Lecture Notes for Math 251: ODE and PDE. Lecture 8: 2.5 Autonomous Equations and Population Dynamics

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1 Autonomous Equations with Population Dynamics

Last Time: We focused on the differences between linear and nonlinear equations as well as identifying intervals of validity without solving any initial value problems (IVP).

1.1 Autonomous Equations

First order differential equations relate the slope of a function to the values of the function and the independent variable. We can visualize this using direction fields. This in principle can be very complicated and it might be hard to determine which initial values correspond to which outcomes. However, there is a special class of equations, called **autonomous equations**, where this process is simplified. The first thing to note is autonomous equations do not depend on t

$$y' = f(y) \tag{1}$$

REMARK: Notice that all autonomous equations are separable.

What we need to know to study the equation qualitatively is which values of y make y' zero, positive, or negative. The values of y making $y' = 0$ are the **equilibrium solutions**. They are constant solutions and are indicated on the ty-plane by horizontal lines.

After we establish the equilibrium solutions we can study the positivity of $f(y)$ on the intermediate intervals, which will tell us whether the equilibrium solutions attract nearby initial conditions (in which case they are called **asymptotically stable**), repel them (**unstable**), or some combination of them (**semi-stable**).

Example 1. Consider

$$y' = y^2 - y - 2 \tag{2}$$

Start by finding the equilibrium solutions, values of y such that $y' = 0$. In this case we need to solve $y^2 - y - 2 = (y-2)(y+1) = 0$. So the equilibrium solutions are $y = -1$ and $y = 2$. There are constant solutions and indicated by horizontal lines. We want to understand their stability. If we plot $y^2 - y - 2$ versus y, we can see that on the interval $(-\infty, -1), f(y) > 0$. On the interval $(-1, 2), f(y) < 0$ and on $(2, \infty), f(y) > 0$. Now consider the initial condition.

(1) If the IC $y(t_0) = y_0 < -1, y' = f(y) > 0$ and $y(t)$ will increase towards -1.
(2) If the IC $-1 < y_0 < 2, y' = f(y) < 0$, so the solution will decrease towards -1. Since the solutions below -1 go to -1 and the solutions above -1 go to -1, we conclude $y(t) = -1$ is an asymptotically stable equilibrium.
(3) If $y_0 > 2, y' = f(y) > 0$, so the solution increases away from 2. So at $y(t) = 2$ above and below solutions move away so this is an unstable equilibrium.

Example 2. Consider

$$y' = (y - 4)(y + 1)^2 \tag{3}$$

The equilibrium solutions are $y = -1$ and $y = 4$. To classify them, we graph $f(y) = (y - 4)(y + 1)^2$.
(1) If $y < -1$, we can see that $f(y) < 0$, so solutions starting below -1 will tend towards $-\infty$.
(2) If $-1 < y_0 < 4, f(y) < 0$, so solutions starting here tend downwards to -1. So $y(t) = 1$ is semistable.
(3) If $y > 4, f(y) > 0$, solutions starting above 4 will asymptotically increase to ∞, so $y(t) = 4$ is unstable since no nearby solutions converge to it.

1.2 Populations

The best examples of autonomous equations come from population dynamics. The most naive model is the ”Population Bomb” since it grows without any deaths

$$P'(t) = rP(t) \tag{4}$$

with $r > 0$. The solution to this differential equation is $P(t) = P_0 e^{rt}$, which indicates that the population would increase exponentially to ∞. This is not realistic at all.

A better and more accurate model is the ”Logistic Model”

$$P'(t) = rP(1 - \frac{P}{N}) = rP - \frac{r}{N}P^2 \tag{5}$$

where $N > 0$ is some constant. With this model we have a birth rate of rP and a mortality rate of

$\frac{r}{N}P^2$. The equation is separable so let's solve it.

$$\begin{aligned}
\frac{dP}{P(1-\frac{P}{N})} &= rdt && (6)\\
\int(\frac{1}{P}+\frac{1/N}{1-P/N})dP &= \int rdt && (7)\\
\ln|P|-\ln|1-\frac{P}{N}| &= rt+c && (8)\\
\frac{P}{1-\frac{P}{N}} &= Ae^{rt} && (9)\\
P &= Ae^{rt}=\frac{1}{N}Ae^{rt}P && (10)\\
P(t) &= \frac{Ae^{rt}}{1+\frac{A}{N}e^{rt}}=\frac{AN}{Ne^{-rt}+A} && (11)
\end{aligned}$$

if $P(0)=P_0$, then $A=\frac{P_0N}{N-P_0}$ to yield

$$P(t)=\frac{P_0N}{(N-P_0)e^{-rt}+P_0} \tag{12}$$

In its present form its hard to analyze what is going on so let's apply the methods from the first section to analyze the stability.

Looking at the logistic equation, we can see that our equilibrium solutions are $P=0$ and $P=N$. Graphing $f(P)=rP(1-\frac{N}{P})$, we see that

(1) If $P<0, f(P)<0$

(2) If $0<P<N, f(P)>0$

(3) If $P>N, f(P)<0$

Thus 0 is unstable while while N is asymptotically stable, so we can conclude for initial population $P_0>0$

$$\lim_{t\to\infty}P(t)=N \tag{13}$$

So what is N? It is the carrying capacity for the environment. If the population exists, it will grow towards N, but the closer it gets to N the slower the population will grow. If the population starts off greater then the carrying capacity for the environment $P_0>N$, then the population will die off until it reaches that stable equilibrium position. And if the population starts off at N, the births and deaths will balance out perfectly and the population will remain exactly at $P_0=N$.

Note: It is possible to construct similar models that have unstable equilibria above 0.

EXERCISE: Show that the equilibrium population $P(t)=N$ is unstable for the autonomous equation

$$P'(t)=rP(\frac{P}{N}-1). \tag{14}$$

HW 2.5 # 1, 3, 4, 9, 10, 12, 22