Lecture Notes for Math 251: ODE and PDE. Lecture 35: 10.8 Laplace's Equation

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Spring 2012

Last Time: We studied another fundamental equation in the study of partial differential equations, which is the wave equation. Today we will look at the final fundamental equation, which is Laplace's Equation.

1 Laplace's Equation

We will consider the two-dimensional and three-dimensional Laplace Equations

$$(2D): \quad u_{xx} + u_{yy} = 0, \tag{1}$$
$$(3D): \quad u_{xx} + u_{yy} + u_{zz} = 0. \tag{2}$$

1.1 Dirichlet Problem for a Rectangle

We want to find the function u satisfying Laplace's Equation

$$u_{xx} + u_{yy} = 0 \tag{3}$$

in the rectangle $0 < x < a, 0 < y < b$, and satisfying the boundary conditions

$$u(x,0) = 0, \quad u(x,b) = 0, \quad 0 < x < a, \tag{4}$$
$$u(0,y) = 0, \quad u(a,y) = f(y), \quad 0 \leq y \leq b. \tag{5}$$

We need four boundary conditions for the four spatial derivatives.

Start by using Separation of Variables and assume $u(x,y) = X(x)Y(y)$. Substitute u into Equation (**??**). This yields

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda, \tag{6}$$

where λ is a constant. We obtain the following system of ODEs

$$X'' - \lambda X = 0 \tag{7}$$
$$Y'' + \lambda Y = 0. \tag{8}$$

From the boundary conditions we find

$$X(0) = 0 \tag{9}$$
$$Y(0) = 0, Y(b) = 0. \tag{10}$$

We first solve the ODE for Y, which we have seen numerous times before. Using the BCs we find there are nontrivial solutions if and only if λ is an eigenvalue

$$\lambda = \left(\frac{n\pi}{b}\right)^2, \quad n = 1, 2, 3, ... \tag{11}$$

and $Y_n(y) = \sin(\frac{n\pi y}{b})$, the corresponding eigenfunction. Now substituting in for λ we want to solve the ODE for X. This is another problem we have seen regularly and the solution is

$$X_n(x) = c_1 \cosh\left(\frac{n\pi x}{b}\right) + c_2 \sinh\left(\frac{n\pi x}{b}\right) \tag{12}$$

The BC implies that $c_1 = 0$. So the fundamental solution to the problem is

$$u_n(x, y) = \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right). \tag{13}$$

By linear superposition the general solution is

$$u(x, y) = \sum_{n=1}^{\infty} c_n u_n(x, y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right). \tag{14}$$

Using the last boundary condition $u(a, y) = f(y)$ solve for the coefficients c_n.

$$u(a, y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi a}{b}\right) \sin\left(\frac{n\pi y}{b}\right) = f(y) \tag{15}$$

Using the Fourier Since Series coefficients we find

$$c_n = \frac{2}{b \sinh(\frac{n\pi a}{b})} \int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy. \tag{16}$$

1.2 Dirichlet Problem For A Circle

Consider solving Laplace's Equation in a circular region $r < a$ subject to the boundary condition

$$u(a, \theta) = f(\theta) \tag{17}$$

where f is a given function on $0 \leq \theta \leq 2\pi$. In polar coordinates Laplace's Equation becomes

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0. \tag{18}$$

Try Separation of Variables in Polar Coordinates

$$u(r, \theta) = R(r)\Theta(\theta), \tag{19}$$

plug into the differential equation, Equation (**??**). This yields

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \tag{20}$$

or

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda \tag{21}$$

where λ is a constant. We obtain the following system of ODEs

$$r^2R'' + rR' - \lambda R \;\; = \;\; 0, \tag{22}$$

$$\Theta'' + \lambda\theta \;\; = \;\; 0. \tag{23}$$

Since we have no homogeneous boundary conditions we must use instead the fact that the solutions must be bounded and also periodic in Θ with period 2π. It can be shown that we need λ to be real. Consider the three cases when $\lambda < 0, \lambda = 0, \lambda > 0$.

If $\lambda < 0$, let $\lambda = -\mu^2$, where $\mu > 0$. So we find the equation for Θ becomes $\Theta'' - \mu^2\Theta = 0$. So

$$\Theta(\theta) = c_1e^{\mu\theta} + c_2e^{-\mu\theta} \tag{24}$$

Θ can only be periodic if $c_1 = c_2 = 0$, so λ cannot be negative (Since we do not get any nontrivial solutions.

If $\lambda = 0$, then the equation for Θ becomes $\Theta'' = 0$ and thus

$$\Theta(\theta) = c_1 + c_2\theta \tag{25}$$

For Θ to be periodic $c_2 = 0$. Then the equation for R becomes

$$r^2R'' + rR' = 0. \tag{26}$$

This equation is an Euler equation and has solution

$$R(r) = k_1 + k_2\ln(r) \tag{27}$$

Since we also need the solution bounded as $r \to \infty$, then $k_2 = 0$. So $u(r, \theta)$ is a constant, and thus proportional to the solution $u_0(r, \theta) = 1$.

If $\lambda > 0$, we let $\lambda = \mu^2$, where $\mu > 0$. Then the system of equations becomes

$$r^2R'' + rR' - \mu^2R \;\; = \;\; 0 \tag{28}$$

$$\Theta'' + \mu^2\Theta = 0 \tag{29}$$

The equation for R is an Euler equation and has the solution

$$R(r) = k_1r^{\mu} + k_2r^{-\mu} \tag{30}$$

and the equation for Θ has the solution

$$\Theta(\theta) = c_1 \sin(\mu\theta) + c_2 \cos(\mu\theta). \tag{31}$$

For Θ to be periodic we need μ to be a positive integer n, so $\mu = n$. Thus the solution $r^{-\mu}$is unbounded as $r \to 0$. So $k_2 = 0$. So the solutions to the original problem are

$$u_n(r,\theta) = r^n \cos(n\theta), \quad v_n(r,\theta) = r^n \sin(n\theta), \quad n = 1, 2, 3, ... \tag{32}$$

Together with $u_0(r,\theta) = 1$, by linear superposition we find

$$u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n (c_n \cos(n\theta) + k_n \sin(n\theta)). \tag{33}$$

Using the boundary condition from the beginning

$$u(a,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} a^n (c_n \cos(n\theta) + k_n \sin(n\theta)) = f(\theta) \tag{34}$$

for $0 \le \theta \le 2\pi$. We compute to coefficients by using our previous Fourier Series equations

$$c_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \cos(n\theta) d\theta, \quad n = 1, 2, 3, ... \tag{35}$$

$$k_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \sin(n\theta) d\theta, \quad n = 1, 2, 3, ... \tag{36}$$

Note we need both terms since sine and cosine terms remain throughout the general solution.

1.3 HW 10.8 # 2

Find the solution $u(x,y)$ of Laplace's Equation in the rectangle $0 < x < a$, $0 < y < b$, that satisfies the boundary conditions

$$u(0,y) = 0, \quad u(a,y) = 0, \quad 0 < y < b \tag{37}$$

$$u(x,0) = h(x), \quad u(x,b) = 0, \quad 0 \le x \le a \tag{38}$$

Answer: Using the method of Separation of Variables, write $u(x,y) = X(x)Y(y)$. We get the following system of ODEs

$$X'' + \lambda X = 0, \quad X(0) = X(a) = 0 \tag{39}$$

$$Y'' - \lambda Y = 0, \quad Y(b) = 0 \tag{40}$$

It follows that $\lambda_n = (\frac{n\pi}{a})^2$ and $X_n(x) = \sin(\frac{n\pi x}{a})$. The solution of the second ODE gives

$$Y(y) = d_1 \cosh(\lambda(b-y)) + d_2 \sinh(\lambda(b-y)). \tag{41}$$

Using $y(b) = 0$, we find that $d_1 = 0$. Therefore the fundamental solutions are

$$u_n(x, y) = \sin(\frac{n\pi x}{a}) \sinh(\lambda_n(b - y)), \tag{42}$$

and the general solution is

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{a}) \sinh(\frac{n\pi(b - y)}{a}). \tag{43}$$

Using another boundary condition

$$h(x) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{a}) \sinh(\frac{n\pi b}{a}). \tag{44}$$

The coefficients are calculated using the equation from the Fourier Sine Series

$$c_n = \frac{2}{a \sinh(\frac{n\pi b}{a})} \int_0^a h(x) \sin(\frac{n\pi x}{a}) dx. \tag{45}$$

1.4 HW 10.8 # 10a

Consider the problem of finding a solution $u(x, y)$ of Laplace's Equation in the rectangle $0 < x < a$, $0 < y < b$, that satisfies the boundary conditions

$$u_x(0, y) = 0, \quad u_x(a, y) = f(y), \quad 0 < y < b, \tag{46}$$

$$u_y(x, 0) = 0, \quad u_y(x, b) = 0, \quad 0 \leq x \leq a \tag{47}$$

This is an example of a Neumann Problem. We want to find the fundamental set of solutions.

$$X'' - \lambda X = 0, \quad X'(0) = 0 \tag{48}$$

$$Y'' + \lambda Y = 0, \quad Y'(0) = Y'(b) = 0. \tag{49}$$

The solution to the equation for Y is

$$Y(y) = c_1 \cos(\lambda^{1/2} y) + c_2 \sin(\lambda^{1/2} y), \tag{50}$$

with $Y'(y) = -c_1 \lambda^{1/2} \sin(\lambda^{1/2} y) + c_2 \lambda^{1/2} \cos(\lambda^{1/2} y)$. Using the boundary conditions we find $c_2 = 0$ and the eigenvalues are $\lambda_n = \frac{n^2\pi^2}{b^2}$, for $n = 1, 2, 3, ...$ The corresponding Eigenfunctions are $Y(y) = \cos(\frac{n\pi y}{b})$ for $n = 1, 2, 3, ...$ The solution of the equation for X becomes $X(x) = d_1 \cosh(\frac{n\pi x}{b}) + d_2 \sinh(\frac{n\pi x}{b})$, with

$$X'(x) = d_1 \frac{n\pi}{b} \sinh(\frac{n\pi x}{b}) + d_2 \frac{n\pi}{b} \cosh(\frac{n\pi x}{b}). \tag{51}$$

Using the boundary conditions, $X(x) = d_1 \cosh(\frac{n\pi x}{b})$.So the fundamental set of solutions is

$$u_n(x,y) = \cosh(\frac{n\pi x}{b})\cos(\frac{n\pi y}{b}), \quad n = 1, 2, 3, ... \tag{52}$$

The general solution is given by

$$u(x,y) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cosh(\frac{n\pi x}{b})\cos(\frac{n\pi y}{b}) \tag{53}$$