Lecture Notes for Math 251: ODE and PDE. Lecture 10: 3.1 Homogeneous Equations with Constant Coefficients

Shawn D. Ryan

Spring 2012

1 Second Order Linear Differential Equations

Last Time: We studied exact equations, which were our last type of first order differential equations and the method for solving them. Now we start Chapter 3: Second Order Linear Equations.

1.1 Basic Concepts

The example of a second order equation which we have seen many times before is Newton's Second Law when expressed in terms of position $s(t)$ is

$$m\frac{d^2s}{dt^2} = F(t, s', s) \tag{1}$$

One of the most basic 2nd order equations is $y'' = -y$. By inspection, we might notice that this has two obvious nonzero solutions: $y_1(t) = \cos(t)$ and $y_2(t) = \sin(t)$. But consider $9\cos(t) - 2\sin(t)$? This is also a solution. Anything of the form $y(t) = c_1\cos(t) + c_2\sin(t)$, where c_1 and c_2 are arbitrary constants. Every solution if no conditions are present has this form.

Example 1. Find all of the solutions to $y'' = 9y$

We need a function whose second derivative is 9 times the original function. What function comes back to itself without a sign change after two derivatives? Always think of the exponential function when situations like this arise. Two possible solutions are $y_1(t) = e^{3t}$ and $y_2(t) = e^{-3t}$. In fact so are any combination of the two. This is the principal of **linear superposition**. So $y(t) = c_1e^{3t} + c_2e^{-3t}$ are infinitely many solutions.

EXERCISE: Check that $y_1(t) = e^{3t}$ and $y_2(t) = e^{-3t}$ are solutions to $y'' = 9y$.

The general form of a second order linear differential equation is

$$p(t)y'' + q(t)y' + r(t)y = g(t). \tag{2}$$

We call the equation **homogeneous** if $g(t) = 0$ and **nonhomogeneous** if $g(t) \neq 0$.

Theorem 2. (Principle of Superposition) *If $y_1(t)$ and $y_2(t)$ are solutions to a second order linear homogeneous differential equation, then so is any linear combination*

$$y(t) = c_1 y_1(t) + c_2 y_2(t). \tag{3}$$

This follows from the homogeneity and the fact that a derivative is a linear operator. So given any two solutions to a homogeneous equation we can find infinitely more by combining them. The main goal is to be able to write down a general solution to a differential equation, so that with some initial conditions we could uniquely solve an IVP. We want to find $y_1(t)$ and $y_2(t)$ so that the general solution to the differential equation is $y(t) = c_1 y_1(t) + c_2 y_2(t)$. By different we mean solutions which are not constant multiples of each other.

Now reconsider $y'' = -y$. We found two different solutions $y_1(t) = \cos(t)$ and $y_2(t) = \sin(t)$ and any solution to this equation can be written as a linear combination of these two solutions, $y(t) = c_1 \cos(t) + c_2 \sin(t)$. Since we have two constants and a 2nd order equation we need two initial conditions to find a particular solution. We are generally given these conditions in the form of y and y' defined at a particular t_0. So a typical problem might look like

$$p(t)y'' + q(t)y' + r(t)y = 0, \quad y'(t_0) = y_0', \quad y(t_0) = y_0 \tag{4}$$

Example 3. Find a particular solution to the initial value problem

$$y'' + y = 0, \quad y(0) = 2, \quad y'(0) = -1 \tag{5}$$

We have established the general solution to this equation is

$$y(t) = c_1 \cos(t) + c_2 \sin(t) \tag{6}$$

To apply the initial conditions, we'll need to know the derivative

$$y'(t) = -c_1 \sin(t) + c_2 \cos(t) \tag{7}$$

Plugging in the initial conditions yields

$$2 = c_1 \tag{8}$$
$$-1 = c_2 \tag{9}$$

so the particular solution is

$$y(t) = 2\cos(t) - \sin(t). \tag{10}$$

Sometimes when applying initial conditions we will have to solve a system of equations, other times it is as easy as the previous example.

1.2 Homogeneous Equations With Constant Coefficients

We will start with the easiest class of second order linear homogeneous equations, where the coefficients $p(t)$, $q(t)$, and $r(t)$ are constants. The equation will have the form

$$ay'' + by' + c = 0. \tag{11}$$

How do we find solutions to this equation? From calculus we can find a function that is linked to its derivatives by a multiplicative constant, $y(t) = e^{rt}$. Now that we have a candidate plug it into the differential equation. First calculate the derivatives $y'(t) = re^{rt}$ and $y''(t) = r^2e^{rt}$.

$$a(r^2e^{rt}) + b(re^{rt}) + ce^{rt} = 0 \tag{12}$$
$$e^{rt}(ar^2 + br + c) = 0 \tag{13}$$

What can we conclude? If $y(t) = e^{rt}$ is a solution to the differential equation, then $e^{rt}(ar^2 + br + c) = 0$. Since $e^{rt} \neq 0$, then $y(t) = e^{rt}$ will solve the differential equation as long as r is a solution to

$$ar^2 + br + c = 0. \tag{14}$$

This equation is called the **characteristic equation** for $ay'' + by' + c = 0$.

Thus, to find a solution to a linear second order homogeneous constant coefficient equation, we begin by writing down the characteristic equation. Then we find the roots r_1 and r_2 (not necessarily distinct or real). So we have the solutions

$$y_1(t) = e^{r_1 t}, \quad y_2(t) = e^{r_2 t}. \tag{15}$$

Of course, it is also possible these are the same, since we might have a repeated root. We will see in a future section how to handle these. In fact, we have three cases.

Example 4. Find two solutions to the differential equation $y'' - 9y = 0$ (Example 1). The characteristic equation is $r^2 - 9 = 0$, and this has roots $r = \pm 3$. So we have two solutions $y_1(t) = e^{3t}$ and $y_2(t) = e^{-3t}$, which agree with what we found earlier.

The three cases are the same as the three possibilities for types of roots of quadratic equations:
(1) Real, distinct roots $r_1 \neq r_2$.
(2) Complex roots $r_1, r_2 = \alpha \pm \beta i$.
(3) A repeated real root $r_1 = r_2 = r$.
We'll look at each case more closely in the lectures to come.

HW 3.1 # 1, 3, 5, 8, 10, 12, 15, 16, 20, 23, 24