Lecture Notes for Math 251: ODE and PDE. Lecture 7: 2.4 Differences Between Linear and Nonlinear Equations

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1 Existence and Uniqueness

Last Time: We developed 1st Order ODE models for physical systems and solved them using the methods of Integrating Factor and Separable Equations.

In Section 1.3 we noted three common questions we would be concerned with this semester.
1. (Existence) Given an IVP, does a solution exist?
2. (Uniqueness) If a solution exists, is it unique?
3. If a solution exists, how do we find it?

We have spent a lot of time on developing methods, now we will spend time on the first two questions. **Without Solving** an IVP, what information can we derive about the existence and uniqueness of solutions? Also we will note strong differences between linear and nonlinear equations.

1.1 Linear Equations

While we will focus on first order linear equations, the same basic ideas work for higher order linear equations.

Theorem 1. (Fundamental Theorem of Existence and Uniqueness for Linear Equations) *Consider the IVP*

$$y' + p(t)y = q(t), \quad y(t_0) = y_0. \tag{1}$$

If $p(t)$ and $q(t)$ are continuous functions on an open interval $\alpha < t_0 < \beta$, then there exists a unique solution to the IVP defined on the interval (α, β).

REMARK: The same result holds for general IVPs. If we have the IVP

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \dots + a_1(t)y' + a_0(t)y = g(t), \quad y(t_0) = y_0, \dots, y^{(n-1)}(t_0) = y_0^{(n-1)} \tag{2}$$

then if $a_i(t)$ (for $i = 0, ..., n - 1$) and $g(t)$ are continuous on an open interval $\alpha < t_0 < \beta$, there exists a unique solution to the IVP defined on the interval (α, β).

What does Theorem 1 tell us?
(1) If the given linear differential equation is nice, not only do we know **EXACTLY ONE** solution exists. In most applications knowing a solution is unique is more important than knowing a solution exists.
(2) If the interval (α, β) is the largest interval on which $p(t)$ and $q(t)$ are continuous, then (α, β) is the interval of validity to the unique solution guaranteed by the theorem. Thus given a "nice" IVP there is no need to solve the equation to find the interval of validity. The interval only depends on t_0 since the interval must contain it, but does not depend on y_0.

Example 2. Without solving, determine the interval of validity for the solution to the following IVP

$$(t^2 - 9)y' + 2y = \ln|20 - 4t|, \quad y(4) = -3 \tag{3}$$

Ans: If we look at Theorem 1, we need to write our equation in the form given in Theorem 1 (i.e. coefficient of y' is 1). So rewrite as

$$y' + \frac{2}{t^2 - 9} = \frac{\ln|20 - 4t|}{t^2 - 9} \tag{4}$$

Next we identify where either of the two other coefficients are discontinuous. By removing those points we find all intervals of validity. Then the last step is to identify which interval of validity contains t_0.

Using the notation in Theorem 1, $p(t)$ is discontinuous when $t = \pm 3$, since at those points we are dividing by zero. $q(t)$ is discontinuous at $t = 5$, since the natural log of 0 does not exists (only defined on $(0, \infty)$). This yields four intervals of validity where both $p(t)$ and $q(t)$ are continuous

$$(-\infty, -3), \quad (-3, 3), \quad (3, 5), \quad (5, \infty) \tag{5}$$

Notice the endpoints are where $p(t)$ and $q(t)$ are discontinuous, guaranteeing within each interval **both** are continuous. Now all that is left is to identify which interval contains $t_0 = 4$. Thus our interval of validity is $(3, 5)$.

REMARK: The other intervals of validity we found are intervals of validity for the same differential equation, but for different initial conditions. For example, if our IC was $y(2) = 5$ then the interval of validity must contain 2, so the answer would be $(-3, 3)$.

What happens if our IC is at one of the bad points where $p(t)$ and $q(t)$ are discontinuous? Unfortunately we are unable to conclude anything, since the theorem does not apply. On the other hand we cannot say that a solution does not exist just because the hypothesis are not met, so the bottom line is that we cannot conclude anything.

Example 3. Without solving, find the interval of validity for the following IVP

$$\cos(x)y' = \sin(x)y - \sqrt{x - 1}, \quad y(\frac{3}{2}) = 0 \tag{6}$$

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First we need to put the equation in the form of Theorem 1

$$y' - \tan(x)y = -\frac{\sqrt{x-1}}{\cos(x)} \tag{7}$$

Using the notation in Theorem 1, $p(t)$ is discontinuous at $x = \frac{n\pi}{2}$ for odd integers n and $q(t)$ is discontinuous there and for any $x < 1$. Thus we can list the possible intervals of validity

$$(1, \frac{\pi}{2}), (\frac{\pi}{2}, \frac{3\pi}{2}), ..., (\frac{(2n+1)}{2}, \frac{(2n+3)\pi}{2}) \tag{8}$$

for all positive integers n. Since the IC is $y(\frac{3}{2}) = 0$, then the I.O.V. must contain $\frac{3}{2}$. Therefore the answer is $(1, \frac{\pi}{2})$.

1.2   Nonlinear Equations

We saw in the linear case every "nice enough" equation has a unique solution except for if the initial conditions are ill-posed. But even this seemingly simple nonlinear equation

$$(\frac{dt}{dx})^2 + x^2 + 1 = 0 \tag{9}$$

has no real solutions.

So we have the following revision of Theorem 1 that applies to nonlinear equations as well. Since this is applied to a broader class the conclusions are expected to be weaker.

Theorem 4. *Consider the IVP*

$$y' = f(t, y), \quad y(t_0) = y_0. \tag{10}$$

If f and $\frac{\partial f}{\partial y}$ are continuous functions on some rectangle $\alpha < t_0 < \beta, \gamma < y_0 < \delta$ containing the point (t_0, y_0), then there is a unique solution to the IVP defined on some interval (a, b) satisfying $\alpha < a < t_0 < b \leq \beta$.

OBSERVATION:
(1) Unlike Theorem 1, Theorem 2 does not tell us the interval of a unique solution guaranteed by it. Instead, it tells us the largest possible interval that the solution will exist in, we would need to actually solve the IVP to get the interval of validity.
(2) For nonlinear differential equations, the value of y_0 may affect the interval of validity, as we will see in a later example. We want our IC to NOT lie on the boundary of a region where f or its partial derivative are discontinuous. Then we find the largest t-interval on the line $y = y_0$ containing t_0 where everything is continuous.

REMARK: Theorem 2 refers to **partial derivative** $\frac{\partial f}{\partial y}$ of the function of two variables $f(t, y)$. We will talk extensively about this later, but for now we treat t as a constant and take a normal derivative with respect to y. For example

$$f(t, y) = t^2 - 2y^3t, \quad \text{then} \quad \frac{\partial f}{\partial y} = -6y^2t. \tag{11}$$

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Example 5. Determine the largest possible interval of validity for the IVP

$$y' = x \ln(y), \quad y(2) = e \tag{12}$$

We have $f(x, y) = x \ln(y)$, so $\frac{\partial f}{\partial y} = \frac{x}{y}$. f is discontinuous when $y \leq 0$, and f_y (partial derivative with respect to y) is discontinuous when $y = 0$. Since our IC $y(2) = e > 0$ there is no problem since y_0 is never in the discontinuous region. Since there are no discontinuities involving x, then the rectangle is $-\infty < x_0 < \infty, 0 < y_0 < \infty$. Thus the theorem concludes that the unique solution exists somewhere inside $(-\infty, \infty)$.

REMARK: Note that this basically told us nothing, and nonlinear problems are quite harder to deal with than linear.

Example 6. Determine the largest possible interval of validity for the IVP

$$y' = \sqrt{y - t^2}, \quad y(0) = 1. \tag{13}$$

$f(t, y) = \sqrt{y - t^2}$ and $f_y = \frac{1}{2\sqrt{y-t^2}}$. The region of discontinuities is given by $y \leq t^2$. Our IC is $y(0) = 1$ does not lie in this region, so we can continue. The line $y = 1$ is continuous for $-1 < t < 1$, so our conclusion is that the interval of validity of the guaranteed unique solution is contained somewhere within $(-1, 1)$.

What can happen if the conditions of Theorem 2 are NOT met?

Example 7. Determine all possible solutions to the IVP

$$y' = y^{\frac{1}{3}}, \quad y(0) = 0. \tag{14}$$

First note this does not satisfy the conditions of the theorem, since $f_y = \frac{1}{3y^{2/3}}$ is not continuous at $y_0 = y = 0$. Now solve the equation it is separable. Notice the equilibrium solution is $y = 0$. This satisfies the IC, but let's solve the equation.

$$\int y^{-1/3} dy = \int dt \tag{15}$$

$$\frac{3}{2} y^{2/3} = t + c \tag{16}$$

$$y(0) = 0 \tag{17}$$

$$y(t) = \pm \left(\frac{2}{3} t\right)^{\frac{3}{2}} \tag{18}$$

$$\tag{19}$$

The IC does not rule out either of these possibilities, so we end up with three possible solutions (these two and the equilibrium solution $y(t) \equiv 0$).

In our class we will be mostly dealing with nice equations and unique solutions, but be aware this is not always the case. Consider the next example which illustrates the dependence of the interval of validity on y_0.

Example 8. Determine the interval of validity for the IVP

$$y' = y^2, \quad y(0) = y_0 \tag{20}$$

First notice its nonlinear so Theorem 1 does not apply. y^2 is continuous everywhere, so for every y_0 there will be a unique solution. It is defined somewhere in $(-\infty, \infty)$. So let's solve. Notice first the equilibrium solution if $y_0 = 0$ we have $y \equiv 0$. So assume $y_0 \neq 0$.

$$\int \frac{1}{y^2} dy = \int dt \tag{21}$$

$$-\frac{1}{y} = t + c \tag{22}$$

$$c = -\frac{1}{y_0} \tag{23}$$

$$-\frac{1}{y} = t - \frac{1}{y_0} \tag{24}$$

$$y(t) = \frac{y_0}{1 - y_0 t} \tag{25}$$

What is the interval of validity? The only point of discontinuity is $t = \frac{1}{y_0}$. So the two possible intervals of validity are

$$\left(-\infty, \frac{1}{y_0}\right), \quad \left(\frac{1}{y_0}, \infty\right) \tag{26}$$

The correct choice will be the interval containing $t_0 = 0$. But this will depend on y_0. If $y_0 > 0$, 0 will be contained in the interval $(-\infty, \frac{1}{y_0})$ and so this is the interval of validity. On the other hand, if $y_0 < 0$, 0 is contained inside $(\frac{1}{y_0}, \infty)$ and so this is the interval of validity. Thus we have the following possible intervals of validity, depending on y_0.
(1) If $y_0 > 0$, $(-\infty, \frac{1}{y_0})$ is the interval of validity
(2) If $y_0 = 0$, $(-\infty, \infty)$ is the interval of validity
(3) If $y_0 < 0$, $(-\frac{1}{y_0}, \infty)$ is the interval of validity

1.3 Summary

We established conditions for existence and uniqueness of solutions to first order ODEs. Intervals of validity for linear equations do not depend on the initial choice of y_0, while nonlinear equations may. Secondly, we can find intervals of validity for solutions for linear equations without having to solve the equation. For a nonlinear equation, we would need to solve the equation to get the actual interval of validity. But we can still find all places where the interval of validity definitely will not be defined.

HW 2.4 # 1, 3, 5, 7, 10