Lecture Notes for Math 251: ODE and PDE. Lecture 5: 2.2 Separable Equations

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1 Separable Equations

Last Time: We used integration to solve first order equations of the form

$$\frac{dy}{dt} = a(t)y + b(t) \tag{1}$$

the method of integrating factor only works on an equation of this form, but we want to handle a more general class of equations

$$\frac{dy}{dt} = f(t, y) \tag{2}$$

We want to solve **separable** equations which have the form

$$\frac{dy}{dx} = f(y)g(x) \tag{3}$$

The General Solution Method:

$$\text{Step 1: (Separate)} \quad \frac{1}{f(y)}dy = g(x)dx \tag{4}$$

$$\text{Step 2: (Integrate)} \quad \int \frac{1}{f(y)}dy = \int g(x)dx \tag{5}$$

$$\text{Step 2: (Solve for } y\text{)} \quad F(y) = G(x) + c \tag{6}$$

Note only need a constant of integration of one side, could just combine the constants we get on each side. Also, we only solve for y if it is possible, if not leave in **implicit form**.

Definition 1. An **equilibrium solution** is the value of y which makes $dy/dx = 0$, y remains this constant forever.

Example 2. *(Newton's Law of Cooling)* Consider the ODE, where E is a constant:

$$\frac{dB}{dt} = \kappa(E - B) \tag{7}$$

with initial condition (IC) $B(0) = B_0$. This is separable

$$\begin{aligned}
\int \frac{dB}{E - B} &= \int \kappa dt && (8)\\
-\ln|E - B| &= \kappa t + c && (9)\\
E - B &= e^{-\kappa t + c} = Ae^{-\kappa t} && (10)\\
B(t) &= E - Ae^{-\kappa t} && (11)\\
B(0) &= E - A && (12)\\
A &= E - B_0 && (13)\\
B(t) &= E - \frac{E - B_0}{e^{\kappa t}} && (14)
\end{aligned}$$

Example 3.

$$\frac{dy}{dt} = 6y^2x, \quad y(1) = \frac{1}{3}. \tag{15}$$

Separate and Solve:

$$\begin{aligned}
\int \frac{dy}{y^2} &= \int 6x dx && (16)\\
-\frac{1}{y} &= 3x^2 + c && (17)\\
y(1) &= 1/3 && (18)\\
-3 &= 3(1) + c \Rightarrow c = -6 && (19)\\
-\frac{1}{y} &= 3x^2 - 6 && (20)\\
y(x) &= \frac{1}{6 - 3x^2} && (21)
\end{aligned}$$

What is the **interval of validity** for this solution? Problem when $6 - 3x^2 = 0$ or when $x = \pm\sqrt{2}$. So possible intervals of validity: $(-\infty, -\sqrt{2}), (-\sqrt{2}, \sqrt{2}), (\sqrt{2}, \infty)$. We want to choose the one containing the initial value for x, which is x = 1, so the interval of validity is $(-\sqrt{2}, \sqrt{2})$.

Example 4.

$$y' = \frac{3x^2 + 2x - 4}{2y - 2}, \quad y(1) = 3 \tag{22}$$

There are no equilibrium solutions.

$$\begin{aligned}
\int 2y-2dy &= \int 3x^2+2x-4dx && (23)\\
y^2-2y &= x^3+x^2-4x+c && (24)\\
y(1) &= 3 \Rightarrow c=5 && (25)\\
y^2-2y+1 &= x^3+x^2-4x+6 \quad \text{(Complete the Square)} && (26)\\
(y-1)^2 &= x^3+x^2-4x+6 && (27)\\
y(x) &= 1 \pm \sqrt{x^3+x^2-4x+6} && (28)
\end{aligned}$$

There are two solutions we must choose the appropriate one. Use the IC to determine only the positive solution is correct.

$$y(x) = 1+\sqrt{x^3+x^2-4x+6} \tag{29}$$

We need the terms under the square root to be positive, so the interval of validity is values of x where $x^3+x^2-4x+6 \geq 0$. Note $x=1$ is in here so IC is in interval of validity.

Example 5.

$$\frac{dy}{dx} = \frac{xy^3}{1+x^2}, \quad y(0)=1 \tag{30}$$

One equilibrium solution, $y(x)=0$, which is not our case (since it does not meet the IC). So separate:

$$\begin{aligned}
\int \frac{dy}{y^3} &= \int \frac{x}{1+x^2}dx && (31)\\
-\frac{1}{2y^2} &= \frac{1}{2}\ln(1+x^2)+c && (32)\\
y(0) &= 1 \Rightarrow c=-\frac{1}{2} && (33)\\
y^2 &= \frac{1}{1-\ln(1+x^2)} && (34)\\
y(x) &= \frac{1}{\sqrt{1-\ln(1+x^2)}} && (35)
\end{aligned}$$

Determine the interval of validity. Need

$$\ln(1+x^2) < 1 \Rightarrow x^2 < e-1 \tag{36}$$

So the interval of validity is $-\sqrt{e-1} < x < \sqrt{e-1}$.

Example 6.

$$\frac{dy}{dx} = \frac{y-1}{x^2+1} \tag{37}$$

The equilibrium solution is $y(x)=1$ and our IC is $y(0)=1$, so in this case the solution is the constant function $y(s)=1$.

Example 7.

$$\textit{(Review IBP)}\frac{dy}{dt} = e^{y-t}\sec(y)(1+t^2), \quad y(0) = 0 \tag{38}$$

Separate by rewriting, and using Integration By Parts (IBP)

$$\frac{dy}{dt} = \frac{e^y e^{-t}}{\cos(y)}(1+t^2) \tag{39}$$

$$\int e^{-y}\cos(y)dy = \int e^{-t}(1+t^2)dt \tag{40}$$

$$\frac{e^{-y}}{2}(\sin(y) - \cos(y)) = -e^{-t}(t^2+2t+3) + \frac{5}{2} \tag{41}$$

Won't be able to find an explicit solution so leave in implicit form. In the implicit form it is difficult to find the interval of validity so we will stop here.

HW 2.2 # 2, 3, 5, 6, 8, 10, 13, 14

Hint (#5): Recall $\frac{d}{dx}[\tan(u(x))] = \sec^2(u)\frac{du}{dx}$, and $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$.