Lecture Notes for Math 251: ODE and PDE. Lecture 31: 10.5 Separation of Variables; Solutions of Heat Conduction Problems

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Last Time: We studied basic solutions to Two-Point Boundary Value Problems and studied the eigenvalues and eigenvectors of such problems.

1 The Heat Equation

We will soon see that partial differential equations can be far more complicated than ordinary differential equations. For PDEs, there is no general theory, the methods need to be adapted for smaller groups of equations. This course will only do an introduction, you can find out much more in advanced courses. We will be focusing on a single solution method called **Separation of Variables**, which is pervasive in engineering and mathematics.

The first partial differential equation to consider is the famous **heat equation** which models the temperature distribution in some object. We will focus on the one-dimensional heat equation, where we want to find the temperature distributions in a one-dimensional bar of length l. In particular we will assume that our bar corresponds to the interval $(0, l)$ on the real line.

The assumption is made purely for simplicity. If we assume we have a real bar, the one-dimensional assumption is equivalent to assuming at every lateral cross-section and every instant of time, the temperature is constant. While this is unrealistic it is not a terrible assumption. Also, if the length is much larger than the width in advanced mathematics one can assume the width is 0 since it is such a small fraction of the length. We are also assuming the bar is perfectly insulated, so the only way heat can enter or leave the bar is through the ends $x = 0$ and $x = l$. So any heat transfer will be one-dimensional.

1.1 Derivation of the Heat Equation

Many PDEs come from basic physical laws. Let $u(x, t)$ denote the temperature at a point x at time t. c will be the specific heat of the material the bar is made from (which is the amount of heat needed to raise one unit of mass of this material by one temperature unit) and ρ is the density of
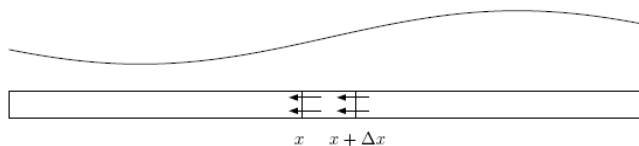
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Figure 1: Heat Flux across the boundary of a small slab with length Δx. The graph is the graph of temperature at a given time t. In accordance with Fourier's Law, the heat leaves or enters the boundary by flowing from hot to cold; hence at x the flux is opposing the sign of u_x, while at $x + \Delta x$ it is agreeing.

the rod. Note that in general, the specific heat and density of the rods do not have to be constants, they may vary with x. We greatly simplify the problem by allowing them to be constant.

Let's consider a small slab of length Δx. We will let $H(t)$ be the amount of heat contained in this slab. The mass of the slab is $\rho \Delta x$ and the heat energy contained in this small region is given by

$$H(t) = cu\rho\Delta x \tag{1}$$

On the other hand, within the slab, heat will flow from hot to cold (this is **Fourier's Law**). The only way heat can leave is by leaving through the boundaries, which are at x and $x + \Delta x$ (This is the **Law of Conservation of Energy**). So the change of heat energy of the slab is equal to the heat flux across the boundary. If κ is the conductivity of the bar's material

$$\frac{dH}{dt} = \kappa u_x(x + \Delta x, t) - \kappa u_x(x, t) \tag{2}$$

This is illustrated in Figure 1.1. Setting the derivative of $H(t)$ from above equal to the previous equations we find

$$(cu(x,t)\rho\Delta x)_t = \kappa u_x(x + \Delta x, t) - \kappa u_x(x, t) \tag{3}$$

or

$$c\rho u_t(x,t) = \frac{\kappa u_x(x + \Delta x, t) - \kappa u_x(x, t)}{\Delta x}. \tag{4}$$

If we take the limit as $\Delta x \to 0$, the right hand side is just the x-derivative of $\kappa u_x(x, t)$ or

$$c\rho u_t(x,t) = \kappa u_{xx}(x, t). \tag{5}$$

Setting $k = \frac{\kappa}{c\rho} > 0$, we have the heat equation

$$u_t = k u_{xx}. \tag{6}$$

Notice that the heat equation is a linear PDE, since all of the derivatives of u are only multiplied by constants. What is the constant k? It is called the **Thermal Diffusivity** of the bar and is a measure of how quickly heat spreads through a given material.

How do we interpret the heat equation? Graph the temperature of the bar at a fixed time. Suppose it looks like Figure 2. On the left side the bar is concave up. If the graph is concave up,
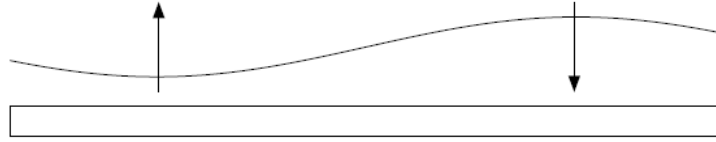
Figure 2: Temperature versus position on a bar. The arrows show time dependence in accordance with the heat equation. The temperature graph is concave up, so the left side of the bar is warming up. While on the right the temperature is concave down and so th right side is cooling down..

that means that the second derivative of the temperature (with respect to position x) is positive. The heat equation tells us that the time derivative of the temperature at any of the points on the left side of the bar will be increasing. The left side of the bar will be warming up. Similarly, on the right side of the bar, the graph is concave down. Thus the second x-derivative of the temperature is negative, and so will be the first t-derivative, and we can conclude that the right side of the bar is cooling down.

2 Separation of Variables and Heat Equation IVPs

2.1 Initial Value Problems

Partial Differential Equations generally have a lot of solutions. To specify a unique one, we will need additional conditions. These conditions are motivated by physics and are initial or boundary conditions. An IVP for a PDE consists for the heat equation, initial conditions, and boundary conditions.

An initial condition specifies the physical state at a given time t_0. For example, and initial condition for the heat equation would be the starting temperature distribution

$$u(x, 0) = f(x) \tag{7}$$

This is the only condition required because the heat equation is first order with respect to time. The wave equation, considered in a future section is second order in time and needs two initial conditions.

PDEs are only valid on a given domain. Boundary conditions specify how the solution behaves on the boundaries of the given domain. These need to be specified, because the solution does not exist on one side of the boundary, we might have problems with differentiability there.

Our heat equation was derived for a one-dimensional bar of length l, so the relevant domain in question can be taken to be the interval $0 < x < l$ and the boundary consists of the two points $x = 0$ and $x = l$. We could have derived a two-dimensional heat equation, for example, in which case the domain would be some region in the xy-plane with the boundary being some closed curve.

It will be clear from the physical description of the problem what the appropriate boundary conditions are. We might know at the endpoints $x = 0$ and $x = l$, the temperature $u(0, t)$ and $u(l, t)$ are fixed. Boundary conditions that give the value of the solution are called **Dirichlet**

Boundary Conditions. Or we might insulate the ends of the bar, meaning there should be no heat flow out of the boundary. This would yield the boundary conditions $u_x(0, t) = u_x(l, t) = 0$. If the boundary conditions specify the derivative at the boundary, they are called **Neumann Conditions**. If the boundary conditions specify that we have one insulated end and at the other we control the temperature. This is an example of a **Mixed Boundary Condition**.

As we have seen, changing boundary conditions can significantly change the solution. Initially, we will work with homogeneous Dirichlet conditions $u(0, t) = u(l, t) = 0$, giving us the following initial value problem

$$(DE): \quad u_t = k u_x x \tag{8}$$
$$(BC): \quad u(0, t) = u(l, t) = 0 \tag{9}$$
$$(IC): u(x, 0) = f(x) \tag{10}$$

After we have seen the general method, we will see what happens with homogeneous Neumann conditions. We will discuss nonhomogeneous equations later.

2.2 Separation of Variables

Above we have derived the heat equation for the bar of length L. Suppose we have an initial value problem such as Equation (8)-(10). How should we proceed? We want to try to build a general solution out of smaller solutions which are easier to find.

We start by assuming we have a **separated solution**, where

$$u(x, t) = X(x)T(t). \tag{11}$$

Our solution is the product of a function that depends only on x and a function that depends only on t. We can then try to write down an equation depending only on x and another solution depending only on t before using our knowledge of ODEs to try and solve them.

It should be noted that this is a very special situation and will not occur in general. Even when we can use it sometimes it is hard to move beyond the first step. However, it works for all equations we will be considering in this class, and is a good starting point.

How does this method work? Plug the separated solution into the heat equation.

$$\frac{\partial}{\partial t}[X(x)T(t)] = k\frac{\partial^2}{\partial x^2}[X(x)T(t)] \tag{12}$$
$$X(x)T'(t) = kX''(x)T(t) \tag{13}$$

Now notice that we can move everything depending on x to one side and everything depending on t to the other.

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} \tag{14}$$

This equation should says that both sides are equal for any x or t we choose. Thus they both must be equal to a constant. Since if what they equal depended on x or t both sides would not be equal for all x and t. So

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda \tag{15}$$

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We have written the minus sign for convenience. It will turn out that $\lambda > 0$.

The equation above contains a pair of separate ordinary differential equations

$$X'' + \lambda X = 0 \tag{16}$$
$$T' + \lambda k T = 0. \tag{17}$$

Notice that our boundary conditions becomes $X(0) = 0$ and $X(l) = 0$. Now the second equation can easily be solved, since we have $T' = -\lambda k T$, so that

$$T(t) = A e^{-\lambda k t}. \tag{18}$$

The first equation gives a boundary value problem

$$X'' + \lambda X = 0 \quad X(0) = 0 \quad X(l) = 0 \tag{19}$$

This should look familiar. The is the basic eigenfunction problem studied in section 10.1. As in that example, it turns out our eigenvalues have to be positive. Let $\lambda = \mu^2$ for $\mu > 0$, our general solution is

$$X(x) = B\cos(\mu x) + C\sin(\mu x). \tag{20}$$

The first boundary condition says $B = 0$. The second condition says that $X(l) = C\sin(\mu l) = 0$. To avoid only having the trivial solution, we must have $\mu l = n\pi$. In other words,

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad \text{and} \quad X_n(x) = \sin\left(\frac{n\pi x}{l}\right) \tag{21}$$

for $n = 1, 2, 3, ...$

So we end up having found infinitely many solutions to our boundary value problem, one for each positive integer n. They are

$$u_n(x,t) = A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\left(\frac{n\pi x}{l}\right). \tag{22}$$

The heat equation is linear and homogeneous. As such, the Principle of Superposition still holds. So a linear combination of solutions is again a solution. So any function of the form

$$u(x,t) = \sum_{n=0}^{N} A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\left(\frac{n\pi x}{l}\right) \tag{23}$$

is also a solution to our problem.

Notice we have not used our initial condition yet. We have

$$f(x) = u(x,0) = \sum_{n=0}^{N} A_n \sin\left(\frac{n\pi x}{l}\right). \tag{24}$$

So if our initial condition has this form, the result of superposition Equation (23) is in a good form to use the IC. The coefficients A_n just being the associated coefficients from $f(x)$.

Example 1. Find the solutions to the following heat equation problem on a rod of length 2.

$$u_t = u_{xx} \tag{25}$$
$$u(0, t) = u(2, t) = 0 \tag{26}$$
$$u(x, 0) = \sin(\frac{3\pi x}{2}) - 5\sin(3\pi x) \tag{27}$$

In this problem, we have $k = 1$. Now we know that our solution will have the form like Equation (23), since our initial condition is just the difference of two sine functions. We just need to figure out which terms are represented and what the coefficients A_n are.

Our initial condition is

$$f(x) = \sin(\frac{3\pi x}{2}) - 5\sin(3\pi x) \tag{28}$$

Looking at (23) with $l = 2$, we can see that the first term corresponds to $n = 3$ and the second $n = 6$, and there are no other terms. Thus we have $A_3 = 1$ and $A_6 = -5$, and all other $A_n = 0$. Our solution is then

$$u(x, t) = e^{-(\frac{9\pi^2}{4})t}\sin(\frac{3\pi x}{2}) - 5e^{(-9\pi^2)t}\sin(3\pi x). \tag{29}$$

There is no reason to suppose that our initial distribution is a finite sum of sine functions. Physically, such situations are special. What do we do if we have a more general initial temperature distribution?

Let's consider what happens if we take an **infinite** sum of our separated solutions. Then our solution is

$$u(x, t) = \sum_{n=0}^{\infty} A_n e^{-(\frac{n\pi}{l})^2 kt}\sin(\frac{n\pi x}{l}). \tag{30}$$

Now the initial condition gives

$$f(x) = \sum_{n=0}^{\infty} A_n \sin(\frac{n\pi x}{l}). \tag{31}$$

This idea is due to the French Mathematician Joseph Fourier and is called the **Fourier Sine Series** for $f(x)$.

There are several important questions that arise. Why should we believe that our initial condition $f(x)$ ought to be able to be written as an infinite sum of sines? why should we believe that such a sum would converge to anything?

2.3 Neumann Boundary Conditions

Now let's consider a heat equation problem with homogeneous Neumann conditions

$$(DE): \quad u_t = u_{xx} \tag{32}$$
$$(BC): \quad u_x(0, t) = u_x(l, t) = 0 \tag{33}$$
$$(IC): \quad u(x, 0) = f(x) \tag{34}$$

We will start by again supposing that our solution to Equation (32) is separable, so we have $u(x, t) = X(x)T(t)$ and we obtain a pair of ODEs, which are the same as before

$$
\begin{align}
X'' + \lambda X &= 0 \tag{35}\\
T' + \lambda k T &= 0. \tag{36}
\end{align}
$$

The solution to the first equation is still

$$
T(t) = Ae^{-\lambda k t} \tag{37}
$$

Now we need to determine the boundary conditions for the second equation. Our boundary conditions are $u_x(0, t)$ and $u_x(l, t)$. Thus they are conditions for $X'(0)$ and $X'(l)$, since the t-derivative is not controlled at all. So we have the boundary value problem

$$
X'' + \lambda X = 0 \quad X'(0) = 0 \quad X'(l) = 0. \tag{38}
$$

Along the lines of the analogous computation last lecture, this has eigenvalues and eigenfunctions

$$
\begin{align}
\lambda_n &= \left(\frac{n\pi}{l}\right)^2 \tag{39}\\
y_n(x) &= \cos\left(\frac{n\pi x}{l}\right) \tag{40}
\end{align}
$$

for $n = 0, 1, 2, ...$ So the individual solutions to Equation (32) have the form

$$
u(x, t) = A_n e^{\left(\frac{n\pi}{l}\right)^2 kt} \cos\left(\frac{n\pi x}{l}\right). \tag{41}
$$

Taking finite linear combinations of these work similarly to the Dirichlet case (and is the solution to Equation (32) when $f(x)$ is a finite linear combination of constants and cosines, but in general we are interested in knowing when we can take infinite sums, i.e.

$$
u(x, t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \cos\left(\frac{n\pi x}{l}\right) \tag{42}
$$

Notice how we wrote the $n = 0$ case, as $\frac{1}{2}A_0$. The reason will be clear when talking about Fourier Series. The initial conditions means we need

$$
f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right). \tag{43}
$$

An expression of the form above is called the **Fourier Cosine Series** of $f(x)$.

2.4 Other Boundary Conditions

It is also possible for certain boundary conditions to require the "full" Fourier Series of the initial data, this is an expression of the form

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{n\pi x}{l}\right) + B_n \sin\left(\frac{n\pi x}{l}\right)\right). \tag{44}$$

but in most cases we will work with Dirichlet or Neumann conditions. However, in the process of learning about Fourier sine and cosine series, we will also learn how to compute the full Fourier series of a function.

HW 10.5 # 2,4,7,8