Lecture Notes for Math 251: ODE and PDE. Lecture 6: 2.3 Modeling With First Order Equations

Shawn D. Ryan

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1 Modeling With First Order Equations

Last Time: We solved separable ODEs and now we want to look at some applications to real world situations

There are two key questions to keep in mind throughout this section:
1. How do we write a differential equation to model a given situation?
2. What can the solution tell us about that situation?

Example 1. *(Radioactive Decay)*

$$\frac{dN}{dt} = -\lambda N(t), \tag{1}$$

where $N(t)$ is the number of atoms of a radioactive isotope and $\lambda > 0$ is the decay constant. The equation is separable, and if the initial data is $N(0) = N_0$, the solution is

$$N(t) = N_0 e^{-\lambda t}. \tag{2}$$

so we can see that radioactive decay is exponential.

Example 2. *(Newton's Law of Cooling)* If we immerse a body in an environment with a constant temperature E, then if $B(t)$ is the temperature of the body we have

$$\frac{dB}{dt} = \kappa(E - B), \tag{3}$$

where $\kappa > 0$ is a constant related to the material of the body and how it conducts heat. This equation is separable. We solved it before with the initial condition $B(0) = B_0$ to get

$$B(t) = E - \frac{E - B_0}{e^{\kappa t}}. \tag{4}$$

Approaches to writing down a model describing a situation:
1. Remember the derivative is the **rate of change**. It's possible that the description of the problem tells us directly what the rate of change is. Newton's Law of Cooling tells us the rate of change of the body's temperature was proportional to the difference in temperature between the body and the environment. All we had to do was set the relevant terms equal.

2. There are also cases where we are not explicitly given the formula for the rate of change. But we may be able to use the physical description to define the rate of change and then set the derivative equal to that. Note: The derivative = increase - decrease. This type of thinking is only applicable to first order equations since higher order equations are not formulated as rate of change equals something.

3. We may just be adapting a known differential equation to a particular situation, i.e. Newton's Second Law $F = ma$. It is either a first or second order equation depending on if you define it for position for velocity. Combine all forces and plug in value for F to yield the differential equation. Used for falling bodies, harmonic motion, and pendulums.

4. The last possibility is to determine two different expressions for the same quantity and set the equal to derive a differential equation. Useful when discussing PDEs later in the course.

The first thing one must do when approaching a modeling problem is determining which of the four situations we are in. It is crucial to practice this identification now it will be useful on exams and later sections. Secondly, your differential equation should not depend on the initial condition. The IC only tells the starting position and should not effect how a system evolves.

Type I: (Interest)

Suppose there is a bank account that gives $r\%$ interest per year. If I withdraw a constant w dollars per month, what is the differential equation modeling this?

Ans: Let t be time in years, and denote the balance after t years as $B(t)$. $B'(t)$ is the rate of change of my account balance from year to year, so it will be the difference between the amount added and the amount withdrawn. The amount added is interest and the amount withdrawn is $12w$. Thus

$$B'(t) = \frac{r}{100}B(t) - 12w \tag{5}$$

This is a linear equation, so we can solve by integrating factor. Note: UNITS ARE IMPORTANT, w is withdrawn each month, but $12w$ is withdrawn per year.

Example 3. Bill wants to take out a 25 year loan to buy a house. He knows that he can afford maximum monthly payments of \$$400$. If the going interest rate on housing loans is 4%, what is the largest loan Bill can take out so that he will be able to pay it off in time?

Ans: Measure time t in years. The amount Bill owes will be $B(t)$. We want $B(25) = 0$. The

4% interest rate will take the form of $.04B$ added. He can make payments of $12 \times 400 = 4800$ each year. So the IVP will be

$$B'(t) = .04B(t) - 4800, \quad B(25) = 0 \tag{6}$$

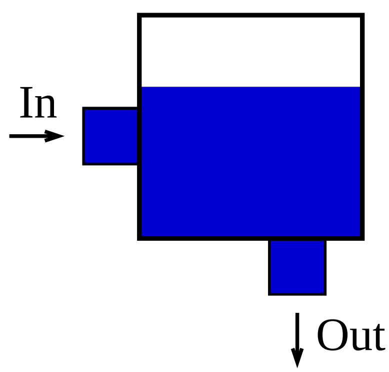
This is a linear equation in standard form, use integrating factor

$$\begin{aligned}
B'(t) - .04B(t) &= -4800 && (7)\\
\mu(t) &= e^{\int -.04dt} = e^{-.04t} && (8)\\
(e^{-\frac{4}{100}t}B(t))' &= -4800e^{-\frac{4}{100}t} && (9)\\
e^{-\frac{4}{100}t}B(t) &= -4800\int e^{-\frac{4}{100}t}dt = 120000e^{-\frac{4}{100}t} + c && (10)\\
B(t) &= 120000 + ce^{\frac{4}{100}t} && (11)\\
B(25) &= 0 = 120000 + ce \Rightarrow c = -120000e^{-1} && (12)\\
B(t) &= 120000 - 120000e^{\frac{4}{100}(t-25)} && (13)
\end{aligned}$$

We want the size of the loan, which is the amount Bill begins with $B(0)$:

$$B(0) = 120000 - 120000e^{-1} = 120000(1 - e^{-1}) \tag{14}$$

Type II: (Mixing Problems)



We have a mixing tank containing some liquid inside. Contaminant is being added to the tank at some constant rate and the mixed solution is drained out at a (possibly different) rate. We will want to find the amount of contaminant in the tank at a given time.
How do we write the DE to model this process? Let $P(t)$ be the amount of pollutant (Note: Amount of pollutant, not the concentration) in the tank at time t. We know the amount of pollutant that is entering and leaving the tank each unit of time. So we can use the second approach

$$\text{Rate of Change of} P(t) = \text{Rate of entry of contaminant} - \text{Rate of exit of contaminant} \tag{15}$$

The **rate of entry** can be defined in different ways. 1. Directly adding contaminant i.e. pipe adding food coloring to water. 2. We might be adding solution with a known concentration of contaminant

to the tank (amount = concentration x volume).
What is the **rate of exit**? Suppose that we are draining the tank at a rate of r_{out}. The amount of contaminant leaving the tank will be the amount contained in the drained solution, that is given by rate x concentration. We know the rate, and we need the concentration. This will just be the concentration of the solution in the tank, which is in turn given by the amount of contaminant in the tank divided by the volume.

$$\text{Rate of exit of contaminant} = \text{Rate of drained solution} \times \frac{\text{Amount of Contaminant}}{\text{Volume of Tank}} \tag{16}$$

or

$$\text{Rate of exit of contaminant} = r_{out}\frac{P(t)}{V(t)}. \tag{17}$$

What is $V(t)$? The Volume is decreasing by r_{out} at each t. Is there anything being added to the volume? That depends if we are adding some solution to the tank at a certain rate r_{in}, that will add to the in-tank volume. If we directly add contaminant not in solution, nothing is added. So determine which situation by reading the problem. In the first case if the initial volume is V_0, we'll get $V(t) = V_0 + t(r_{in} - r_{out})$, and in the second, $V(t) = V_0 - tr_{out}$.

Example 4. Suppose a 120 gallon well-mixed tank initially contains 90 lbs. of salt mixed with 90 gal. of water. Salt water (with a concentration of 2 lb/gal) comes into the tank at a rate of 4 gal/min. The solution flows out of the tank at a rate of 3 gal/min. How much salt is in the tank when it is full?

Ans: We can immediately write down the expression for volume $V(t)$. How much liquid is entering each minute? 4 gallons. How much is leaving the tank in the same minute? 3 gallons. So each minute the Volume increases by 1 gallon, and we have $V(t) = 90 + (4-3)t = 90 + t$. This tells us the tank will be full at $t = 30$.
We let $P(t)$ be the amount of salt (in pounds) in the tank at time t. Ultimately, we want to determine $P(30)$, since this is when the tank will be full. We need to determine the rates at which salt is entering and leaving the tank. How much salt is entering? 4 gallons of salt water enter the tank each minute, and each of those gallons has 2lb. of salt dissolved in it. Hence we are adding 8 lbs. of salt to the tank each minute. How much is exiting the tank? 3 gallons leave each minute, and the concentration in each of those gallons is $P(t)/V(t)$. Recall

$$\text{Rate of Change of} P(t) = \text{Rate of entry of contaminant} - \text{Rate of exit of contaminant} \tag{18}$$

$$\text{Rate of exit of contaminant} = \text{Rate of drained solution} \times \frac{\text{Amount of Contaminant}}{\text{Volume of Tank}} \tag{19}$$

$$\frac{dP}{dt} = (4gal/min)(2lb/gal) - (3gal/min)(\frac{P(t)lb}{V(t)gal}) = 8 - \frac{3P(t)}{90+t} \tag{20}$$

This is the ODE for the salt in the tank, what is the IC? P(0) = 90 as given by the problem. Now we have an IVP so solve (since linear) using integrating factor

$$\begin{aligned}
\frac{dP}{dt} + \frac{3}{90+t}P(t) &= 8 && (21)\\
\mu(t) &= e^{\int \frac{3}{90+t}dt} = e^{3\ln(90+t)} = (90+t)^3 && (22)\\
((90+t)^3 P(t))' &= 8(90+t)^3 && (23)\\
(90+t)^3 P(t) &= \int 8(90+t)^3 dt = 2(90+t)^4 + c && (24)\\
P(t) &= 2(90+t) + \frac{c}{(90+t)^3} && (25)\\
P(0) &= 90 = 2(90) + \frac{c}{90^3} \Rightarrow c = -(90)^4 && (26)\\
P(t) &= 2(90+t) - \frac{90^4}{(90+t)^3} && (27)
\end{aligned}$$

Remember we wanted $P(30)$ which is the amount of salt when the tank is full. So

$$P(30) = 240 - \frac{90^4}{120^3} = 240 - 90(\frac{3}{4})^3 = 240 - 90(\frac{27}{64}). \quad (28)$$

We could ask for amount of salt at anytime before overflow and all would be the same besides last step where we replace 30 with the time wanted.

Exercise: What is the concentration of the tank when the tank is full?

Example 5. A full 20 liter tank has 30 grams of yellow food coloring dissolved in it. If a yellow food coloring solution (with concentration of 2 grams/liter) is piped into the tank at a rate of 3 liters/minute while the well mixed solution is drained out of the tank at a rate of 3 liters/minute, what is the limiting concentration of yellow food coloring solution in the tank?

Ans: The ODE would be

$$\frac{dP}{dt} = (3L/min)(2g/L) - (3L/min)\frac{P(t)g}{V(t)L} = 6 - \frac{3P}{20} \quad (29)$$

Note that volume is constant since we are adding and removing the same amount at each time step. Use the method of integrating factor.

$$\begin{aligned}
\mu(t) &= e^{\int \frac{3}{20}dt} = e^{\frac{3}{20}t} && (30)\\
(e^{\frac{3}{20}t}P(t))' &= 6e^{\frac{3}{20}t} && (31)\\
e^{\frac{3}{20}t}P(t) &= \int 6e^{\frac{3}{20}t}dt = 40e^{\frac{3}{20}t} + c && (32)\\
P(t) &= 40 + \frac{c}{e^{\frac{3}{20}t}} && (33)\\
P(0) &= 20 = 40 + c \Rightarrow c = -20 && (34)\\
P(t) &= 40 - \frac{20}{e^{\frac{3}{20}t}}. && (35)
\end{aligned}$$

Now what will happen to the concentration in the limit, or as $t \to \infty$. We know the volume will always be 20 liters.

$$\lim_{t\to\infty} \frac{P(t)}{V(t)} = \lim_{t\to\infty} \frac{40 - 20e^{-\frac{3}{20}t}}{20} = 2 \tag{36}$$

So the limiting concentration is $2g/L$. Why does this make physical sense? After a period of time the concentration of the mixture will be exactly the same as the concentration of the incoming solution. It turns out that the same process will work if the concentration of the incoming solution is variable.

Example 6. A 150 gallon tank has 60 gallons of water with 5 pounds of salt dissolved in it. Water with a concentration of $2+\cos(t)$ lbs/gal comes into the tank at a rate of 9 gal/hr. If the well mixed solution leaves the tank at a rate of 6 gal/hour, how much salt is in the tank when it overflows?

Ans: The only difference is the incoming concentration is variable. Given the Volume starts at 600 gal and increases at a rate of 3 gal/min

$$\frac{dP}{dt} = 9(2+\cos(t)) - \frac{6P}{60+3t} \tag{37}$$

Our IC is $P(0) = 5$ and use the method of integrating factor

$$\begin{aligned}
\mu(t) &= e^{\int \frac{6}{60+3t}dt} = e^{2\ln(20+t)} = (20+t)^2. && (38)\\
((20+t)^2P(t))' &= 9(2+\cos(t))(20+t)^2 && (39)\\
(20+t)^2P(t) &= \int 9(2+\cos(t))(20+t)^2 dt && (40)\\
&= 9(\frac{2}{3}(20+t)^3 + (20+t)^2\sin(t) + 2(20+t)\cos(t) - 2\sin(t)) + c && (41)\\
P(t) &= 9(\frac{2}{3}(20+t) + \sin(t) + \frac{2\cos(t)}{20+t} - \frac{2\sin(t)}{(20+t)^2}) + \frac{c}{(20+t)^2} && (42)\\
P(0) &= 5 = 9(\frac{2}{3}(20) + \frac{2}{20}) + \frac{c}{400} = 120 + \frac{9}{10} + \frac{c}{400} && (43)\\
c &= -46360 && (44)
\end{aligned}$$

We want to know how much salt is in the tank when it overflows. This happens when the volume hits 150, or at $t = 30$.

$$P(30) = 300 + 9\sin(30) + \frac{18\cos(30)}{50} - \frac{18\sin(30)}{2500} - \frac{46360}{2500} \tag{45}$$

So $P(t) \approx 272.63$ pounds.

We could make the problem more complicated by assuming that there will be a change in the situation if the solution ever reached a critical concentration. The process would still be the same, we would just need to solve two different but limited IVPs.

Type III: (Falling Bodies)
Lets consider an object falling to the ground. This body will obey Newton's Second Law of Motion,

$$m\frac{dv}{dt} = F(t, v) \tag{46}$$

where m is the object's mass and F is the net force acting on the body. We will look at the situation where the only forces are air resistance and gravity. It is crucial to be careful with the signs. Throughout this course **downward displacements and forces are positive**. Hence the force due to gravity is given by $F_G = mg$, where $g \approx 10m/s^2$ is the gravitational constant.
Air Resistance acts against velocity. If the object is moving up air resistance works downward, always in opposite direction. We will assume air resistance is linearly dependant on velocity (ie $F_A = \alpha v$, where F_A is the force due to air resistance). This is not realistic, but it simplifies the problem. So $F(t, v) = F_G + F_A = 10 - \alpha v$, and our ODE is

$$m\frac{dv}{dt} = 10m - \alpha v \tag{47}$$

Example 7. A 50 kg object is shot from a cannon straight up with an initial velocity of 10 m/s off the very tip of a bridge. If the air resistance is given by $5v$, determine the velocity of the mass at any time t and compute the rock's terminal velocity.

Ans: Two parts: 1. When the object is moving upwards and 2. When the object is moving downwards. If we look at the forces it turns out we get the same DE

$$50v' = 500 - 5v \tag{48}$$

The IC is $v(0) = -10$, since we shot the object upwards. Our DE is linear and we can use integrating factor

$$\begin{aligned}
v' + \frac{1}{10}v &= 10 && (49)\\
\mu(t) &= e^{\frac{t}{10}} && (50)\\
(e^{\frac{t}{10}}v(t))' &= 10e^{\frac{t}{10}} && (51)\\
e^{\frac{t}{10}}v(t) &= \int 10e^{\frac{t}{10}}dt = 100e^{\frac{t}{10}} + c && (52)\\
v(t) &= 100 + \frac{c}{e^{\frac{t}{10}}} && (53)\\
v(0) &= -10 = 100 + c \Rightarrow c = -110 && (54)\\
v(t) &= 100 - \frac{110}{e^{\frac{t}{10}}}. && (55)
\end{aligned}$$

What is the terminal velocity of the rock? The terminal velocity is given by the limit of the velocity as $t \to \infty$, which is 100. We could also have computed the velocity of the rock when it hit the ground if we knew the height of the bridge (integrate to get position).

Example 8. A 60kg skydiver jumps out of a plane with no initial velocity. Assuming the magnitude of air resistance is given by $0.8|v|$, what is the appropriate initial value problem modeling his velocity?

Ans: Air Resistance is an upward force, while gravity is acting downward. So our force should be

$$F(t, v) = mg - .8v \tag{56}$$

thus our IVP is

$$60v' = 60g - .8v, \quad v(0) = 0 \tag{57}$$

HW 2.3 # 2, 3, 8, 10, 13, 21