Lecture Notes for Math 251: ODE and PDE. Lecture 33: 10.6 Other Heat Conduction Problems

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Spring 2012

Last Time: We studied Even and Odd Functions as well as the convergence of Fourier Series.

1 Heat Equation Problems

In the previous lecture on the Heat Equation we saw that the product solutions to the heat equation with homogeneous Dirichlet boundary conditions problem

$$\begin{aligned} u_t &= ku_{xx} & (1)\\ u(0,t) &= u(l,t) = 0 & (2)\\ u(x,0) &= f(x) & (3) \end{aligned}$$

had the form

$$u_n(x,t) = B_n e^{-(\frac{n\pi}{l})kt} \sin(\frac{n\pi x}{l}) \quad n = 1, 2, 3, ... \tag{4}$$

Taking linear combinations of these (over each n) gives a general solution to the above problem.

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-(\frac{n\pi}{l})kt} \sin(\frac{n\pi x}{l}) \tag{5}$$

Setting $t = 0$, this implies that we must have

$$f(x) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{l}) \tag{6}$$

In other words, the coefficients in the general solution for the given initial condition are the **Fourier Sine** coefficients of $f(x)$ on $(0, l)$, which are given by

$$B_n = \frac{2}{l} \int_0^l f(x) \sin(\frac{n\pi x}{l}) dx. \tag{7}$$

We also, saw that if we instead have a problem with homogeneous Neumann boundary conditions

$$\begin{aligned} u_t &= k u_{xx} \quad 0 < x < l, \quad t > 0 && (8) \\ u_x(0,t) &= u_x(l,t) = 0 && (9) \\ u(0,t) &= f(x) && (10) \end{aligned}$$

the product solutions had the form

$$u_n(x,t) = A_n e^{-(\frac{n\pi}{l})^2 kt} \cos\Big(\frac{n\pi x}{l}\Big) \quad n = 1, 2, 3, ... \tag{11}$$

and the general solution has the form

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{l})^2 kt} \cos\Big(\frac{n\pi x}{l}\Big). \tag{12}$$

With $t = 0$ this means that the initial condition must satisfy

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\Big(\frac{n\pi x}{l}\Big). \tag{13}$$

and so the coefficients for a particular initial condition are the **Fourier Cosine** coefficients of $f(x)$, given by

$$A_n = \frac{2}{l}\int_0^l f(x)\cos\Big(\frac{n\pi x}{l}\Big)dx. \tag{14}$$

One way to think about this difference is that given the initial data $u(x,0) = f(x)$, the Dirichlet conditions specify the **odd extension** of $f(x)$ as the desired periodic solution, while the Neumann conditions specify the **even extension**. This should make sense since odd functions must have $f(0) = 0$, while even functions must have $f'(0) = 0$.

So to solve a homogeneous heat equation problem, we begin by identifying the type of boundary conditions we have. If we have Dirichlet conditions, we know our solution will have the form of Equation (**??**). All we then have to do is compute the Fourier Sine coefficients of $f(x)$. Similarly, if we have Neumann conditions, we know the solution has the form of Equation (**??**) and we have to compute the Fourier Cosine coefficients of $f(x)$.

REMARK: Observe that for any homogeneous Dirichlet problem, the temperature distribution (**??**) will go to 0 as $t \to \infty$. This should make sense because these boundary conditions have a physical interpretation where we keep the ends of our rod at freezing temperature without regulating the heat flow in and out of the endpoints. As a result, if the interior of the rod is initially above freezing, that heat will radiate towards the endpoints and into our reservoirs at the endpoints. On the other hand, if the interior of the rod is below freezing, heat will come from the reservoirs at the endpoints and warm it up until the temperature is uniform.

For the Neumann problem, the temperature distribution (**??**) will converge to $\frac{1}{2}A_0$. Again, this should make sense because these boundary conditions correspond to a situation where we have insulated ends, since we are preventing any heat from escaping the bar. Thus all heat energy will move around inside the rod until the temperature is uniform.

1.1 Examples

Example 1. Solve the initial value problem

$$\begin{aligned} u_t &= 3u_{xx} \quad 0 < x < 2, \quad t > 0 && (15)\\ u(0,t) &= u(2,t) = 0 && (16)\\ u(x,0) &= 20. && (17) \end{aligned}$$

This problem has homogeneous Dirichlet conditions, so by (**??**) our general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-3(\frac{n\pi}{2})^2 t} \sin(\frac{n\pi x}{2}). \tag{18}$$

The coefficients for the particular solution are the Fourier Sine coefficients of $u(x,0) = 20$, so we have

$$\begin{aligned} B_n &= \frac{2}{2}\int_0^2 20\sin(\frac{n\pi x}{2})dx && (19)\\ &= [-\frac{40}{n\pi}\cos(\frac{n\pi x}{2})]_0^2 && (20)\\ &= -\frac{40}{n\pi}(\cos(n\pi) - \cos(0)) && (21)\\ &= \frac{40}{n\pi}(1 + (-1)^{n+1}) && (22) \end{aligned}$$

and the solution to the problem is

$$u(x,t) = \frac{40}{\pi}\sum_{n=1}^{\infty} \frac{1 + (-1)^{n+1}}{n} e^{-\frac{3n^2\pi^2}{4}t} \sin(\frac{n\pi x}{2}). \tag{23}$$

Example 2. Solve the initial value problem

$$\begin{aligned} u_t &= 3u_{xx} \quad 0 < x < 2, \quad t > 0 && (24)\\ u_x(0,t) &= u_x(2,t) = 0 && (25)\\ u(x,0) &= 3x. && (26) \end{aligned}$$

This problem has homogeneous Neumann conditions, so by (**??**) our general solution is

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-3(\frac{n\pi}{2})^2 t} \cos(\frac{n\pi x}{2}). \tag{27}$$

The coefficients for the particular solution are the Fourier Cosine coefficients of $u(x,0) = 3x$, so we have

$$
\begin{aligned}
A_0 &= \frac{2}{2}\int_0^2 3x dx = 6 && (28)\\
A_n &= \frac{2}{2}\int_0^2 3x\cos(\frac{n\pi x}{2})dx && (29)\\
&= [-\frac{6x}{n\pi}\cos(\frac{n\pi x}{2}) + \frac{12}{n^2\pi^2}\sin(\frac{n\pi x}{2})]_0^2 && (30)\\
&= -\frac{12}{n\pi}\cos(n\pi) && (31)\\
&= \frac{12}{n\pi}(-1)^{n+1} && (32)
\end{aligned}
$$

and the solution to the problem is

$$
u(x,t) = \frac{3}{2} + \frac{12}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-\frac{3n^2\pi^2}{4}t}\cos(\frac{n\pi x}{2}). \tag{33}
$$

Example 3. Solve the initial value problem

$$
\begin{aligned}
u_t &= 4u_{xx} \quad 0 < x < 2\pi, \quad t > 0 && (34)\\
u(0,t) &= u(2\pi,t) = 0 && (35)\\
u(x,0) &= \begin{cases} 1 & 0 < x < \pi \\ x & \pi < x < 2\pi \end{cases} . && (36)
\end{aligned}
$$

This problem has homogeneous Dirichlet conditions, so our general solution is

$$
u(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2 t}\sin(\frac{nx}{2}). \tag{37}
$$

The coefficients for the particular solution are the Fourier Sine coefficients of $u(x,0)$, so we have

$$
\begin{aligned}
B_n &= \frac{2}{2\pi}\left(\int_0^{\pi}\sin(\frac{nx}{2})dx + \int_{\pi}^{2\pi} x\sin(\frac{nx}{2})dx\right) && (38)\\
&= -\frac{2}{n\pi}\cos(\frac{nx}{2})|_0^{\pi} - \frac{2x}{n\pi}\cos(\frac{nx}{2})|_{\pi}^{2\pi} + \frac{4}{n^2\pi}\sin(\frac{nx}{2})|_{\pi}^{2\pi} && (39)\\
&= -\frac{2}{n\pi}(\cos(\frac{nx}{2}) - \cos(0)) - \frac{4}{n}\cos(n\pi) + \frac{2}{n}\cos(\frac{n\pi}{2}) - \frac{4}{n^2\pi}\sin(\frac{n\pi}{2}) && (40)\\
&= -\frac{2}{n\pi}(\cos(\frac{n\pi}{2}) - 1) + \frac{4}{n}(-1)^{n+1} + \frac{2}{n}\cos(\frac{n\pi}{2}) - \frac{4}{n^2\pi}\sin(\frac{n\pi}{2}) && (41)\\
&= \frac{2}{n}\left(-\frac{1}{\pi}(\cos(\frac{n\pi}{2}) - 1) + 2(-1)^{n+1}\cos(\frac{n\pi}{2}) - \frac{2}{n\pi}\sin(\frac{n\pi}{2})\right) && (42)
\end{aligned}
$$

and the solution to the problem is

$$
u(x,t) = 2\sum_{n=1}^{\infty}\frac{1}{n}\left(-\frac{1}{\pi}(\cos(\frac{n\pi}{2}) - 1) + 2(-1)^{n+1}\cos(\frac{n\pi}{2}) - \frac{2}{n\pi}\sin(\frac{n\pi}{2})\right)e^{-n^2 t}\sin(\frac{nx}{2}). \tag{43}
$$

2 Other Boundary Conditions

So far, we have used the technique of separation of variables to produce solutions to the heat equation

$$u_t = ku_{xx} \tag{44}$$

on $0 < x < l$ with either homogeneous Dirichlet boundary conditions $[u(0,t) = u(l,t) = 0]$ or homogeneous Neumann boundary conditions $[u_x(0,t) = u_x(l,t) = 0]$. What about for some other physically relevant boundary conditions?

2.1 Mixed Homogeneous Boundary Conditions

We could have the following boundary conditions

$$u(0,t) = u_x(l,t) = 0 \tag{45}$$

Physically, this might correspond to keeping the end of the rod where $x = 0$ in a bowl of ice water, while the other end is insulated.

Use Separation of Variables. Let $u(x,t) = X(x)T(t)$, and we get the pair of ODEs

$$\begin{aligned} T' &= -k\lambda T & (46)\\ X'' &= -\lambda X. & (47) \end{aligned}$$

Thus

$$T(t) = Be^{-k\lambda t}. \tag{48}$$

We now have a boundary value problem for X to deal with, where the boundary conditions are $X(0) = X'(l) = 0$. There are only positive eigenvalues, which are given by

$$\lambda_n = \left(\frac{(2n-1)\pi}{2l}\right)^2 \tag{49}$$

and their associated eigenfunctions are

$$X_n(x) = \sin\left(\frac{(2n-1)\pi x}{2l}\right). \tag{50}$$

The separated solutions are then given by

$$u_n(x,t) = B_n e^{-(\frac{(2n-1)\pi}{2l})^2 kt} \sin\left(\frac{(2n-1)\pi x}{2l}\right) \tag{51}$$

and the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-(\frac{(2n-1)\pi}{2l})^2 kt} \sin\left(\frac{(2n-1)\pi x}{2l}\right). \tag{52}$$

with an initial condition $u(x,0) = f(x)$, we have that

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{(2n-1)\pi x}{2l}\right). \tag{53}$$

This is an example of a specialized sort of Fourier Series, the coefficients are given by

$$B_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{(2n-1)\pi x}{2l}\right)dx. \tag{54}$$

REMARK: The convergence for a series like the one above is different than that of our standard Fourier Sine or Cosine series, which converge to the periodic extension of the odd or even extensions of the original function, respectively. Notice that the terms in the sum above are periodic with period $4l$ (as opposed to the $2l$-periodic series we have seen before). In this case, we need to first extend our function $f(x)$, given on $(0,l)$, to a function on $(0,2l)$ symmetric around $x = l$. Then, as our terms are all sines, the convergence on $(-2l, 2l)$ will be to the odd extension of this extended function, and the periodic extension of this will be what the series converges to on the entire real line.

Example 4. Solve the following heat equation problem

$$\begin{aligned} u_t &= 25u_{xx} && (55)\\ u(0,t) &= 0 \quad u_x(10,t) = 0 && (56)\\ u(x,0) &= 5. && (57) \end{aligned}$$

By (**??**) our general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-25(\frac{(2n-1)\pi}{20})^2 t}\sin\left(\frac{(2n-1)\pi x}{20}\right). \tag{58}$$

The coefficients for the particular solution are given by

$$\begin{aligned} B_n &= \frac{2}{10}\int_0^{10} 5\sin\left(\frac{(2n-1)\pi x}{20}\right)dx && (59)\\ &= -\frac{10}{(2n-1)\pi}\cos\left(\frac{(2n-1)\pi x}{20}\right)\Big|_0^{10} && (60)\\ &= -\frac{10}{(2n-1)\pi}\left(\cos\left(\frac{(2n-1)\pi}{2}\right) - \cos(0)\right) && (61)\\ &= \frac{10}{(2n-1)\pi}. && (62) \end{aligned}$$

and the solution to the problem is

$$u(x,t) = \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}e^{-\frac{(2n-1)^2\pi^2}{16}t}\sin\left(\frac{(2n-1)\pi x}{20}\right). \tag{63}$$

2.2 Nonhomogeneous Dirichlet Conditions

The next type of boundary conditions we will look at are Dirichlet conditions, which fix the value of u at the endpoints $x = 0$ and $x = l$. For the heat equation, this corresponds to fixing the temperature at the ends of the rod. We have already looked at homogeneous conditions where the ends of the rod had fixed temperature 0. Now consider the nonhomogeneous Dirichlet conditions

$$u(0,t) = T_1, \quad u(l,t) = T_2 \tag{64}$$

This problem is slightly more difficult than the homogeneous Dirichlet condition problem we have studied. Recall that for separation of variables to work, the differential equations and the boundary conditions must be homogeneous. When we have nonhomogeneous conditions we need to try to split the problem into one involving homogeneous conditions, which we know how to solve, and another dealing with the nonhomogeneity.

REMARK: We used a similar approach when we applied the method of Undetermined Coefficients to nonhomogeneous linear ordinary differential equations.

How can we separate the core homogeneous problem from what is causing the nonhomogeneity? Consider what happens as $t \to \infty$. We should expect that, since we fix the temperatures at the endpoints and allow free heat flux at the boundary, at some point the temperature will stabilize and we will be at equilibrium. Such a temperature distribution would clearly not depend on time, and we can write

$$\lim_{t\to\infty} u(x,t) = v(x) \tag{65}$$

Notice that $v(x)$ must still satisfy the boundary conditions and the heat equation, but we should not expect it to satisfy the initial conditions (since for large t we are far from where we initially started). A solution such as $v(x)$ which does not depend on t is called a **steady-state** or **equilibrium solution**.

For a steady-state solution the boundary value problem becomes

$$0 = kv'' \quad v(0) = T_1 \quad v(l) = T_2. \tag{66}$$

It is easy to see that solutions to this second order differential equation are

$$v(x) = c_1 x + c_2 \tag{67}$$

and applying the boundary conditions, we have

$$v(x) = T_1 + \frac{T_2 - T_1}{l} x. \tag{68}$$

Now, let

$$w(x,t) = u(x,t) - v(x) \tag{69}$$

so that

$$u(x,t) = w(x,t) + v(x). \tag{70}$$

This function $w(x,t)$ represents the **transient** part of $u(x,t)$ (since $v(x)$ is the equilibrium part). Taking derivatives we have

$$u_t = w_t + v_t = w_t \quad \text{and} \quad u_{xx} = w_{xx} + v_{xx} = w_{xx}. \tag{71}$$

Here we use the fact that $v(x)$ is independent of t and must satisfy the differential equation. Also, using the equilibrium equation $v'' = v_{xx} = 0$.

Thus $w(x,t)$ must satisfy the heat equation, as the relevant derivatives of it are identical to those of $u(x,t)$, which is known to satisfy the equation. What are the boundary and initial conditions?

$$w(0,t) = u(0,t) - v(0) = T_1 - T_1 = 0 \tag{72}$$
$$w(l,t) = u(l,t) - v(l) = T_2 - T_2 = 0 \tag{73}$$
$$w(x,0) = u(x,0) - v(x) = f(x) - v(x) \tag{74}$$

where $f(x) = u(x,0)$ is the given initial condition for the nonhomogeneous problem. Now, even though our initial condition is slightly messier, we now have homogeneous boundary conditions, since $w(x,t)$ must solve the problem

$$w_t = kw_{xx} \tag{75}$$
$$w(0,t) = w(l,t) = 0 \tag{76}$$
$$w(x,0) = f(x) - v(x) \tag{77}$$

This is just a homogeneous Dirichlet problem. We know the general solution is

$$w(x,t) = \sum_{n=1}^{\infty} B_n e^{-(\frac{n\pi}{l})^2 kt} \sin(\frac{n\pi x}{l}). \tag{78}$$

where the coefficients are given by

$$B_n = \frac{2}{l} \int_0^l (f(x) - v(x)) \sin(\frac{n\pi x}{l}) dx. \tag{79}$$

Notice that $\lim_{t\to\infty} w(x,t) = 0$, so that $w(x,t)$ is transient.

Thus, the solution to the nonhomogeneous Dirichlet problem

$$u_t = ku_{xx} \tag{80}$$
$$u(0,t) = T_1, \quad u(l,t) = T_2 \tag{81}$$
$$u(x,0) = f(x) \tag{82}$$

is $u(x,t) = w(x,t) + v(x)$, or

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-(\frac{n\pi}{l})^2 lt} \sin\left(\frac{n\pi x}{l}\right) + T_1 + \frac{T_2 - T_1}{l} x \tag{83}$$

with coefficients

$$B_n = \frac{2}{l}\int_0^l (f(x) - T_1 - \frac{T_2 - T_1}{l}x)\sin\big(\frac{n\pi x}{l}\big)dx. \tag{84}$$

REMARK: Do not memorize the formulas but remember what problem $w(x,t)$ has to solve and that the final solution is $u(x,t) = v(x) + w(x,t)$. For $v(x)$, it is not a hard formula, but if one is not sure of it, remember $v_{xx} = 0$ and it has the same boundary conditions as $u(x,t)$. This will recover it.

Example 5. Solve the following heat equation problem

$$\begin{aligned} u_t &= 3u_{xx} && (85)\\ u(0,t) &= 20, \quad u(40,t) = 100 && (86)\\ u(x,0) &= 40 - 3x && (87) \end{aligned}$$

We start by writing

$$u(x,t) = v(x) + w(x,t) \tag{88}$$

where $v(x) = 20 + 2x$. Then $w(x,t)$ must satisfy the problem

$$\begin{aligned} w_t &= 3w_{xx} && (89)\\ w(0,t) &= w(40,t) = 0 && (90)\\ w(x,0) &= 40 - 3x - (20 + 2x) = 20 - x && (91) \end{aligned}$$

This is a homogeneous Dirichlet problem, so the general solution for $w(x,t)$ will be

$$w(x,t) = \sum_{n=1}^{\infty} e^{-3(\frac{n\pi}{40})^2 t}\sin(\frac{n\pi x}{40}). \tag{92}$$

The coefficients are given by

$$\begin{aligned} B_n &= \frac{2}{40}\int_0^{40}(20 - x)\sin\big(\frac{n\pi x}{40}\big)dx && (93)\\ &= \frac{1}{20}[-\frac{40(20-x)}{n\pi}\cos\big(\frac{n\pi x}{40}\big) - \frac{1600}{n^2\pi^2}\sin\big(\frac{n\pi x}{40}\big)]_0^{40} && (94)\\ &= \frac{1}{20}\left(\frac{800}{n\pi}\cos(n\pi) + \frac{800}{n\pi}\cos(0)\right) && (95)\\ &= \frac{40}{n\pi}((-1)^n + 1). && (96) \end{aligned}$$

So the solution is

$$u(x,t) = 20 + 2x + \frac{40}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n + 1}{n}e^{-\frac{3n^2\pi^2}{1600}t}\sin\big(\frac{n\pi x}{40}\big). \tag{97}$$

2.3 Other Boundary Conditions

There are many other boundary conditions one could use, most of which have a physical interpretation. For example the boundary conditions

$$u(0,t) + u_x(0,t) = 0 \quad u(l,t) + u_x(l,t) = 0 \tag{98}$$

say that the heat flux at the end points should be proportional to the temperature. We could also have had nonhomogeneous Neumann conditions

$$u_x(0,t) = F_1 \quad u_x(l,t) = F_2 \tag{99}$$

which would specify allowing a certain heat flux at the boundaries. These conditions are not necessarily well suited for the method of separation of variables though and are left for future classes.

HW 10.6 # 1, 4, 6, 13abc