Lecture Notes for Math 251: ODE and PDE. Lecture 4: 2.1 Linear Equations with Variable Coefficients

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1 Linear Equations; Method of Integrating Factors

Last Time: We classified ODEs and PDEs in terms of Order (the highest derivative taken) and linearity.

Now we start Chapter 2: First Order Differential Equations
All equations in this chapter will have the form

$$\frac{dy}{dt} = f(t, y) \tag{1}$$

If f depends linearly on y the equation will be a first order linear equation.

Consider the general equation

$$\frac{dy}{dt} + p(t)y = g(t) \tag{2}$$

We said in Chapter 1 if $p(t)$ and $g(t)$ are constants we can solve the equation explicitly. Unfortunately this is not the case when they are not constants. We need the method of **integrating factor** developed by Leibniz, who also invented calculus, where we multiply (2) by a certain function $\mu(t)$, chosen so the resulting equation is integrable. $\mu(t)$ is called the **integrating factor**. The challenge of this method is finding it.

Summary of Method:
1. Rewrite the equation as (MUST BE IN THIS FORM)

$$y' + ay = f \tag{3}$$

2. Find an integrating factor, which is any function

$$\mu(t) = e^{\int a(t)dt}. \tag{4}$$

3. Multiply both sides of (3) by the integrating factor.

$$\mu(t)y' + a\mu(t)y = f\mu(t) \tag{5}$$

4. Rewrite as a derivative

$$(\mu y)' = \mu f \tag{6}$$

5. Integrate both sides to obtain

$$\mu(t)y(t) = \int \mu(t)f(t)dt + C \tag{7}$$

and thus

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)f(t)dt + \frac{C}{\mu(t)} \tag{8}$$

Now lets see some examples:

Example 1. Find the general solution of

$$y' = y + e^{-t} \tag{9}$$

Step 1:

$$y' - y = e^{t} \tag{10}$$

Step 2:

$$\mu(t) = e^{-\int 1dt} = e^{-t} \tag{11}$$

Step 3:

$$e^{-t}(y' - y) = e^{-2t} \tag{12}$$

Step 4:

$$(e^{-t}y)' = e^{-2t} \tag{13}$$

Step 5:

$$e^{-t}y = \int e^{-2t}dt = -\frac{1}{2}e^{-2t} + C \tag{14}$$

Solve for y

$$y(t) = -\frac{1}{2}e^{-t} + Ce^{t} \tag{15}$$

Example 2. Find the general solution of

$$y' = y\sin t + 2te^{-\cos t} \tag{16}$$

and $y(0) = 1$.
Step 1:

$$y' - y\sin t = 2te^{-\cos t} \tag{17}$$

Step 2:

$$\mu(t) = e^{-\int \sin t dt} = e^{\cos t} \tag{18}$$

Step 3:
$$e^{\cos t}(y' - y\sin t) = 2t \tag{19}$$
Step 4:
$$(e^{\cos t}y)' = 2t \tag{20}$$
Step 5:
$$e^{\cos t}y = t^2 + C \tag{21}$$
So the general solution is:
$$y(t) = (t^2 + C)e^{-\cos t} \tag{22}$$
With IC
$$y(t) = (t^2 + e)e^{-\cos t} \tag{23}$$

Example 3. Find General Solution to

$$y' = y\tan t + \sin t \tag{24}$$

with $y(0) = 2$. Note Integrating factor

$$\mu(t) = e^{-\int \tan dt} = e^{\ln(\cos t)} = \cos t \tag{25}$$

Final Answer
$$y(t) = -\frac{\cos t}{2} + \frac{5}{2\cos t} \tag{26}$$

Example 4. Solve
$$2y' + ty = 2 \tag{27}$$
with $y(0) = 1$. Integrating Factor
$$\mu(t) = e^{t^2/4} \tag{28}$$
Final Answer
$$y(t) = e^{-t^2/4}\int_0^t e^{s^2/4}ds + e^{-t^2/4}. \tag{29}$$

1.1 REVIEW: Integration By Parts

This is the most important integration technique learned in Calculus 2. We will derive the method. Consider the product rule for two functions of t.

$$\frac{d}{dt}\Big[uv\Big] = u\frac{dv}{dt} + v\frac{du}{dt} \tag{30}$$

Integrate both sides from a to b

$$uv\Big|_a^b = \int_a^b u\frac{dv}{dt} + \int_a^b v\frac{du}{dt} \tag{31}$$

Rearrange the resulting terms

$$\int_a^b u\frac{dv}{dt} = uv\Big|_a^b - \int_a^b v\frac{du}{dt} \tag{32}$$

Practicing this many times will be helpful on the homework. Consider two examples.

Example 5. Find the integral $\int_1^9 \ln(t)dt$. First define $u, du, dv,$ and v.

$$u = \ln(t) \qquad dv = dt \tag{33}$$

$$du = \frac{1}{t}dt \qquad v = t \tag{34}$$

Thus

$$\int_1^9 \ln(t)dt = t\ln(t)\Big|_1^9 - \int_1^9 1dt \tag{35}$$

$$= 9\ln(9) - t\Big|_1^9 \tag{36}$$

$$= 9\ln(9) - 9 + 1 \tag{37}$$

$$= 9\ln(9) - 8 \tag{38}$$

Example 6. Find the integral $\int e^x \cos(x)dx$. First define $u, du, dv,$ and v.

$$u = \cos(x) \qquad dv = e^x dx \tag{39}$$

$$du = -\sin(x)dx \qquad v = e^x \tag{40}$$

Thus

$$\int e^x \cos(x)dx = e^x\cos(x) - \int e^x \sin(x)dx \tag{41}$$

Do Integration By Parts Again

$$u = \sin(x) \qquad dv = e^x dx \tag{42}$$

$$du = \cos(x)dx \qquad v = e^x \tag{43}$$

So

$$\int e^x \cos(x)dx = e^x\cos(x) - \int e^x \sin(x)dx \tag{44}$$

$$= e^x\cos(x) + e^x\sin(x) - \int e^x\cos(x)dx \tag{45}$$

$$2\int e^x\cos(x)dx = e^x\big(\cos(x) + \sin(x)\big) \tag{46}$$

$$\int e^x\cos(x)dx = \frac{1}{2}e^x\big(\cos(x) + \sin(x)\big) + C \tag{47}$$

Notice when we do not have limits of integration we need to include the arbitrary constant of integration C.

HW 2.1 # 1, 3c, 4c, 7c, 8c, 12c, 13, 16, 19, 28 (solve)