Lecture Notes for Math 251: ODE and PDE. Lecture 34: 10.7 Wave Equation and Vibrations of an Elastic String

Shawn D. Ryan

Spring 2012

Last Time: We studied other Heat Equation problems with various other boundary conditions.

1 The Wave Equation

1.1 Derivation of the Wave Equation

Consider a completely flexible string of length l and constant density ρ. We will assume that the string will only undergo relatively small vertical vibrations, so that points do not move from side to side. An example might be a plucked guitar string. Thus we can let $u(x,t)$ be its displacement from equilibrium at time t. The assumption of complete flexibility means that the tension force is tangent to the string, and the string itself provides no resistance to bending. This means the tension force only depends on the slope of the string.

Take a small piece of string going from x to $x + \Delta x$. Let $\Theta(x,t)$ be the angle from the horizontal of the string. Our goal is to use Newton's Second Law $F = ma$ to describe the motion. What forces are acting on this piece of string?
(a) Tension pulling to the right, which has magnitude $T(x + \Delta x, t)$ and acts at an angle of $\Theta(x + \Delta x, t)$ from the horizontal.
(b) Tension pulling to the left, which has magnitude $T(x,t)$ and acts at an angle of $\Theta(x,t)$ from the horizontal.
(c) Any external forces, which we denote by $F(x,t)$.
Initially, we will assume that $F(x,t) = 0$. The length of the string is essentially $\sqrt{(\Delta x)^2 + (\Delta u)^2}$, so the vertical component of Newton's Law says that

$$\rho\sqrt{(\Delta x)^2 + (\Delta u)^2}u_{tt}(x,t) = T(x + \Delta x, t)\sin(\Theta(x + \Delta x, t)) - T(x,t)\sin(\Theta(x,t)). \quad (1)$$

Dividing by Δx and taking the limit as $\Delta x \to 0$, we get

$$\rho\sqrt{1 + (u_x)^2}u_{tt}(x,t) = \frac{\partial}{\partial x}[T(x,t)\sin(\Theta(x,t))]. \quad (2)$$

We assumed our vibrations were relatively small. This means that $\Theta(x,t)$ is very close to zero. As a result, $\sin(\Theta(x,t)) \equiv \tan(\Theta(x,t))$. Moreover, $\tan(\Theta(x,t))$ is just the slope of the string

$u_x(x,t)$. We conclude, since $\Theta(x,t)$ is small, that $u_x(x,t)$ is also very small. The above equation becomes

$$\rho u_{tt}(x,t) = (T(x,t)u_x(x,t))_x. \tag{3}$$

We have not used the horizontal component of Newton's Law yet. Since we assume there are only vertical vibrations, our tiny piece of string can only move vertically. Thus the net horizontal force is zero.

$$T(x+\Delta x,t)\cos(\Theta(x+\Delta x,t)) - T(x,t)\cos(\Theta(x,t)) = 0. \tag{4}$$

Dividing by Δx and taking the limit as $\Delta x \to \infty$ yields

$$\frac{\partial}{\partial x}[T(x,t)\cos(\Theta(x,t))] = 0. \tag{5}$$

Since $\Theta(x,t)$ is very close to zero, $\cos(\Theta(x,t))$ is close to one. thus we have that $\frac{\partial T}{\partial x}(x,t)$ is close to zero. So $T(x,t)$ is constant along the string, and independent of x. We will also assume that T is independent of t. Then Equation **(??)** becomes the one-dimensional wave equation

$$u_{tt} = c^2 u_{xx} \tag{6}$$

where $c^2 = \frac{T}{\rho}$.

1.2 The Homogeneous Dirichlet Problem

Now that we have derived the wave equation, we can use Separation of Variables to obtain basic solutions. We will consider homogeneous Dirichlet conditions, but if we had homogeneous Neumann conditions the same techniques would give us a solution. The wave equation is second order in t, unlike the heat equation which was first order in t. We will need to initial conditions in order to obtain a solution, one for the initial displacement and the other for the initial speed.

The relevant wave equation problem we will study is

$$\begin{aligned} u_{tt} &= c^2 u_{xx} & (7)\\ u(0,t) &= u(l,t) = 0 & (8)\\ u(x,0) &= f(x), \quad u_t(x,0) = g(x) & (9) \end{aligned}$$

The physical interpretation of the boundary conditions is that the ends of the string are fixed in place. They might be attached to guitar pegs.

We start by assuming our solution has the form

$$u(x,t) = X(x)T(t). \tag{10}$$

Plugging this into the equation gives

$$T''(t)X(x) = c^2 T(t)X''(x). \tag{11}$$

Separating variables, we have

$$\frac{X''}{X} = \frac{T''}{c^2 T} = -\lambda \tag{12}$$

where λ is a constant. This gives a pair of ODEs

$$
\begin{aligned}
T'' + c^2\lambda T &= 0 \qquad (13)\\
X'' + \lambda X &= 0. \qquad (14)
\end{aligned}
$$

The boundary conditions transform into

$$
\begin{aligned}
u(0,t) &= X(0)T(t) = 0 \quad \Rightarrow \quad X(0) = 0 \qquad (15)\\
u(l,t) &= X(l)T(t) = 0 \quad \Rightarrow \quad X(l) = 0. \qquad (16)
\end{aligned}
$$

This is the same boundary value problem that we saw for the heat equation and thus the eigenvalues and eigenfunctions are

$$
\begin{aligned}
\lambda_n &= \left(\frac{n\pi}{l}\right)^2 \qquad (17)\\
X_n(x) &= \sin\left(\frac{n\pi x}{l}\right) \qquad (18)
\end{aligned}
$$

for $n = 1, 2, ...$ The first ODE (**??**) is then

$$T'' + \left(\frac{cn\pi}{l}\right)^2 T = 0, \qquad (19)$$

and since the coefficient of T is clearly positive this has a general solution

$$T_n(t) = A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right). \qquad (20)$$

There is no reason to think either of these are zero, so we end up with separated solutions

$$u_n(x,t) = \left[A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right)\right] \sin\left(\frac{n\pi x}{l}\right) \qquad (21)$$

and the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right)\right] \sin\left(\frac{n\pi x}{l}\right). \qquad (22)$$

We can directly apply our first initial condition, but to apply the second we will need to differentiate with respect to t. This gives us

$$u_t(x,t) = \sum_{n=1}^{\infty} \left[-\frac{n\pi c}{l} A_n \sin\left(\frac{n\pi ct}{l}\right) + \frac{n\pi c}{l} B_n \cos\left(\frac{n\pi ct}{l}\right)\right] \sin\left(\frac{n\pi x}{l}\right) \qquad (23)$$

Plugging in the initial condition then yields the pair of equations

$$
\begin{aligned}
u(x,0) &= f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) \qquad (24)\\
u_t(x,0) &= g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \sin\left(\frac{n\pi x}{l}\right). \qquad (25)
\end{aligned}
$$

These are both Fourier Sine series. The first is directly the Fourier Since series for $f(x)$ on $(0, l)$. The second equation is the Fourier Sine series for $g(x)$ on $(0, l)$ with a slightly messy coefficient. The Euler-Fourier formulas then tell us that

$$\begin{aligned} A_n &= \frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi x}{l})dx && (26)\\ \frac{n\pi c}{l}B_n &= \frac{2}{l}\int_0^l g(x)\sin(\frac{n\pi x}{l})dx && (27)\\ A_n &= \frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi x}{l})dx && (28)\\ B_n &= \frac{2}{n\pi c}\int_0^l g(x)\sin(\frac{n\pi x}{l})dx. && (29) \end{aligned}$$

1.3 Examples

Example 1. Find the solution (displacement $u(x, t)$) for the problem of an elastic string of length L whose ends are held fixed. The string has no initial velocity ($u_t(x, 0) = 0$) from an initial position

$$u(x,0) = f(x) = \begin{cases} \frac{4x}{L} & 0 \le x \le \frac{L}{4} \\ 1 & \frac{L}{4} < x < \frac{3L}{4} \\ \frac{4(L-x)}{L} & \frac{3L}{4} \le x \le L \end{cases} \tag{30}$$

By the formulas above we see if we separate variables we have the following equation for T

$$T'' + (\frac{cn\pi}{L})^2 T = 0 \tag{31}$$

with the general solution

$$T_n(t) = A_n\cos(\frac{n\pi ct}{L}) + B_n\sin(\frac{n\pi ct}{L}). \tag{32}$$

since the initial speed is zero, we find $T'(0) = 0$ and thus $B_n = 0$. Therefore the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n\cos(\frac{n\pi ct}{L})\sin(\frac{n\pi x}{L}). \tag{33}$$

where the coefficients are the Fourier Sine coefficients of $f(x)$. So

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx && (34)\\ &= \frac{2}{L}\left[\int_0^{L/4}\frac{4x}{L}\sin(\frac{n\pi x}{L})dx + \int_{L/4}^{3L/4}\sin(\frac{n\pi x}{L})dx + \int_{3L/4}^{L}\frac{4L-4x}{L}\sin(\frac{n\pi x}{L})dx\right] && (35)\\ &= 8\frac{\sin(\frac{n\pi}{4}) + \sin(\frac{3n\pi}{4})}{n^2\pi^2} && (36) \end{aligned}$$

Thus the displacement of the string will be

$$u(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin(\frac{n\pi}{4}) + \sin(\frac{3n\pi}{4})}{\pi^2}\cos(\frac{n\pi ct}{L})\sin(\frac{n\pi x}{L}). \tag{37}$$

Example 2. Find the solution (displacement $u(x,t)$) for the problem of an elastic string of length L whose ends are held fixed. The string has no initial velocity ($u_t(x,0) = 0$) from an initial position

$$u(x,0) = f(x) = \frac{8x(L-x)^2}{L^3} \tag{38}$$

By the formulas above we see if we separate variables we have the following equation for T

$$T'' + (\frac{cn\pi}{L})^2 T = 0 \tag{39}$$

with the general solution

$$T_n(t) = A_n\cos(\frac{n\pi ct}{L}) + B_n\sin(\frac{n\pi ct}{L}). \tag{40}$$

since the initial speed is zero, we find $T'(0) = 0$ and thus $B_n = 0$. Therefore the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n\cos(\frac{n\pi ct}{L})\sin(\frac{n\pi x}{L}). \tag{41}$$

where the coefficients are the Fourier Sine coefficients of $f(x)$. So

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx && (42)\\ &= \frac{2}{L}\int_0^L \frac{8x(L-x)^2}{L^3}\sin(\frac{n\pi x}{L})dx && (43)\\ &= 32\frac{2+\cos(n\pi)}{n^3\pi^3} \quad \text{Integrate By Parts} && (44) \end{aligned}$$

Thus the displacement of the string will be

$$u(x,t) = \frac{32}{\pi^3}\sum_{n=1}^{\infty}\frac{2+\cos(n\pi)}{n^3}\cos(\frac{n\pi ct}{L})\sin(\frac{n\pi x}{L}). \tag{45}$$

Example 3. Problem 12 is a great exercise.

HW 10.7 # 4a, 5a, 7a, 8a, 12

Go through the Separation of Variables it will be important for Exams.