Lecture Notes for Math 251: ODE and PDE. Lecture 32: 10.3 The Fourier Convergence Theorem

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Last Time: We found Fourier Series for given functions and now we want to discuss if these Fourier Series converge to anything.

1 Convergence of Fourier Series

Last class we derived the Euler-Fourier formulas for the coefficients of the Fourier Series of a given function $f(x)$. For the **Fourier Sine Series** of $f(x)$ on the interval $(0, l)$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l}), \tag{1}$$

we have

$$A_n = \frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi x}{l})dx, \tag{2}$$

where $n = 1, 2, 3, ...$ For the **Fourier Cosine Series** of $f(x)$ on $(0, l)$,

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{l}), \tag{3}$$

with the coefficients given by

$$A_n = \frac{2}{l}\int_0^l f(x)\cos(\frac{n\pi x}{l})dx, \tag{4}$$

where $n = 1, 2, 3, ...$ Finally, for the full **Fourier Series** of $f(x)$, which is valid on the interval $(-l, l)$ (Note: Note the same interval as the previous two cases!),

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{l}) + B_n \sin(\frac{n\pi x}{l}), \tag{5}$$

the coefficients are given by

$$A_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos(\frac{n\pi x}{l})dx \quad n = 1, 2, 3, ... \tag{6}$$

$$B_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin(\frac{n\pi x}{l})dx \quad n = 1, 2, 3, ... \tag{7}$$

Example 1. Compute the Fourier Series for $f(x) = \begin{cases} 2 & -2 \leq x < -1 \\ 1-x & -1 \leq x < 2 \end{cases}$ on the interval $(-2, 2)$.

We start by using the Euler-Fourier Formulas. For the Cosine terms we find

$$\begin{aligned} A_0 &= \frac{1}{2}\int_{-2}^{2} f(x)dx \\ &= \frac{1}{2}\left(\int_{-2}^{-1} 2dx + \int_{-1}^{2} 1 - xdx\right) \\ &= \frac{1}{2}(2 + \frac{3}{2}) = \frac{7}{4} \end{aligned}$$

and

$$\begin{aligned} A_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos(\frac{n\pi x}{2})dx \\ &= \frac{1}{2}\left(\int_{-2}^{-1} 2\cos(\frac{n\pi x}{2})dx + \int_{-1}^{2}(1-x)\cos(\frac{n\pi x}{2})dx\right) \\ &= \frac{1}{2}\left(\frac{4}{n\pi}\sin(\frac{n\pi x}{2})|_{-2}^{-1} + \frac{2(1-x)}{n\pi}\sin(\frac{n\pi x}{2})|_{-1}^{2} - \frac{4}{n^2\pi^2}\left(\cos(\frac{n\pi x}{2})|_{-1}^{2}\right)\right) \\ &= \frac{1}{2}\left(-\frac{4}{n\pi}\sin(\frac{n\pi}{2}) + \frac{4}{n\pi}\sin(\frac{n\pi}{2}) - \frac{4}{n^2\pi^2}(\cos(n\pi) - \cos(\frac{n\pi}{2}))\right) \\ &= \begin{cases} \frac{2}{n^2\pi^2} & n\text{odd} \\ 0 & n = 4m \\ -\frac{4}{n^2\pi^2} & n = 4m+2 \end{cases}. \end{aligned}$$

Also, for the sine terms

$$
\begin{aligned}
B_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin(\frac{n\pi x}{2})dx \\
&= \frac{1}{2}\left(\int_{-2}^{-1} 2\sin(\frac{n\pi x}{2})dx + \int_{-1}^{2}(1-x)\sin(\frac{n\pi x}{2})dx\right) \\
&= \frac{1}{2}\left(-\frac{4}{n\pi}\cos(\frac{n\pi x}{2})|_{-2}^{-1} - \frac{2(1-x)}{n\pi}\cos(\frac{n\pi x}{2})|_{-1}^{2} - \frac{4}{n^2\pi^2}(\sin(\frac{n\pi x}{2})|_{-1}^{2}\right) \\
&= \frac{1}{2}\left(\frac{6}{n\pi}\cos(n\pi) - \frac{4}{n^2\pi^2}\sin(\frac{n\pi}{2})\right) \\
&= \begin{cases} \frac{3}{n\pi} & n\text{even} \\ -\frac{3}{n\pi} - \frac{2}{n^2\pi^2} & n = 4m+1 \\ -\frac{3}{n\pi} + \frac{2}{n^2\pi^2} & n = 4m+3 \end{cases} .
\end{aligned}
$$

So we have

$$
\begin{aligned}
f(x) = \frac{7}{8} &+ \sum_{m=1}^{\infty} \frac{2}{(4m+1)^2\pi^2}\cos\left(\frac{(4m+1)\pi x}{2}\right) + \left(-\frac{3}{(4m+1)\pi} - \frac{2}{(4m+1)^2\pi^2}\right)\sin\left(\frac{(4m+1)\pi x}{2}\right) \\
&- \frac{4}{(4m+2)^2\pi^2}\cos\left(\frac{(4m+2)\pi x}{2}\right) + \frac{3}{(4m+2)\pi}\sin\left(\frac{(4m+2)\pi x}{2}\right) \\
&+ \frac{2}{(4m+3)^2\pi^2}\cos\left(\frac{(4m+3)\pi x}{2}\right) + \left(-\frac{3}{(4m+3)\pi} + \frac{2}{(4m+3)^2\pi^2}\right)\sin\left(\frac{(4m+3)\pi x}{2}\right) \\
&+ \frac{3}{4m\pi}\sin\left(\frac{4m\pi x}{2}\right).
\end{aligned}
$$

This example represents a worst case scenario. There are a lot of Fourier coefficients to keep track of. Notice that for each value of m, the summand specifies four different Fourier terms (for $4m, 4m+1, 4m+2, 4m+3$). This can often happen and depending on l, even more terms maybe required.

1.1 Convergence of Fourier Series

So we know that if a function $f(x)$ is to have a Fourier Series on an appropriate interval, the coefficients have to be in the form of a Fourier Sine series (1) on $(0, l)$, a Fourier Cosine Series (3) on $(0, l)$, or the full Fourier Series (5) on $(-l, l)$. What do these series converge to? First consider the full Fourier Series.

We require that $f(x)$ is **piecewise smooth**. This is even stronger than the piecewise continuity we saw with Laplace Transforms. We want to divide $(-l, l)$ into a finite number of subintervals so that both $f(x)$ and its derivative $f'(x)$ are continuous on each interval. We also require that the only discontinuities at the boundary points of the subintervals are jump discontinuities (not asymptotically approaching infinity).

Example 2. Any continuous function with continuous derivative on the desired interval is automatically piecewise smooth. This is proven later in an advanced analysis class.

Example 3. Consider the function from Example 1

$$f(x) = \begin{cases} 2 & -2 \le x < -1 \\ 1 - x & -1 \le x \le 2 \end{cases} . \tag{8}$$

$f(x)$ is continuous for all x in $(-2, 2)$, but the derivative $f'(x)$ has a discontinuity at $x = -1$. This is a jump discontinuity, with $\lim_{x \to -1^-} f'(x) = 0$ and $\lim_{x \to -1^+} f'(x) = -1$. Thus, $f(x)$ is piecewise smooth.

The next thing to note is that even though we only need $f(x)$ to be defined on $(-l, l)$ to compute the Fourier Series, the Fourier Series itself is defined for all x. Also, all of the terms in a Fourier Series are $2l$-periodic. They are either constants of have the form $\sin(\frac{n\pi x}{l})$ or $\cos(\frac{n\pi x}{l}$. So we can regard the Fourier Series either as the expansion of a function on $(-l, l)$ or as the expansion of a $2l$-periodic function on $-\infty < x < \infty$.

What will this $2l$-periodic function be? It will have to coincide on $(-l, l)$ with $f(x)$ (since it is also the expansion of $f(x)$ on that interval) and still be $2l$-periodic. Define the **periodic extension** of $f(x)$ to be

$$f_{per}(x) = f(x - 2lm) \quad \text{for} \quad -l + 2lm < x < l + 2lm \tag{9}$$

for all integers m.

REMARK: The definition (9) does not specify what the periodic extension is at the endpoints $x = l + 2lm$. This is because the extension will, in general, have jumps at these points. This happens when $f(-l^+) \neq f(l^-)$.

No what does the Fourier Series of $f(x)$ converge to?

Theorem 4. (Fourier Convergence Theorem) *Suppose* $f(x)$ *is piecewise smooth on* $(-l, l)$. *Then at* $x = x_0$, *the Fourier Series of* $f(x)$ *will converge to*
(1) $f_{per}(x_0)$ *if* f_{per} *is continuous at* x_0 *or*
(2) The average of the one sided limits $\frac{1}{2}[f_{per}(x_0^+) + f_{per}(x_0^-)]$ *is* f_{per} *has a jump discontinuity at* $x = x_0$.

Theorem 1 tells us that on the interval $(-l, l)$ the Fourier Series will **almost** converge to the original function $f(x)$, with the only problems occurring at the discontinuities.

Example 5. What does the Fourier Series of $f(x) = \begin{cases} 1 & -3 \le x \le 0 \\ 2x & 0 < x \le 3 \end{cases}$ will converge to at $x = -2, 0, 3, 5, 6$?

The first two points are inside the original interval of definition of $f(x)$, so we can just directly consider $f_{per}(x)$. The only discontinuity of $f(x)$ occurs at $x = 0$. So at $x = -2$, $f(x)$ is nice and continuous. The Fourier Series will converge to $f(-2) = 1$. On the other hand, at $x = 0$ we have a jump discontinuity, so the Fourier Series will converge to the average of the one-sided limits.

$f(0^+) = \lim_{x\to 0^+} f(x) = 0$ and $f(0^-) = \lim_{x\to 0^-} f(x) = 1$, so the Fourier Series will converge to $\frac{1}{2}[f(0^+) + f(0^-)] = \frac{1}{2}$.

What happens at the other points? Here we consider $f_{per}(x)$ and where it has jump discontinuities. these can only occur either at $x = x_0 + 2lm$ where $-l < x_0 < l$ is a jump discontinuity of $f(x)$ or at endpoints $x = \pm l + 2lm$, since the periodic extension might not "sync up" at these points, producing a jump discontinuity.

At $x = 3$, we are at one of these "boundary points" and the left-sided limit is 6 while the right-sided limit is 1. Thus the Fourier Series will converge here to $\frac{6+1}{2} = \frac{7}{2}$. $x = 5$ is a point of continuity for $f_{per}(x)$ and so the Fourier Series will converge to $f_{per}(5) = f(-1) = 1$. $x = 6$, is a jump discontinuity (corresponding to $x = 0$), so the Fourier Series will converge to $\frac{1}{2}$.

Example 6. Where does the Fourier Series for $f(x) = \begin{cases} 2 & -2 \leq x < -1 \\ 1 - x & -1 \leq x \leq 2 \end{cases}$ converge at $x = -7, -1, 6$?

None of the points are inside $(-2, 2)$ where $f(x)$ is discontinuous. The only points where the periodic extension might be discontinuous are the "boundary points" $x = \pm 2 + 4k$. In fact, since $f(-2) \neq f(2)$, these will be points of discontinuity. So $f_{per}(x)$ is continuous at $x = -7$, since it is not a boundary point and we have $f_{per}(-7) = f(1) = 0$, which is what the Fourier Series will converge to. The same for $x = -1$, the Fourier Series will converge to $f(-1) = \frac{2+2}{2} = 2$.

For $x = 6$ we are at an endpoint. The left-sided limit is -1, while the right-sided limit is 2, so the Fourier Series will converge to their average $\frac{1}{2}$.

2 10.4 Even and Odd Functions

Before we can apply the discussion from Section 1 to the Fourier Sine and Cosine Series, we need to review some facts about Even and Odd Functions.

Recall that an **even** function is a function satisfying

$$g(-x) = g(x). \tag{10}$$

This means that the graph $y = g(x)$ is symmetric with respect to the y-axis. An **odd** function satisfies

$$g(-x) = -g(x) \tag{11}$$

meaning that its graph $y = g(x)$ is symmetric with respect to the origin.

Example 7. A monomial x^n is even if n is even and odd if n is odd. $\cos(x)$ is even and $\sin(x)$ is odd. Note $\tan(x)$ is odd.

There are some general rules for how products and sums behave:
(1) If $g(x)$ is odd and $h(x)$ is even, their product $g(x)h(x)$ is odd.
(2) If $g(x)$ and $h(x)$ are either both even or both odd, $g(x)h(x)$ is even.
(3) The sum of two even functions or two odd functions is even or odd, respectively.
To remember the rules consider how many negative signs come out of the arguments. EXERCISE:

Verify these rules.
(4) The sum of an even and an odd function can be anything. In fact, any function on $(-l, l)$ can be written as a sum of an even function, called the **even part**, and an odd function, called the **odd part**.
(5) Differentiation and Integration can change the parity of a function. If $f(x)$ is even, $\frac{df}{dx}$ and $\int_0^x f(s)ds$ are both odd, and vice versa.

The graph of an odd function $g(x)$ must pass through the origin by definition. This also tells us that if $g(x)$ is even, as long as $g'(0)$ exists, then $g'(0) = 0$.

Definite Integrals on symmetric intervals of odd and even functions have useful properties

$$\int_{-l}^{l} (odd)dx = 0quad\text{and} \quad \int_{-l}^{l} (even)dx = 2\int_{-l}^{l} (even)dx \tag{12}$$

Given a function $f(x)$ defined on $(0, l)$, there is only one way to extend it to $(-l, l)$ to an even or odd function. The **even extension** of $f(x)$ is

$$f_{even}(x) = \begin{cases} f(x) & \text{for} 0 < x < l \\ f(-x) & \text{for} -l < x < 0. \end{cases} \tag{13}$$

This is just its reflection across the y-axis. Notice that the even extension is not necessarily defined at the origin.

The **odd extension** of $f(x)$ is

$$f_{odd}(x) = \begin{cases} f(x) & \text{for} 0 < x < l \\ -f(-x) & \text{for} -l < x < 0 \\ 0 & \text{for} x = 0 \end{cases} \quad . \tag{14}$$

This is just its reflection through the origin.

2.1 Fourier Sine Series

Each of terms in the Fourier Sine Series for $f(x)$, $\sin(\frac{n\pi x}{l})$, is odd. As with the full Fourier Series, each of these terms also has period $2l$. So we can think of the Fourier Sine Series as the expansion of an odd function with period $2l$ defined on the entire line which coincides with $f(x)$ on $(0, l)$.

One can show that the full Fourier Series of f_{odd} is the same as the Fourier Sine Series of $f(x)$. Let

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{l}) + B_n \sin(\frac{n\pi x}{l}) \tag{15}$$

be the Fourier Series for $f_{odd}(x)$, with coefficients given by (6)

$$A_n = \frac{1}{l}\int_{-l}^{l} f_{odd}(x)\cos(\frac{n\pi x}{l})dx = 0 \tag{16}$$

But f_{odd} is odd and $\cos$ is even, so their product is again odd.

$$B_n = \frac{1}{l}\int_{-l}^{l} f_{odd}(x)\sin(\frac{n\pi x}{l})dx \tag{17}$$

But both f_{odd} and $\sin$ are odd, so their product is even.

$$\begin{aligned} B_n &= \frac{2}{l}\int_0^l f_{odd}(x)\sin(\frac{n\pi x}{l})dx && (18)\\ &= \frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi x}{l})dx, && (19)\end{aligned}$$

which are just the Fourier Sine coefficients of $f(x)$. Thus, as the Fourier Sine Series of $f(x)$ is the full Fourier Series of $f_{odd}(x)$, the $2l$-periodic odd function that the Fourier Sine Series expands is just the periodic extension f_{odd}.

This goes both ways. If we want to compute a Fourier Series for an odd function on $(-l,l)$ we can just compute the Fourier Sine Series of the function restricted to $(0,l)$. It will **almost** converge to the original function on $(-l,l)$ with the only issues occurring at any jump discontinuities. The **only works for odd functions**. Do not use the formula for the coefficients of the Sine Series, unless you are working with an odd function.

Example 8. Write down the odd extension of $f(x) = l-x$ on $(0,l)$ and compute its Fourier Series.

To get the odd extension of $f(x)$ we will need to see how to reflect f across the origin. What we end up with is the function

$$f_{odd}(x) = \begin{cases} l-x & 0<x<l \\ -l-x & -l<x<0 \end{cases}. \tag{20}$$

Now. what is the Fourier Series of $f_{odd}(x)$? By the previous discussion, we know that is will be identical to the Fourier Sine Series of $f(x)$, as this will converge on $(-l,0)$ to f_{odd}. So we have

$$f_{odd}(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l}), \tag{21}$$

where

$$\begin{aligned} A_n &= \frac{2}{l}\int_0^l (l-x)\sin(\frac{n\pi x}{l})dx && (22)\\ &= \frac{2}{l}[-\frac{l(l-x)}{n\pi}\cos(\frac{n\pi x}{l}) - \frac{l^2}{n^2\pi^2}\sin(\frac{n\pi x}{l})]_0^l && (23)\\ &= \frac{2l}{n\pi}. && (24)\end{aligned}$$

Thus the desired Fourier Series is

$$f_{odd}(x) = \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(\frac{n\pi x}{l}). \tag{25}$$

You might wonder how we were able a few days lectures ago to compute the Fourier Sine Series of a constant function like $f(x) = 1$ which is even. It is important to remember that if we are computing the Fourier Sine Series for $f(x)$, it only needs to converge to $f(x)$ on $(0, l)$, where issues of evenness and oddness do not occur. The Fourier Sine Series will converge to the odd extension of $f(x)$ on $(-l, l)$.

Example 9. Find the Fourier Series for the odd extension of

$$f(x) = \begin{cases} \frac{3}{2} & 0 < x < \frac{3}{2} \\ x - \frac{3}{2} & \frac{3}{2} < x < 3. \end{cases} \tag{26}$$

on $(-3, 3)$.

The Fourier Series for $f_{odd}(x)$ on $(-3, 3)$ will just be the Fourier Sine Series for $f(x)$ on $(0, 3)$. The Fourier Sine coefficients for $f(x)$ are

$$\begin{aligned}
A_n &= \frac{2}{3}\int_0^3 f(x)\sin(\frac{n\pi x}{l})dx && (27)\\
&= \frac{2}{3}\left(\int_0^{\frac{3}{2}} \frac{3}{2}\sin(\frac{n\pi x}{3})dx + \int_{\frac{3}{2}}^3 (x - \frac{3}{2})\sin(\frac{n\pi x}{3})\right) && (28)\\
&= \frac{2}{3}\left(-\frac{9}{2n\pi}\cos(\frac{n\pi x}{3})|_0^{\frac{3}{2}} + \frac{3(x-\frac{3}{2})}{n\pi}\cos(\frac{n\pi x}{3})|_{\frac{3}{2}}^3 + \frac{9}{n^2\pi^2}\sin(\frac{n\pi x}{3})|_{\frac{3}{2}}^3\right) && (29)\\
&= \frac{2}{3}\left(-\frac{9}{2n\pi}(\cos(\frac{n\pi}{2}) - 1) - \frac{9}{2n\pi}\cos(n\pi) - \frac{9}{n^2\pi^2}\sin(\frac{n\pi x}{2})\right) && (30)\\
&= \frac{2}{3}(\frac{9}{2n\pi}(1 - \cos(\frac{n\pi}{2}) + (-1)^{n+1}) - \frac{9}{n^2\pi^2}\sin(\frac{n\pi}{2})\Big) && (31)\\
&= \frac{3}{n\pi}\left(1 - \cos(\frac{n\pi}{2}) + (-1)^{n+1} - \frac{2}{n\pi}\sin(\frac{n\pi}{2})\right) && (32)
\end{aligned}$$

and the Fourier Series is

$$f_{odd}(x) = \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}[1 - \cos(\frac{n\pi}{2}) + (-1)^{n+1} - \frac{2}{n\pi}\sin(\frac{n\pi}{2})]\sin(\frac{n\pi x}{3}). \tag{33}$$

EXERCISE: Sketch the Odd Extension of $f(x)$ given in the previous Example and write down the formula for it.

2.2 Fourier Cosine Series

Now consider what happens for the Fourier Cosine Series of $f(x)$ on $(0, l)$. This is analogous to the Sine Series case. Every term in the Cosine Series has the form

$$A_n\cos(\frac{n\pi x}{l}) \tag{34}$$

and hence is even, so the entire Cosine Series is even. So the Cosine Series must converge on $(-l, l)$ to an even function which coincides on $(0, l)$ with $f(x)$. this must be the even extension

$$f_{even}(x) = \begin{cases} f(x) & 0 < x < l \\ f(-x) & -l < x < 0 \end{cases} \quad . \tag{35}$$

Notice that this definition does not specify the value of the function at zero, the only restriction on an even function at zero is that, if it exists, the derivative should be zero.

It is straight forward enough to show that the Fourier coefficients of $f_{even}(x)$ coincide with the Fourier Cosine coefficients of $f(x)$. The Euler-Fourier formulas give

$$\begin{aligned} A_n &= \frac{1}{l}\int_{-l}^{l} f_{even}(x)\cos(\frac{n\pi x}{l})dx & (36)\\ &= \frac{2}{l}\int_{0}^{l} f_{even}(x)\cos(\frac{n\pi x}{l})dx \quad \text{since} f_{even}(x)\cos(\frac{n\pi x}{l}) \text{is even} & (37)\\ &= \frac{2}{l}\int_{0}^{l} f_{even}(x)\cos(\frac{n\pi x}{l})dx & (38) \end{aligned}$$

which are the Fourier Cosine coefficients of $f(x)$ on $(0, l)$

$$B_n = \frac{1}{l}\int_{-l}^{l} f_{even}(x)\sin(\frac{n\pi x}{l})dx = 0 \tag{39}$$

since $f_{even}(x)\sin(\frac{n\pi x}{l})$ is odd. Thus the Fourier Cosine Series of $f(x)$ on $(0, l)$ can be considered as the Fourier expansion of $f_{even}(x)$ on $(-l, l)$, and therefore also as expansion of the periodic extension of $f_{even}(x)$. It will converge as in the Fourier Convergence Theorem to this periodic extension.

This also means that if we want to compute the Fourier Series of an even function, we can just compute the Fourier Cosine Series of its restriction to $(0, l)$. It is very important that this only be attempted if the function we are starting with is even.

Example 10. Write down the even extension of $f(x) = l - x$ on $(0, l)$ and compute its Fourier Series.

The even extension will be

$$f_{even}(x) = \begin{cases} l - x & 0 < x < l \\ l + x & -l < x < 0 \end{cases} \quad . \tag{40}$$

Its Fourier Series is the same as the Fourier Cosine Series of $f(x)$, by the previous discussion. So we can just compute the coefficients. Thus we have

$$f_{even}(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{l}), \tag{41}$$

where

$$\begin{aligned}
A_0 &= \frac{2}{l}\int_0^l f(x)dx = \frac{2}{l}\int_0^l (l-x)dx = l && (42)\\
A_n &= \frac{2}{l}\int_0^l f(x)\cos(\frac{n\pi x}{l})dx && (43)\\
&= \frac{2}{l}\int_0^l (l-x)\cos(\frac{n\pi x}{l})dx && (44)\\
&= \frac{2}{l}[\frac{l(l-x)}{n\pi}\sin(\frac{n\pi x}{l}) - \frac{l^2}{n^2\pi^2}\cos(\frac{n\pi x}{l})]_0^l && (45)\\
&= \frac{2}{l}\left(\frac{l^2}{n^2\pi^2}(-\cos(n\pi)+\cos(0))\right) && (46)\\
&= \frac{2l}{n^2\pi^2}((-1)^{n+1}+1). && (47)
\end{aligned}$$

So we have

$$f_{even}(x) = \frac{l}{2} + \sum_{n=1}^{\infty}\frac{2l}{n^2\pi^2}((-1)^{n+1}+1). \tag{48}$$

Example 11. Write down the even extension of

$$f(x) = \begin{cases} \frac{3}{2} & 0 \le x < \frac{3}{2} \\ x - \frac{3}{2} & \frac{3}{2} \le x \le 3 \end{cases} \tag{49}$$

and compute its Fourier Series.

Using Equation (35) we see that the even extension is

$$f_{even}(x) = \begin{cases} x - \frac{3}{2} & \frac{3}{2} < x < 3 \\ \frac{3}{2} & 0 \le x < \frac{3}{2} \\ \frac{3}{2} & -\frac{3}{2} < x < 0 \\ -x - \frac{3}{2} & -3 \le x \le -\frac{3}{2} \end{cases} . \tag{50}$$

We just need to compute the Fourier Cosine coefficients of the original $f(x)$ on $(0,3)$.

$$\begin{aligned}
A_0 &= \frac{2}{3}\int_0^3 f(x)dx && (51)\\
&= \frac{2}{3}\left(\int_0^{3/2}\frac{3}{2}dx + \int_{3/2}^3 x - \frac{3}{2}dx\right) && (52)\\
&= \frac{2}{3}(\frac{9}{4}+\frac{9}{8}) = \frac{9}{4} && (53)\\
A_n &= \frac{2}{3}\int_0^3 f(x)\cos(\frac{n\pi x}{3})dx && (54)\\
&= \frac{2}{3}\left(\int_0^{3/2}\frac{3}{2}\cos(\frac{n\pi x}{3})dx + \int_{3/2}^3 (x-\frac{3}{2})\cos(\frac{n\pi x}{3})dx\right) && (55)\\
&= \frac{2}{3}\left(\frac{9}{2n\pi}\sin(\frac{n\pi x}{3})|_0^{3/2} + \frac{3(x-\frac{3}{2})}{n\pi}\sin(\frac{n\pi x}{3})|_{3/2}^3 + \frac{9}{n^2\pi^2}\cos(\frac{n\pi x}{3})|_{3/2}^3\right) && (56)\\
&= \frac{2}{3}\left(\frac{9}{2n\pi}\sin(\frac{n\pi}{2}) + \frac{9}{n^2\pi^2}\left(cos(n\pi) - \cos(\frac{n\pi}{2})\right)\right) && (57)\\
&= \frac{6}{n\pi}\left(\frac{1}{2}\sin(\frac{n\pi}{2}) + \frac{1}{n\pi}\left((-1)^n - \cos(\frac{n\pi}{2})\right)\right) && (58)\\
&= \frac{6}{n\pi}\left(\frac{1}{n\pi}((-1)^n - \cos(\frac{n\pi}{2})) + \frac{1}{2}\sin(\frac{n\pi}{2})\right). && (59)
\end{aligned}$$

So the Fourier Series is

$$f_{even} = \frac{9}{8} + \frac{6}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{n\pi}((-1)^n - \cos(\frac{n\pi}{2})) + \frac{1}{2}\sin(\frac{n\pi}{2})\right)\cos(\frac{n\pi x}{3}). \qquad (60)$$

HW 10.3 # 1,3,6

HW 10.4 # 1,2,8,11,12,16,17