we want to discuss if these Fourier Series converge to anything.

1 Convergence of Fourier Series

Last class we derived the Euler-Fourier formulas for the coefficients of the Fourier Series of a given function $f(x)$. For the **Fourier Sine Series** of $f(x)$ on the interval $(0, l)$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l}), \tag{1}$$

we have

$$A_n = \frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi x}{l})dx, \tag{2}$$

where $n = 1, 2, 3, ...$ For the **Fourier Cosine Series** of $f(x)$ on $(0, l)$,

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{l}), \tag{3}$$

with the coefficients given by

$$A_n = \frac{2}{l}\int_0^l f(x)\cos(\frac{n\pi x}{l})dx, \tag{4}$$

where $n = 1, 2, 3, ...$ Finally, for the full **Fourier Series** of $f(x)$, which is valid on the interval $(-l, l)$ (Note: **Not** the same interval as the previous two cases!),

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{l}) + B_n \sin(\frac{n\pi x}{l}), \tag{5}$$

the coefficients are given by

$$A_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos(\frac{n\pi x}{l})dx \quad n = 1, 2, 3, ... \tag{6}$$

$$B_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin(\frac{n\pi x}{l})dx \quad n = 1, 2, 3, ... \tag{7}$$

Example 1. Compute the Fourier Series for $f(x) = \begin{cases} 2 & -2 \le x < -1 \\ 1-x & -1 \le x < 2 \end{cases}$ on the interval $(-2, 2)$.

We start by using the Euler-Fourier Formulas. For the Cosine terms we find

$$\begin{aligned} A_0 &= \frac{1}{2}\int_{-2}^{2} f(x)dx \\ &= \frac{1}{2}\left(\int_{-2}^{-1} 2dx + \int_{-1}^{2} 1 - xdx\right) \\ &= \frac{1}{2}(2 + \frac{3}{2}) = \frac{7}{4} \end{aligned}$$

and

$$\begin{aligned} A_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos(\frac{n\pi x}{2})dx \\ &= \frac{1}{2}\left(\int_{-2}^{-1} 2\cos(\frac{n\pi x}{2})dx + \int_{-1}^{2}(1-x)\cos(\frac{n\pi x}{2})dx\right) \\ &= \frac{1}{2}\left(\frac{4}{n\pi}\sin(\frac{n\pi x}{2})|_{-2}^{-1} + \frac{2(1-x)}{n\pi}\sin(\frac{n\pi x}{2})|_{-1}^{2} - \frac{4}{n^2\pi^2}\left(\cos(\frac{n\pi x}{2})|_{-1}^{2}\right)\right) \\ &= \frac{1}{2}\left(-\frac{4}{n\pi}\sin(\frac{n\pi}{2}) + \frac{4}{n\pi}\sin(\frac{n\pi}{2}) - \frac{4}{n^2\pi^2}(\cos(n\pi) - \cos(\frac{n\pi}{2}))\right) \\ &= \begin{cases} \frac{2}{n^2\pi^2} & n \text{ odd} \\ 0 & n = 4m \\ -\frac{4}{n^2\pi^2} & n = 4m+2 \end{cases}. \end{aligned}$$

Also, for the sine terms

$$\begin{aligned}
B_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin(\frac{n\pi x}{2})dx \\
&= \frac{1}{2}\left(\int_{-2}^{-1} 2\sin(\frac{n\pi x}{2})dx + \int_{-1}^{2}(1-x)\sin(\frac{n\pi x}{2})dx\right) \\
&= \frac{1}{2}\left(-\frac{4}{n\pi}\cos(\frac{n\pi x}{2})|_{-2}^{-1} - \frac{2(1-x)}{n\pi}\cos(\frac{n\pi x}{2})|_{-1}^{2} - \frac{4}{n^2\pi^2}(\sin(\frac{n\pi x}{2})|_{-1}^{2}\right) \\
&= \frac{1}{2}\left(\frac{6}{n\pi}\cos(n\pi) - \frac{4}{n^2\pi^2}\sin(\frac{n\pi}{2})\right) \\
&= \begin{cases} \frac{3}{n\pi} & n\text{even} \\ -\frac{3}{n\pi} - \frac{2}{n^2\pi^2} & n = 4m+1 \\ -\frac{3}{n\pi} + \frac{2}{n^2\pi^2} & n = 4m+3 \end{cases} .
\end{aligned}$$

So we have

$$\begin{aligned}
f(x) = \frac{7}{8} &+ \sum_{m=1}^{\infty} \frac{2}{(4m+1)^2\pi^2}\cos\left(\frac{(4m+1)\pi x}{2}\right) + \left(-\frac{3}{(4m+1)\pi} - \frac{2}{(4m+1)^2\pi^2}\right)\sin\left(\frac{(4m+1)\pi x}{2}\right) \\
&- \frac{4}{(4m+2)^2\pi^2}\cos\left(\frac{(4m+2)\pi x}{2}\right) + \frac{3}{(4m+2)\pi}\sin\left(\frac{(4m+2)\pi x}{2}\right) \\
&+ \frac{2}{(4m+3)^2\pi^2}\cos\left(\frac{(4m+3)\pi x}{2}\right) + \left(-\frac{3}{(4m+3)\pi} + \frac{2}{(4m+3)^2\pi^2}\right)\sin\left(\frac{(4m+3)\pi x}{2}\right) \\
&+ \frac{3}{4m\pi}\sin\left(\frac{4m\pi x}{2}\right).
\end{aligned}$$

This example represents a worst case scenario. There are a lot of Fourier coefficients to keep track of. Notice that for each value of m, the summand specifies four different Fourier terms (for $4m, 4m+1, 4m+2, 4m+3$). This can often happen and depending on l, even more terms maybe required.

1.1 Convergence of Fourier Series

So we know that if a function $f(x)$ is to have a Fourier Series on an appropriate interval, the coefficients have to be in the form of a Fourier Sine series (**??**) on $(0, l)$, a Fourier Cosine Series (**??**) on $(0, l)$, or the full Fourier Series (**??**) on $(-l, l)$. What do these series converge to? First consider the full Fourier Series.

We require that $f(x)$ is **piecewise smooth**. This is even stronger than the piecewise continuity we saw with Laplace Transforms. We want to divide $(-l, l)$ into a finite number of subintervals so that both $f(x)$ and its derivative $f'(x)$ are continuous on each interval. We also require that the only discontinuities at the boundary points of the subintervals are jump discontinuities (not asymptotically approaching infinity).

Example 2. Any continuous function with continuous derivative on the desired interval is automatically piecewise smooth. This is proven later in an advanced analysis class.

Example 3. Consider the function from Example 1

$$f(x) = \begin{cases} 2 & -2 \leq x < -1 \\ 1 - x & -1 \leq x \leq 2 \end{cases} . \tag{8}$$

$f(x)$ is continuous for all x in $(-2, 2)$, but the derivative $f'(x)$ has a discontinuity at $x = -1$. This is a jump discontinuity, with $\lim_{x \to -1^-} f'(x) = 0$ and $\lim_{x \to -1^+} f'(x) = -1$. Thus, $f(x)$ is piecewise smooth.

The next thing to note is that even though we only need $f(x)$ to be defined on $(-l, l)$ to compute the Fourier Series, the Fourier Series itself is defined for all x. Also, all of the terms in a Fourier Series are $2l$-periodic. They are either constants of have the form $\sin(\frac{n\pi x}{l})$ or $\cos(\frac{n\pi x}{l})$. So we can regard the Fourier Series either as the expansion of a function on $(-l, l)$ or as the expansion of a $2l$-periodic function on $-\infty < x < \infty$.

What will this $2l$-periodic function be? It will have to coincide on $(-l, l)$ with $f(x)$ (since it is also the expansion of $f(x)$ on that interval) and still be $2l$-periodic. Define the **periodic extension** of $f(x)$ to be

$$f_{per}(x) = f(x - 2lm) \quad \text{for} \quad -l + 2lm < x < l + 2lm \tag{9}$$

for all integers m.

REMARK: The definition (**??**) does not specify what the periodic extension is at the endpoints $x = l + 2lm$. This is because the extension will, in general, have jumps at these points. This happens when $f(-l^+) \neq f(l^-)$.

No what does the Fourier Series of $f(x)$ converge to?

Theorem 4. (Fourier Convergence Theorem) *Suppose $f(x)$ is piecewise smooth on $(-l, l)$. Then at $x = x_0$, the Fourier Series of $f(x)$ will converge to*
(1) $f_{per}(x_0)$ if f_{per} is continuous at x_0 or
(2) The average of the one sided limits $\frac{1}{2}[f_{per}(x_0^+) + f_{per}(x_0^-)]$ is f_{per} has a jump discontinuity at $x = x_0$.

Theorem 1 tells us that on the interval $(-l, l)$ the Fourier Series will **almost** converge to the original function $f(x)$, with the only problems occurring at the discontinuities.

Example 5. What does the Fourier Series of $f(x) = \begin{cases} 1 & -3 \leq x \leq 0 \\ 2x & 0 < x \leq 3 \end{cases}$ will converge to at $x = -2, 0, 3, 5, 6$?

The first two points are inside the original interval of definition of $f(x)$, so we can just directly consider $f_{per}(x)$. The only discontinuity of $f(x)$ occurs at $x = 0$. So at $x = -2$, $f(x)$ is nice and continuous. The Fourier Series will converge to $f(-2) = 1$. On the other hand, at $x = 0$ we have a jump discontinuity, so the Fourier Series will converge to the average of the one-sided limits.

$f(0^+) = \lim_{x\to 0^+} f(x) = 0$ and $f(0^-) = \lim_{x\to 0^-} f(x) = 1$, so the Fourier Series will converge to $\frac{1}{2}[f(0^+) + f(0^-)] = \frac{1}{2}$.

What happens at the other points? Here we consider $f_{per}(x)$ and where it has jump discontinuities. These can only occur either at $x = x_0 + 2lm$ where $-l < x_0 < l$ is a jump discontinuity of $f(x)$ or at endpoints $x = \pm l + 2lm$, since the periodic extension might not "sync up" at these points, producing a jump discontinuity.

At $x = 3$, we are at one of these "boundary points" and the left-sided limit is 6 while the right-sided limit is 1. Thus the Fourier Series will converge here to $\frac{6+1}{2} = \frac{7}{2}$. $x = 5$ is a point of continuity for $f_{per}(x)$ and so the Fourier Series will converge to $f_{per}(5) = f(-1) = 1$. $x = 6$, is a jump discontinuity (corresponding to $x = 0$), so the Fourier Series will converge to $\frac{1}{2}$.

Example 6. Where does the Fourier Series for $f(x) = \begin{cases} 2 & -2 \le x < -1 \\ 1 - x & -1 \le x \le 2 \end{cases}$ converge at $x = -7, -1, 6$?

None of the points are inside $(-2, 2)$ where $f(x)$ is discontinuous. The only points where the periodic extension might be discontinuous are the "boundary points" $x = \pm 2 + 4k$. In fact, since $f(-2) \neq f(2)$, these will be points of discontinuity. So $f_{per}(x)$ is continuous at $x = -7$, since it is not a boundary point and we have $f_{per}(-7) = f(1) = 0$, which is what the Fourier Series will converge to. The same for $x = -1$, the Fourier Series will converge to $f(-1) = \frac{2+2}{2} = 2$.

For $x = 6$ we are at an endpoint. The left-sided limit is -1, while the right-sided limit is 2, so the Fourier Series will converge to their average $\frac{1}{2}$.

HW 10.3 # 1,3,6