Lecture Notes for Math 251: ODE and PDE. Lecture 33: 10.4 Even and Odd Functions

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Last Time: We studied what a given Fourier Series converges to if at all.

1 Even and Odd Functions

Before we can apply the discussion from Section 10.3 to the Fourier Sine and Cosine Series, we need to review some facts about Even and Odd Functions.

Recall that an **even** function is a function satisfying

$$g(-x) = g(x). \tag{1}$$

This means that the graph $y = g(x)$ is symmetric with respect to the y-axis. An **odd** function satisfies

$$g(-x) = -g(x) \tag{2}$$

meaning that its graph $y = g(x)$ is symmetric with respect to the origin.

Example 1. A monomial x^n is even if n is even and odd if n is odd. $\cos(x)$ is even and $\sin(x)$ is odd. Note $\tan(x)$ is odd.

There are some general rules for how products and sums behave:
(1) If $g(x)$ is odd and $h(x)$ is even, their product $g(x)h(x)$ is odd.
(2) If $g(x)$ and $h(x)$ are either both even or both odd, $g(x)h(x)$ is even.
(3) The sum of two even functions or two odd functions is even or odd, respectively.
To remember the rules consider how many negative signs come out of the arguments.
EXERCISE: Verify these rules.
(4) The sum of an even and an odd function can be anything. In fact, any function on $(-l, l)$ can be written as a sum of an even function, called the **even part**, and an odd function, called the **odd part**.
(5) Differentiation and Integration can change the parity of a function. If $f(x)$ is even, $\frac{df}{dx}$ and $\int_0^x f(s)ds$ are both odd, and vice versa.

The graph of an odd function $g(x)$ must pass through the origin by definition. This also tells us that if $g(x)$ is even, as long as $g'(0)$ exists, then $g'(0) = 0$.

Definite Integrals on symmetric intervals of odd and even functions have useful properties

$$\int_{-l}^{l} (odd)dx = 0 \quad \text{and} \quad \int_{-l}^{l} (even)dx = 2\int_{0}^{l} (even)dx \tag{3}$$

Given a function $f(x)$ defined on $(0, l)$, there is only one way to extend it to $(-l, l)$ to an even or odd function. The **even extension** of $f(x)$ is

$$f_{even}(x) = \begin{cases} f(x) & \text{for } 0 < x < l \\ f(-x) & \text{for } -l < x < 0. \end{cases} \tag{4}$$

This is just its reflection across the y-axis. Notice that the even extension is not necessarily defined at the origin.

The **odd extension** of $f(x)$ is

$$f_{odd}(x) = \begin{cases} f(x) & \text{for } 0 < x < l \\ -f(-x) & \text{for } -l < x < 0 \\ 0 & \text{for } x = 0 \end{cases} \quad . \tag{5}$$

This is just its reflection through the origin.

1.1 Fourier Sine Series

Each of terms in the Fourier Sine Series for $f(x)$, $\sin(\frac{n\pi x}{l})$, is odd. As with the full Fourier Series, each of these terms also has period $2l$. So we can think of the Fourier Sine Series as the expansion of an odd function with period $2l$ defined on the entire line which coincides with $f(x)$ on $(0, l)$.

One can show that the full Fourier Series of f_{odd} is the same as the Fourier Sine Series of $f(x)$. Let

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{l}) + B_n \sin(\frac{n\pi x}{l}) \tag{6}$$

be the Fourier Series for $f_{odd}(x)$, with coefficients given in Section 10.3

$$A_n = \frac{1}{l}\int_{-l}^{l} f_{odd}(x)\cos(\frac{n\pi x}{l})dx = 0 \tag{7}$$

But f_{odd} is odd and $\cos$ is even, so their product is again odd.

$$B_n = \frac{1}{l}\int_{-l}^{l} f_{odd}(x)\sin(\frac{n\pi x}{l})dx \tag{8}$$

But both f_{odd} and $\sin$ are odd, so their product is even.

$$\begin{aligned} B_n &= \frac{2}{l}\int_{0}^{l} f_{odd}(x)\sin(\frac{n\pi x}{l})dx & (9) \\ &= \frac{2}{l}\int_{0}^{l} f(x)\sin(\frac{n\pi x}{l})dx, & (10) \end{aligned}$$

which are just the Fourier Sine coefficients of $f(x)$. Thus, as the Fourier Sine Series of $f(x)$ is the full Fourier Series of $f_{odd}(x)$, the $2l$-periodic odd function that the Fourier Sine Series expands is just the periodic extension f_{odd}.

This goes both ways. If we want to compute a Fourier Series for an odd function on $(-l, l)$ we can just compute the Fourier Sine Series of the function restricted to $(0, l)$. It will **almost** converge to the original function on $(-l, l)$ with the only issues occurring at any jump discontinuities. The **only works for odd functions**. Do not use the formula for the coefficients of the Sine Series, unless you are working with an odd function.

Example 2. Write down the odd extension of $f(x) = l - x$ on $(0, l)$ and compute its Fourier Series.

To get the odd extension of $f(x)$ we will need to see how to reflect f across the origin. What we end up with is the function

$$f_{odd}(x) = \begin{cases} l - x & 0 < x < l \\ -l - x & -l < x < 0 \end{cases} . \tag{11}$$

Now. what is the Fourier Series of $f_{odd}(x)$? By the previous discussion, we know that is will be identical to the Fourier Sine Series of $f(x)$, as this will converge on $(-l, 0)$ to f_{odd}. So we have

$$f_{odd}(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l}), \tag{12}$$

where

$$\begin{aligned} A_n &= \frac{2}{l} \int_0^l (l - x) \sin(\frac{n\pi x}{l}) dx & (13) \\ &= \frac{2}{l} [-\frac{l(l-x)}{n\pi} \cos(\frac{n\pi x}{l}) - \frac{l^2}{n^2\pi^2} \sin(\frac{n\pi x}{l})]_0^l & (14) \\ &= \frac{2l}{n\pi}. & (15) \end{aligned}$$

Thus the desired Fourier Series is

$$f_{odd}(x) = \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(\frac{n\pi x}{l}). \tag{16}$$

You might wonder how we were able a few lectures ago to compute the Fourier Sine Series of a constant function like $f(x) = 1$ which is even. It is important to remember that if we are computing the Fourier Sine Series for $f(x)$, it only needs to converge to $f(x)$ on $(0, l)$, where issues of evenness and oddness do not occur. The Fourier Sine Series will converge to the odd extension of $f(x)$ on $(-l, l)$.

Example 3. Find the Fourier Series for the odd extension of

$$f(x) = \begin{cases} \frac{3}{2} & 0 < x < \frac{3}{2} \\ x - \frac{3}{2} & \frac{3}{2} < x < 3. \end{cases} \tag{17}$$

on $(-3,3)$.

The Fourier Series for $f_{odd}(x)$ on $(-3,3)$ will just be the Fourier Sine Series for $f(x)$ on $(0,3)$. The Fourier Sine coefficients for $f(x)$ are

$$\begin{aligned}
A_n &= \frac{2}{3}\int_0^3 f(x)\sin(\frac{n\pi x}{l})dx && (18)\\
&= \frac{2}{3}\left(\int_0^{\frac{3}{2}}\frac{3}{2}\sin(\frac{n\pi x}{3})dx + \int_{\frac{3}{2}}^3 (x-\frac{3}{2})\sin(\frac{n\pi x}{3})\right) && (19)\\
&= \frac{2}{3}\left(-\frac{9}{2n\pi}\cos(\frac{n\pi x}{3})|_0^{\frac{3}{2}} + \frac{3(x-\frac{3}{2})}{n\pi}\cos(\frac{n\pi x}{3})|_{\frac{3}{2}}^3 + \frac{9}{n^2\pi^2}\sin(\frac{n\pi x}{3})|_{\frac{3}{2}}^3\right) && (20)\\
&= \frac{2}{3}\left(-\frac{9}{2n\pi}(\cos(\frac{n\pi}{2})-1) - \frac{9}{2n\pi}\cos(n\pi) - \frac{9}{n^2\pi^2}\sin(\frac{n\pi x}{2})\right) && (21)\\
&= \frac{2}{3}(\frac{9}{2n\pi}(1-\cos(\frac{n\pi}{2}) + (-1)^{n+1}) - \frac{9}{n^2\pi^2}\sin(\frac{n\pi}{2})\Big) && (22)\\
&= \frac{3}{n\pi}\left(1-\cos(\frac{n\pi}{2}) + (-1)^{n+1} - \frac{2}{n\pi}\sin(\frac{n\pi}{2})\right) && (23)
\end{aligned}$$

and the Fourier Series is

$$f_{odd}(x) = \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}[1-\cos(\frac{n\pi}{2}) + (-1)^{n+1} - \frac{2}{n\pi}\sin(\frac{n\pi}{2})]\sin(\frac{n\pi x}{3}). \qquad (24)$$

EXERCISE: Sketch the Odd Extension of $f(x)$ given in the previous Example and write down the formula for it.

1.2 Fourier Cosine Series

Now consider what happens for the Fourier Cosine Series of $f(x)$ on $(0,l)$. This is analogous to the Sine Series case. Every term in the Cosine Series has the form

$$A_n\cos(\frac{n\pi x}{l}) \qquad (25)$$

and hence is even, so the entire Cosine Series is even. So the Cosine Series must converge on $(-l,l)$ to an even function which coincides on $(0,l)$ with $f(x)$. this must be the even extension

$$f_{even}(x) = \begin{cases} f(x) & 0<x<l \\ f(-x) & -l<x<0 \end{cases} . \qquad (26)$$

Notice that this definition does not specify the value of the function at zero, the only restriction on an even function at zero is that, if it exists, the derivative should be zero.

It is straight forward enough to show that the Fourier coefficients of $f_{even}(x)$ coincide with the Fourier Cosine coefficients of $f(x)$. The Euler-Fourier formulas give

$$\begin{aligned} A_n &= \frac{1}{l}\int_{-l}^{l} f_{even}(x)\cos(\frac{n\pi x}{l})dx & (27)\\ &= \frac{2}{l}\int_{0}^{l} f_{even}(x)\cos(\frac{n\pi x}{l})dx \quad \text{since } f_{even}(x)\cos(\frac{n\pi x}{l}) \text{ is even} & (28)\\ &= \frac{2}{l}\int_{0}^{l} f_{even}(x)\cos(\frac{n\pi x}{l})dx & (29)\end{aligned}$$

which are the Fourier Cosine coefficients of $f(x)$ on $(0,l)$

$$B_n = \frac{1}{l}\int_{-l}^{l} f_{even}(x)\sin(\frac{n\pi x}{l})dx = 0 \tag{30}$$

since $f_{even}(x)\sin(\frac{n\pi x}{l})$ is odd. Thus the Fourier Cosine Series of $f(x)$ on $(0,l)$ can be considered as the Fourier expansion of $f_{even}(x)$ on $(-l,l)$, and therefore also as expansion of the periodic extension of $f_{even}(x)$. It will converge as in the Fourier Convergence Theorem to this periodic extension.

This also means that if we want to compute the Fourier Series of an even function, we can just compute the Fourier Cosine Series of its restriction to $(0,l)$. It is very important that this only be attempted if the function we are starting with is even.

Example 4. Write down the even extension of $f(x) = l - x$ on $(0,l)$ and compute its Fourier Series.

The even extension will be

$$f_{even}(x) = \begin{cases} l - x & 0 < x < l \\ l + x & -l < x < 0 \end{cases} . \tag{31}$$

Its Fourier Series is the same as the Fourier Cosine Series of $f(x)$, by the previous discussion. So we can just compute the coefficients. Thus we have

$$f_{even}(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{l}), \tag{32}$$

where

$$\begin{aligned}
A_0 &= \frac{2}{l}\int_0^l f(x)dx = \frac{2}{l}\int_0^l (l-x)dx = l && (33)\\
A_n &= \frac{2}{l}\int_0^l f(x)\cos(\frac{n\pi x}{l})dx && (34)\\
&= \frac{2}{l}\int_0^l (l-x)\cos(\frac{n\pi x}{l})dx && (35)\\
&= \frac{2}{l}[\frac{l(l-x)}{n\pi}\sin(\frac{n\pi x}{l}) - \frac{l^2}{n^2\pi^2}\cos(\frac{n\pi x}{l})]_0^l && (36)\\
&= \frac{2}{l}\left(\frac{l^2}{n^2\pi^2}(-\cos(n\pi)+\cos(0))\right) && (37)\\
&= \frac{2l}{n^2\pi^2}((-1)^{n+1}+1). && (38)
\end{aligned}$$

So we have

$$f_{even}(x) = \frac{l}{2} + \sum_{n=1}^{\infty}\frac{2l}{n^2\pi^2}((-1)^{n+1}+1). \tag{39}$$

Example 5. Write down the even extension of

$$f(x) = \begin{cases} \frac{3}{2} & 0 \le x < \frac{3}{2} \\ x - \frac{3}{2} & \frac{3}{2} \le x \le 3 \end{cases} \tag{40}$$

and compute its Fourier Series.

Using Equation (**??**) we see that the even extension is

$$f_{even}(x) = \begin{cases} x - \frac{3}{2} & \frac{3}{2} < x < 3 \\ \frac{3}{2} & 0 \le x < \frac{3}{2} \\ \frac{3}{2} & -\frac{3}{2} < x < 0 \\ -x - \frac{3}{2} & -3 \le x \le -\frac{3}{2} \end{cases} . \tag{41}$$

We just need to compute the Fourier Cosine coefficients of the original $f(x)$ on $(0,3)$.

$$\begin{aligned}
A_0 &= \frac{2}{3}\int_0^3 f(x)dx && (42)\\
&= \frac{2}{3}\left(\int_0^{3/2}\frac{3}{2}dx + \int_{3/2}^3 x - \frac{3}{2}dx\right) && (43)\\
&= \frac{2}{3}(\frac{9}{4}+\frac{9}{8}) = \frac{9}{4} && (44)\\
A_n &= \frac{2}{3}\int_0^3 f(x)\cos(\frac{n\pi x}{3})dx && (45)\\
&= \frac{2}{3}\left(\int_0^{3/2}\frac{3}{2}\cos(\frac{n\pi x}{3})dx + \int_{3/2}^3 (x-\frac{3}{2})\cos(\frac{n\pi x}{3})dx\right) && (46)\\
&= \frac{2}{3}\left(\frac{9}{2n\pi}\sin(\frac{n\pi x}{3})|_0^{3/2} + \frac{3(x-\frac{3}{2})}{n\pi}\sin(\frac{n\pi x}{3})|_{3/2}^3 + \frac{9}{n^2\pi^2}\cos(\frac{n\pi x}{3})|_{3/2}^3\right) && (47)\\
&= \frac{2}{3}\left(\frac{9}{2n\pi}\sin(\frac{n\pi}{2}) + \frac{9}{n^2\pi^2}\left(cos(n\pi) - \cos(\frac{n\pi}{2})\right)\right) && (48)\\
&= \frac{6}{n\pi}\left(\frac{1}{2}\sin(\frac{n\pi}{2}) + \frac{1}{n\pi}\left((-1)^n - \cos(\frac{n\pi}{2})\right)\right) && (49)\\
&= \frac{6}{n\pi}\left(\frac{1}{n\pi}((-1)^n - \cos(\frac{n\pi}{2})) + \frac{1}{2}\sin(\frac{n\pi}{2})\right). && (50)
\end{aligned}$$

So the Fourier Series is

$$f_{even} = \frac{9}{8} + \frac{6}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{n\pi}\big((-1)^n - \cos(\frac{n\pi}{2})\big) + \frac{1}{2}\sin(\frac{n\pi}{2})\right)\cos(\frac{n\pi x}{3}). \qquad (51)$$

HW 10.4 # 1,2,8,11,12,16,17