Lecture Notes for Math 251: ODE and PDE. Lecture 11: 3.2 Fundamental Solutions of Linear Homogeneous Equations

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1 Solutions of Linear Homogeneous Equations and the Wronskian

Last Time: We studied linear homogeneous equations, the principle of linear superposition, and the characteristic equation.

1.1 Existence and Uniqueness

Given an initial value problem involving a linear second order equation, when does a solution exist? We had a theroem in the previous chapter for the first order case so the following theorem will cover second order equations.

Theorem 1. *Consider the initial value problem*

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0'. \tag{1}$$

If $p(t)$, $q(t)$, and $g(t)$ are all continuous on some interval (a, b) such that $a < t_0 < b$, then the initial value problem has a unique solution defined on (a, b).

1.2 Wronskian

Let's suppose we are given the initial value problem

$$p(t)y'' + q(t)y' + r(t)y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0' \tag{2}$$

and that we know two solutions $y_1(t)$ and $y_2(t)$. Since the differential equation is linear and homogeneous, the Principle of Superposition says that any linear combination

$$y(t) = c_1y_1(t) + c_2y_2(t) \tag{3}$$

is also a solution. When is this the general solution? For this to be the case it must satisfy its initial conditions. As long as t_0 does not make any of the coefficient discontinuous, Theorem 1 says $y(t)$ meeting the initial conditions is the general solution. Start by differentiating our candidate $y(t)$ and using the initial conditions

$$\begin{aligned} y_0 = y(t_0) &= c_1 y_1(t_0) + c_2 y_2(t_0) && (4) \\ y_0' = y'(t_0) &= c_1 y_1'(t_0) + c_2 y_2'(t_0) && (5) \end{aligned}$$

Solve this system of equations to get

$$c_1 = \frac{y_0 - c_2 y_2(t_0)}{y_1(t_0)} \tag{6}$$

Thus

$$\begin{aligned} y_0' &= \frac{y_0 y_1'(t_0) - c_2 y_2(t_0) y_1'(t_0)}{y_1(t_0)} + c_2 y_2'(t_0) && (7) \\ &= \frac{y_0 y_1'(t_0) - c_2 y_2(t_0) y_1'(t_0) + c_2 y_2'(t_0) y_1(t_0)}{y_1(t_0)} && (8) \end{aligned}$$

and we compute

$$\begin{aligned} c_2 &= \frac{y_0' y_1(t_0) - y_0 y_1'(t_0)}{y_1(t_0) y_2'(t_0) - y_2(t_0) y_1'(t_0)} && (9) \\ c_1 &= \frac{y_0' y_2(t_0) - y_0 y_2'(t_0)}{y_1(t_0) y_2'(t_0) - y_2(t_0) y_1'(t_0)} && (10) \\ & && (11) \end{aligned}$$

Notice that c_1 and c_2 have the same quantity in their denominators, so the only time we can solve for c_1 and c_2 is when this quantity is NOT zero.

Definition 2. The quantity

$$W(y_1, y_2)(t_0) = y_1(t_0) y_2'(t_0) - y_2(t_0) y_1'(t_0) \tag{12}$$

is called the **Wronskian** of y_1 and y_2 at t_0.

REMARK:
(1) When it's clear what the two functions are, we will often denote the Wronskian by W.
(2) We can think of the Wronskian, $W(y_1, y_2)(t)$, as a function of t and can be evaluated at any t where y_1 and y_2 are defined. For any two solutions satisfying the initial conditions we need the Wronskian $W(y_1, y_2)$ to be nonzero at any value t_0 where Theorem 1 applies.
(3) We could have solved the system of equations for $y(t_0)$ and $y'(t_0)$ by Cramer's Rule from Linear Algebra and we have the following formula for the Wronskian

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}. \tag{13}$$

We will generally represent the Wronskian as a determinant.

Two solutions will form the general solution if they satisfy the general initial conditions. The above computation showed that this will be the case so long as

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0) \neq 0 \tag{14}$$

If $y_1(t)$ and $y_2(t)$ are solutions to our second order equation and $W(y_1, y_2) \neq 0$, then the two solutions are said to be a **fundamental set of solutions** and the general solution is

$$y(t) = c_1 y_1(t) + c_2 y_2(t). \tag{15}$$

In other words, two solutions are ”different” enough to form a general solution if they are a fundamental set of solutions.

Example 3. If r_1 and r_2 are distinct real roots of the characteristic equation for $ay'' + by' + cy = 0$, check that

$$y_1(t) = e^{r_1 t} \quad \text{and} \quad y_2(t) = e^{r_2 t} \tag{16}$$

form a fundamental set of solutions.

To show this, we compute the Wronskian

$$W = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = r_2 e^{(r_1 + r_2)t} - r_1 e^{(r_2 + r_1)} = (r_2 - r_1) e^{(r_2 + r_1)t} \tag{17}$$

Since the exponentials are never zero and $r_2 \neq r_1$, we conclude that $W \neq 0$ and so as claimed y_1 and y_2 form a fundamental set of solutions for the differential equation and the general solution is

$$y(t) = c_1 y_1(t) + c_2 y_2(t). \tag{18}$$

Example 4. Consider

$$2t^2 y'' + ty' - 3y = 0. \tag{19}$$

given that $y_1(t) = t^{-1}$ is a solution. Show $y_2(t) = t^{3/2}$ form a fundamental set of solutions. To do this, we compute the Wronskian

$$W = \begin{vmatrix} t^{-1} & t^{3/2} \\ -t^{-2} & \frac{3}{2} t^{1/2} \end{vmatrix} = \frac{3}{2} t^{-\frac{1}{2}} + t^{-\frac{1}{2}} = \frac{5}{2\sqrt{t}} \tag{20}$$

Thus $W \neq 0$, so they are a fundamental set of solutions. Notice we cannot plug in $t = 0$, but this is OK since we cannot plug $t = 0$ into the solution anyway since it would make the coefficients in standard for discontinuous. So the general solution is

$$y(t) = c_1 t^{-1} + c_2 t^{\frac{3}{2}} \tag{21}$$

Example 5. Consider

$$t^2y'' + 2ty' - 2y = 0. \tag{22}$$

We are given that $y_1(t) = t$ is a solution and want to test $y_2(t) = t^{-2}$ as our other solution. Check the Wronskian

$$W = \begin{vmatrix} t & t^{-2} \\ 1 & -2t^{-3} \end{vmatrix} = -2t^{-2} - t^{-2} = -3t^{-2} \neq 0. \tag{23}$$

So the solutions are a fundamental set of solutions, and the general solution is

$$y(t) = c_1t + c_2t^{-2} \tag{24}$$

The last question is how we know if a fundamental set of solutions will exist for a given differential equation. The following theorem has the answer.

Theorem 6. *Consider the differential equation*

$$y'' + p(t)y' + q(t) = 0 \tag{25}$$

where $p(t)$ and $q(t)$ are continuous on some interval (a, b). Suppose $a < t_0 < b$. If $y_1(t)$ is a solution satisfying the initial conditions

$$y(t_0) = 1, \quad y'(t_0) = 0 \tag{26}$$

and $y_2(t)$ is a solution satisfying

$$y(t_0) = 0, \quad y'(t_0) = 1, \tag{27}$$

then $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions.

We cannot use this to compute our fundamental set of solutions, but the importance is it assures us that as long as $p(t)$ and $q(t)$ are continuous, then a fundamental set of solutions will exist.

1.3 Linear Independence

Consider two functions $f(t)$ and $g(t)$ and the equation

$$c_1f(t) + c_2g(t) = 0. \tag{28}$$

Notice that $c_1 = 0$ and $c_2 = 0$ always solve this equation, regardless of what f and g are.

Definition 7. If there are nonzero constants c_1 and c_2 such that the above equation is satisfied for all t, then f and g are said to be **linearly dependent**. On the other hand, if the only constants for which the equation holds are $c_1 = c_2 = 0$, then f and g are said to be **linearly independent**.

REMARK: Two functions are linearly dependent when they are constant multiples of each other. So there are nonzero c_1 and c_2 such that

$$f(t) = -\frac{c_2}{c_1}g(t) \tag{29}$$

Example 8. Determine if the following pairs of functions are linearly dependent or independent.
(1) $f(x) = 9\cos(2x), g(x) = 2\cos^2(x) - 2\sin^2(x)$
(2) $f(t) = 2t^2, g(t) = t^4$

(1) Consider
$$9c_1\cos(2x) + 2c_2(\cos^2(x) - \sin^2(x)) = 0. \tag{30}$$
We want to determine if there are nonzero constants c_1 and c_2 so this equation is true. Note the trig identity $\cos(2x) = \cos^2(x) - \sin^2(x)$. So our equation becomes

$$\begin{aligned} 9c_1\cos(2x) + 2c_2\cos(2x) &= 0 & (31)\\ (9c_1 + 2c_2)\cos(2x) &= 0 & (32)\end{aligned}$$

This equation is true for $c_1 = 2$ and $c_2 = -9$, thus f and g are linearly dependent.
(2) Consider
$$2c_1t^2 + c_2t^4 = 0 \tag{33}$$
If this is true differentiate both sides and it will still be true

$$4c_1t + 4c_2t^3 = 0 \tag{34}$$

Solve for c_1 and c_2. The second equation tells us $c_1 = -c_2t^2$. Plug into the first equation to get $-c_2t^4 = 0$, which is only true when $c_2 = 0$. If $c_2 = 0$, then $c_1 = 0$, so f and g are linearly independent.

This can be involved and sometimes it is unclear how to proceed. The Wronskian helps identify when two functions are linearly independent.

Theorem 9. *Given two functions $f(t)$ and $g(t)$ which are differentiable on some interval (a, b),*
(1) If $W(f, g)(t_0) \neq 0$ for some $a < t_0 < b$, then $f(t)$ and $g(t)$ are linearly independent on (a, b) and
(2) If $f(t)$ and $g(t)$ are linearly dependent on (a, b), then $W(f, g)(t) = 0$ for all $a < t < b$.

REMARK: BE CAREFUL, this theorem DOES NOT say that if $W(f, g)(x) = 0$ then f and g are linearly dependent. It's possible for two linearly independent functions to have a zero Wronskian.

Let's use the theorem to check an earlier example.

Example 10. (1) $f(t) = 9\cos(2x), g(x) = 2\cos^2(x) - 2\sin^2(x)$.

$$\begin{aligned} W &= \begin{vmatrix} 9\cos(2x) & 2\cos^2(x) - 2\sin^2(x) \\ -18\sin(2x) & -4\cos(x)\sin(x) - 4\cos(x)\sin(x) \end{vmatrix} & (35)\\ &= \begin{vmatrix} 9\cos(2x) & 2\cos(2x) \\ -18\sin(2x) & -4\sin(2x) \end{vmatrix} & (36)\\ &= -36\cos(2x)\sin(2x) + 36\cos(2x)\sin(2x) = 0 & (37)\end{aligned}$$

We get zero which we expected since the two functions are linearly dependent.
(2) Now let's take $f(t) = 2t^2$ and $g(t) = t^4$.

$$W = \begin{vmatrix} 2t^2 & t^4 \\ 4t & 4t^3 \end{vmatrix} \tag{38}$$

$$= 8t^5 - 4t^5 = 4t^5 \tag{39}$$

The Wronskian will be nonzero so long as $t = 0$, which is OK, we just do not want it to be zero for all t.

1.4 More On The Wronskian

We have established when the Wronskian is nonzero the two functions are linearly independent. We also have seen when y_1 and y_2 are solutions to the linear homogeneous equation

$$p(t)y'' + q(t)y' + r(t)y = 0 \tag{40}$$

$W(y_1, y_2)(t) \neq 0$ is precisely the condition for the general solution of the differential equation to be

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \tag{41}$$

where y_1 and y_2 form a fundamental set of solutions.

1.5 Abel's Theorem

Through the discussion of the Wronskian we have yet to use the differential equation. If y_1 and y_2 are solutions to a linear homogeneous equation we can say more about the Wronskian.

Theorem 11. *Suppose $y_1(t)$ and $y_2(t)$ solve the linear homogeneous equation*

$$y''(t) + p(t)y' + q(t)y = 0, \tag{42}$$

where $p(t)$ and $q(t)$ are continuous on some interval (a, b). Then, for $a < t < b$, their Wronskian is given by

$$W(y_1, y_2)(t) = W(y_1, y_2)(t_0)e^{-\int_{t_0}^{t} p(x)dx}, \tag{43}$$

where t_0 is in (a, b).

If $W(y_1, y_2)(t_0) \neq 0$ at some point t_0 in the interval (a, b), then Abel's Theorem tell us that the Wronskian can't be zero for any t in (a, b), since exponentials are never zero. We can thus change our initial data without worry that our general solution will change.

Another advantage to Abel's Theorem is that it lets us compute the general form of the Wronskian of any two solutions to the differential equation without knowing them explicitly. The formulation in the theorem is not computationally useful, but we might not have a precise t_0 in mind. But applying the Fundamental Theorem of Calculus

$$W(y_1, y_2)(t) = W(y_1, y_2)(t_0)e^{-\int_{t_0}^{t} p(x)dx} = ce^{-\int p(t)dt}. \tag{44}$$

What is c? We only care that $c \neq 0$.

Example 12. Compute, up to a constant, the Wronskian of two solutions y_1 and y_2 of the differential equation

$$t^4y'' - 2t^3y' - t^3y = 0. \tag{45}$$

First we put the equation in the form of Abel's Theorem.

$$y'' - \frac{2}{t}y' - t^4y = 0 \tag{46}$$

So, Abel's Theorem tells us

$$W = ce^{-\int -\frac{2}{t}dt} = ce^{2\ln(t)} = ct^2 \tag{47}$$

The main reason this is important is it is an alternative way to compute the Wronskian. We know by Abel's Theorem

$$W(y_1, y_2)(t) = ce^{-\int p(t)dt}. \tag{48}$$

On the other hand, by definition

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t) \tag{49}$$

Setting these equal, if we know one solution $y_1(t)$, we're left with a first order differential equation for y_2 that we can then solve.

Example 13. Suppose we want to find a general solution to $2t^2y'' + ty' - 3y = 0$ and we're given that $y_1(t) = t^{-1}$ is a solution. We need to find a second solution that will form a fundamental set of solutions with y_1. Let's compute the Wronskian both ways.

$$\begin{aligned} ce^{-\int \frac{1}{2t}dt} &= W(t^{-1}, y_2)(t) = y_2't^{-1} + y_2t^{-2} && (50) \\ y_2't^{-1} + y_2t^{-2} &= ce^{-\frac{1}{2}\ln(t)} = ct^{-\frac{1}{2}} && (51) \end{aligned}$$

This is a first order linear equation with integrating factor $\mu(t) = e^{\int t^{-1}dt} = e^{\ln(t)} = t$. Thus

$$\begin{aligned} (ty_2)' &= ct^{\frac{3}{2}} && (52) \\ ty_2 &= \frac{2}{5}ct^{\frac{5}{2}} + k && (53) \\ y_2(t) &= \frac{2}{5}ct^{\frac{3}{2}} + kt^{-1} && (54) \end{aligned}$$

Now we can choose constants c and k. Notice that k is the coefficient of t^{-1}, which is just $y_1(t)$. So we do not have to worry about that term, and we can take $k = 0$. We can similarly take $c = \frac{5}{2}$, and so we'll get $y_2(t) = t^{\frac{3}{2}}$.

HW 3.2 # 1, 4, 7, 11, 18, 23, 25