Lecture Notes for Math 251: ODE and PDE. Lecture 9: 2.6 Exact Equations

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Spring 2012

1 Exact Equations

Last Time: We solved problems involving population dynamics, plotted phase portraits, and determined the stability of equilibrium solutions.

The final category of first order differential equations we will consider are **Exact Equations**. These nonlinear equations have the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{1}$$

where $y = y(x)$ is a function of x and find the

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{2}$$

where these two derivatives are **partial derivatives**.

1.1 Multivariable Differentiation

If we want a partial derivative of $f(x,y)$ with respect to x we treat y as a constant and differentiate normally with respect to x. On the other hand, if we want a partial derivative of $f(x,y)$ with respect to y we treat x as a constant and differentiate normally with respect to y.

Example 1. Let $f(x,y) = x^2y = y^2$. Then

$$\frac{\partial f}{\partial x} = 2xy \tag{3}$$

$$\frac{\partial f}{\partial y} = x^2 + 2y. \tag{4}$$

Example 2. Let $f(x,y) = y\sin(x)$

$$\frac{\partial f}{\partial x} = y\cos(y) \tag{5}$$

$$\frac{\partial f}{\partial y} = \sin(x) \tag{6}$$

We also need the crucial tool of the multivariable chain rule. If we have a function $\Phi(x, y(x))$ depending on some variable x and a function y depending on x, then

$$\frac{d\Phi}{dx} = \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{dy}{dx} = \Phi_x + \Phi_y y' \tag{7}$$

1.2 Exact Equations

Start with an example to illustrate the method.

Example 3. Consider

$$2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0 \tag{8}$$

The first step in solving an exact equation is to find a certain function $\Phi(x, y)$. Finding $\Phi(x, y)$ is most of the work. For this example it turns out

$$\Phi(x, y) = y^2 + (x^2 + 1)y - 3x^3 \tag{9}$$

Notice if we compute the partial derivatives of Φ, we obtain

$$\Phi_x(x, y) = 2xy - 9x^2 \tag{10}$$

$$\Phi_y(x, y) = 2y + x^2 + 1. \tag{11}$$

Looking back at the differential equation, we can rewrite it as

$$\Phi_x + \Phi_y\frac{dy}{dx} = 0. \tag{12}$$

Thinking back to the chain rule we can express as

$$\frac{d\Phi}{dx} = 0 \tag{13}$$

Thus if we integrate, $\Phi = c$, where c is a constant. So the general solution is

$$y^2 + (x^2 + 1)y - 3x^3 = c \tag{14}$$

for some constant c. If we had an initial condition, we could use it to find the particular solution to the initial value problem.

Let's investigate the last example further. An exact equation has the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{15}$$

with $M_y(x, y) = N_x(x, y)$. The key is to construct $\Phi(x, y)$ such that the DE turns into

$$\frac{d\Phi}{dx} = 0 \tag{16}$$

by using the multivariable chain rule. Thus we require $\Phi(x, y)$ satisfy

$$\Phi_x(x,y) = M(x,y) \tag{17}$$
$$\Phi_y(x,y) = N(x,y) \tag{18}$$

REMARK: A standard fact from multivariable calculus is that mixed partial derivatives commute. That is why we want $M_y = N_x$, so $M_y = \Phi_{xy}$ and $N_x = \Phi_{yx}$, and so these should be equal for Φ to exist. Make sure you check the function is exact before wasting time on the wrong solution process.

Once we have found Φ, then $\frac{d\Phi}{dx} = 0$, and so

$$\Phi(x,y) = c \tag{19}$$

yielding an implicit general solution to the differential equation.

So the majority of the work is computing $\Phi(x, y)$. How can we find this desired function, let's retry Example 3, filling in the details.

Example 4. Solve the initial value problem

$$2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0, \quad y(0) = 2 \tag{20}$$

Let's begin by checking the equation is in fact exact.

$$M(x,y) = 2xy - 9x^2 \tag{21}$$
$$N(x,y) = 2y + x^2 + 1 \tag{22}$$
$$\tag{23}$$

Then $M_y = 2x = N_x$, so the equation is exact.

Now how do we find $\Phi(x, y)$? We have $\Phi_x = M$ and $\Phi_y = N$. Thus we could compute Φ in one of two ways

$$\Phi(x,y) = \int M dx \quad \text{or} \quad \Phi(x,y) = \int N dy. \tag{24}$$

In general it does not usually matter which you choose, one may be easier to integrate than the other. In this case

$$\Phi(x,y) = \int 2xy - 9x^2 dx = x^2y - 3x^3 + h(y). \tag{25}$$

Notice since we only integrate with respect to x we can have an arbitrary function only depending on y. If we differentiate $h(y)$ with respect to x we still get 0 like an arbitrary constant c. So in order to have the highest accuracy we take on an arbitrary function of y. Note if we integrated N with respect to y we would get an arbitrary function of x. DO NOT FORGET THIS!

Now all we need is to find $h(y)$. We know if we differentiate Φ with respect to x, then $h(y)$ will vanish which is unhelpful. So instead differentiate with respect to y, since $\Phi_y = N$ in order to be exact. so any terms in N that aren't in Φ_y must be $h'(y)$.

So $\Phi_y = x^2 + h'(y)$ and $N = x^2 + 2y + 1$. Since these are equal we have $h'(y) = 2y + 1$, an so

$$h(y) = \int h'(y)dy = y^2 + y \tag{26}$$

REMARK: We will drop the constant of integration we get from integrating h since it will combine with the constant c that we get in the solution process.

Thus, we have

$$\Phi(x, y) = x^2 y - 3x^3 + y^2 + y = y^2 + (x^2 + 1)y - 3x^3, \tag{27}$$

which is precisely the Φ that we used in Example 3. Observe

$$\frac{d\Phi}{dx} = 0 \tag{28}$$

and thus $\Phi(x, y) = y^2 + (x^2 + 1)y - 3x^3 = c$ for some constant c. To compute c, we'll use our initial condition $y(0) = 2$

$$2^2 + 2 = c \Rightarrow c = 6 \tag{29}$$

and so we have a particular solution of

$$y^2 + (x^2 + 1)y - 3x^3 = 6 \tag{30}$$

This is a quadratic equation in y, so we can complete the square or use quadratic formula to get an explicit solution, which is the goal when possible.

$$\begin{aligned}
y^2 + (x^2 + 1)y - 3x^3 &= 6 && (31)\\
y^2 + (x^2 + 1)y + \frac{(x^2 + 1)^2}{4} &= 6 + 3x^3 + \frac{(x^2 + 1)^2}{4} && (32)\\
(y + \frac{x^2 + 1}{2})^2 &= \frac{x^4 + 12x^3 + 2x^2 + 25}{4} && (33)\\
y(x) &= \frac{-(x^2 + 1) \pm \sqrt{x^4 + 12x^3 + 2x^2 + 25}}{2} && (34)
\end{aligned}$$

Now we use the initial condition to figure out whether we want the $+$ or $-$ solution. Since $y(0) = 2$ we have

$$2 = y(0) = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2} = 2, -3 \tag{35}$$

Thus we see we want the $+$ so our particular solution is

$$y(x) = \frac{-(x^2 + 1) + \sqrt{x^4 + 12x^3 + 2x^2 + 25}}{2} \tag{36}$$

Example 5. Solve the initial value problem

$$2xy^2 + 2 = 2(3 - x^2 y)y', \quad y(-1) = 1. \tag{37}$$

First we need to put it in the standard form for exact equations

$$2xy^2 + 2 - 2(3 - x^2y)y' = 0. \tag{38}$$

Now, $M(x,y) = 2xy^2 + 2$ and $N(x,y) = -2(3 - x^2y)$. So $M_y = 4xy = N_x$ and the equation is exact.

The next step is to compute $\Phi(x,y)$. We choose to integrate N this time

$$\Phi(x,y) = \int N dy = \int 2x^2y - 6dy = x^2y^2 - 6y + h(x). \tag{39}$$

To find $h(x)$, we compute $\Phi_x = 2xy^2 + h'(x)$ and notice that for this to be equal to M, $h'(x) = 2$. Hence $h(x) = 2x$ and we have an implicit solution of

$$x^2y^2 - 6y + 2x = c. \tag{40}$$

Now, we use the IC $y(-1) = 1$:

$$1 - 6 - 2 = c \Rightarrow c = -7 \tag{41}$$

So our implicit solution is

$$x^2y^2 - 6y + 2x + 7 = 0. \tag{42}$$

Again complete the square or use quadratic formula

$$\begin{aligned} y(x) &= \frac{6 \pm \sqrt{36 - 4x^2(2x+7)}}{2x^2} && (43) \\ &= \frac{3 \pm \sqrt{9 - 2x^3 - 7x^2}}{x^2} && (44) \end{aligned}$$

and using the IC, we see that we want $-$ solution, so the explicit particular solution is

$$y(x) = \frac{3 - \sqrt{9 - 2x^3 - 7x^2}}{x^2} \tag{45}$$

Example 6. Solve the IVP

$$\frac{2ty}{t^2+1} - 2t - (4 - \ln(t^2+1))y' = 0, \quad y(2) = 0 \tag{46}$$

and find the solution's interval of validity.

This is already in the right form. Check if it is exact, $M(t,y) = \frac{2ty}{t^2+1} - 2t$ and $N(t,y) = \ln(t^2+1) - 4$, so $M_y = \frac{2t}{t^2+1} = N_t$. Thus the equation is exact. Now compute $\Phi(x,y)$. Integrate M

$$\begin{aligned} \Phi = \int M dt &= \int \frac{2ty}{t^2+1} dt = y\ln(t^2+1) - t^2 + h(y). && (47) \\ \Phi_y &= \ln(t^2+1) + h'(y) = \ln(t^2+1) - 4 = N && (48) \end{aligned}$$

so we conclude $h'(y) = -4$ and thus $h(y) = -4y$. So our implicit solution is then

$$y \ln(t^2 + 1) - t^2 - 4y = c \tag{49}$$

and using the IC we find $c = -4$. Thus the particular solution is

$$y \ln(t^2 + 1) - t^2 - 4y = -4 \tag{50}$$

Solve explicitly to obtain

$$y(x) = \frac{t^2 - 4}{\ln(t^2 + 1) - 4}. \tag{51}$$

Now let's find the interval of validity. We do not have to worry about the natural log since $t^2 + 1 > 0$ for all t. Thus we want to avoid division by 0.

$$\begin{aligned} \ln(t^2 + 1) - 4 &= 0 && (52)\\ \ln(t^2 + 1) &= 4 && (53)\\ t^2 &= e^4 - 1 && (54)\\ t &= \pm\sqrt{e^4 - 1} && (55) \end{aligned}$$

So there are three possible intervals of validity, we want the one containing $t = 2$, so $(-\sqrt{e^4 - 1}, \sqrt{e^4 - 1})$.

Example 7. Solve

$$3y^3 e^{3xy} - 1 + (2ye^{3xy} + 3xy^2 e^{3xy})y' = 0, \quad y(1) = 2 \tag{56}$$

We have

$$M_y = 9y^2 e^{3xy} + 9xy^3 e^{3xy} = N_x \tag{57}$$

Thus the equation is exact. Integrate M

$$\Phi = \int M dx = \int 3y^3 e^{3xy} - 1 = y^2 e^{3xy} - x + h(y) \tag{58}$$

and

$$\Phi_y = 2ye^{3xy} + 3xy^2 e^{3xy} + h'(y) \tag{59}$$

Comparing Φ_y to N, we see that they are already identical, so $h'(y) = 0$ and $h(y) = 0$. So

$$y^2 e^{3xy} - x = c \tag{60}$$

and using the IC gives $c = 4e^6 - 1$. Thus our implicit particular solution is

$$y^2 e^{3xy} - x = 4e^6 - 1, \tag{61}$$

and we are done because we will not be able to solve this explicitly.

HW 2.6 # 1, 3, 9, 11, 15