Lecture Notes for Math 251: ODE and PDE. Lecture 12: 3.3 Complex Roots of the Characteristic Equation

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1 Complex Roots of the Characteristic Equation

Last Time: We considered the Wronskian and used it to determine when we have solutions to a second order linear equation or if given one solution we can find another which is linearly independent.

1.1 Review Real, Distinct Roots

Recall that a second order linear homogeneous differential equation with constant coefficients

$$ay'' + by' + cy = 0 \tag{1}$$

is solved by $y(t) = e^{rt}$, where r solves the **characteristic equation**

$$ar^2 + br + c = 0 \tag{2}$$

So when there are two distinct roots $r_1 \neq r_2$, we get two solutions $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$. Since they are distinct we can immediately conclude the general solution is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \tag{3}$$

Then given initial conditions we can solve c_1 and c_2.

Exercises:

(1) $y'' + 3y' - 18y = 0, \quad y(0) = 0, \quad y'(0) = -1.$

ANS: $y(t) = \frac{1}{9}e^{-6t} - \frac{1}{9}e^{3t}$.

(2) $y'' - 7y' + 10y = 0, \quad y(0) = 3, \quad y(0) = 2$

ANS: $y(t) = -\frac{4}{3}e^{5t} + \frac{13}{3}e^{2t}$

(3) $2y'' - 5y' + 2y = 0, \quad y(0) = -3, \quad y'(0) = 3$

ANS: $y(t) = -6e^{\frac{1}{2}t} + 3e^{2t}$.

(4) $y'' + 5y' = 0, \quad y(0) = 2, \quad y'(0) = -5$

ANS: $y(t) = 1 + e^{-5t}$

(5) $y'' - 2y' - 8 = 0, \quad y(2) = 1, \quad y'(2) = 0$
ANS: $y(t) = \frac{1}{3e^8}e^{4t} + \frac{2e^4}{3}e^{-2t}$
(6) $y'' + y' - 3y = 0$
ANS: $y(t) = c_1 e^{\frac{-1+\sqrt{13}}{2}t} + c_2 e^{\frac{-1-\sqrt{13}}{2}t}$.

1.2 Complex Roots

Now suppose the characteristic equation has complex roots of the form $r_{1,2} = \alpha \pm i\beta$. This means we have two solutions to our differential equation

$$y_1(t) = e^{(\alpha+i\beta)t}, \quad y_2(t) = e^{(\alpha-i\beta)t} \tag{4}$$

This is a problem since $y_1(t)$ and $y_2(t)$ are complex-valued. Since our original equation was both simple and had real coefficients, it would be ideal to find two real-valued ”different” enough solutions so that we can form a real-valued general solution. There is a way to do this.

Theorem 1. (Euler's Formula)

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{5}$$

In other words, we can write an imaginary exponential as a sum of $\sin$ and $\cos$. How do we establish this fact? There are two ways:

(1) **Differential Equations**: First we want to write $e^{i\theta} = f(\theta) + ig(\theta)$. We also have

$$f' + ig' = \frac{d}{d\theta}[e^{i\theta}] = ie^{i\theta} = if - g. \tag{6}$$

Thus $f' = -g$ and $g' = f$, so $f'' = -f$ and $g'' = -g$. Since $e^0 = 1$, we know that $f(0) = 1$ and $g(0) = 0$. We conclude that $f(\theta) = \cos(\theta)$ and $g(\theta) = \sin(\theta)$, so

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{7}$$

(2) **Taylor Series**: Recall that the Taylor series for e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \tag{8}$$

while the Taylor series for $\sin(x)$ and $\cos(x)$ are

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... \tag{9}$$

$$\cos(x) = \sum_{n-0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... \tag{10}$$

(11)

If we set $x = i\theta$ in the first series, we get

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} && (12)\\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - ... && (13)\\ &= (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ...) + i(\theta - \frac{\theta^3}{3!} + \frac{i\theta^5}{5!} - ...) && (14)\\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n!)} + i \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} && (15)\\ &= \cos(\theta) + i\sin(\theta) && (16) \end{aligned}$$

So we can write our two complex exponentials as

$$\begin{aligned} e^{(\alpha+i\beta)t} &= e^{\alpha t} e^{i\beta t} = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t)) && (17)\\ e^{(\alpha-i\beta)t} &= e^{\alpha t} e^{-i\beta t} = e^{\alpha t}(\cos(\beta t) - i\sin(\beta t)) && (18) \end{aligned}$$

where the minus sign pops out of the sign in the second equation since $\sin$ is odd and $\cos$ is even. Notice our new expression is still complex-valued. However, by the Principle of Superposition, we can obtain the following solutions

$$\begin{aligned} y_1(t) &= \frac{1}{2}(e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))) + \frac{1}{2}(e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))) = e^{\alpha t}\cos(\beta t) && (19)\\ y_2(t) &= \frac{1}{2i}(e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))) - \frac{1}{2i}(e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))) = e^{\alpha t}\sin(\beta t) && (20) \end{aligned}$$

EXERCISE: Check that $y_1(t) = e^{\alpha t}\cos(\beta t)$ and $y_2(t) = e^{\alpha t}\sin(\beta t)$ are in fact solutions to the beginning differential equation when the roots are $\alpha \pm i\beta$.

So now we have two real-valued solutions $y_1(t)$ and $y_2(t)$. It turns out they are linearly independent, so if the roots of the characteristic equation are $r_{1,2} = \alpha \pm i\beta$, we have the general solution

$$y(t) = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t) \qquad (21)$$

Let's consider some examples:

Example 2. Solve the IVP

$$y'' - 4y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = -2 \qquad (22)$$

The characteristic equation is

$$r^2 - 4r + 9 = 0 \qquad (23)$$

which has roots $r_{1,2} = 2 \pm i\sqrt{5}$. Thus the general solution and its derivatives are

$$\begin{aligned} y(t) &= c_1 e^{2t}\cos(\sqrt{5}t) + c_2 e^{2t}\sin(\sqrt{5}t) && (24)\\ y'(t) &= 2c_1 e^{2t}\cos(\sqrt{5}t) - \sqrt{5}c_1 e^{2t}\sin(\sqrt{5}t) + 2c_2 e^{2t}\sin(\sqrt{5}t) + \sqrt{5}c_2 e^{2t}\cos(\sqrt{5}t). && (25) \end{aligned}$$

If we apply the initial conditions, we get

$$\begin{aligned} 0 &= c_1 \quad (26) \\ -2 &= 2c_1 + \sqrt{5}c_2 \quad (27) \end{aligned}$$

which is solved by $c_1 = 0$ and $c_2 = -\frac{2}{\sqrt{5}}$. So the particular solution is

$$y(t) = -\frac{2}{\sqrt{5}}e^{2t}\sin(\sqrt{5}t). \quad (28)$$

Example 3. Solve the IVP

$$y'' - 8y' + 17y = 0, \quad y(0) = 2, \quad y'(0) = 5. \quad (29)$$

The characteristic equation is

$$r^2 - 8r + 17 = 0 \quad (30)$$

which has roots $r_{1,2} = 4 \pm i$. Hence the general solution and its derivatives are

$$\begin{aligned} y(t) &= c_1e^{4t}\cos(t) = c_2e^{4t}\sin(t) \quad (31) \\ y'(t) &= 4c_1e^{4t}\cos(t) - c_1e^{4t}\sin(t) + 4c_2e^{4t}\sin(t) + c_2e^{4t}\cos(t) \quad (32) \end{aligned}$$

and plugging in initial conditions yields the system

$$\begin{aligned} 2 &= c_1 \quad (33) \\ 5 &= 4c_1 + c_2 \quad (34) \end{aligned}$$

so we conclude $c_1 = 2$ and $c_2 = -3$ and the particular solution is

$$y(t) = 2e^{4t}\cos(t) - 3e^{4t}\sin(t) \quad (35)$$

Example 4. Solve the IVP

$$4y'' + 12y' + 10y = 0, \quad y(0) = -1, \quad y'(0) = 3 \quad (36)$$

The characteristic equation is

$$4r^2 + 12r + 10 = 0 \quad (37)$$

which has roots $r_{1,2} = -\frac{3}{2} \pm \frac{1}{2}i$. So the general solution and its derivative are

$$\begin{aligned} y(t) &= c_1e^{\frac{3}{2}t}\cos(\frac{t}{2}) + c_2e^{\frac{3}{2}t}\sin(\frac{t}{2}) \quad (38) \\ y'(t) &= \frac{3}{2}c_1e^{\frac{3}{2}t}\cos(\frac{t}{2}) - \frac{1}{2}c_1e^{\frac{3}{2}t}\sin(\frac{t}{2}) + \frac{3}{2}c_2e^{\frac{3}{2}t}\sin(\frac{t}{2}) + \frac{1}{2}c_2e^{\frac{3}{2}t}\cos(\frac{t}{2}) \quad (39) \end{aligned}$$

Plugging in the initial condition yields

$$\begin{aligned} -1 &= c_1 \quad (40) \\ 3 &= \frac{3}{2}c_1 + \frac{1}{2}c_2 \quad (41) \end{aligned}$$

which has solution $c_1 = -1$ and $c_2 = 9$. The particular solution is

$$y(t) = -e^{\frac{3}{2}t}\cos(\frac{t}{2}) + 9e^{\frac{3}{2}t}\sin(\frac{t}{2}) \quad (42)$$

Example 5. Solve the IVP

$$y'' + 4y = 0, \quad y(\frac{\pi}{4}) = -10, \quad y'(\frac{\pi}{4}) = 4. \tag{43}$$

The characteristic equation is

$$r^2 + 4 = 0 \tag{44}$$

which has roots $r_{1,2} = \pm 2i$. The general solution and its derivatives are

$$y(t) = c_1 \cos(2t) + c_2 \sin(2t) \tag{45}$$

$$y'(t) = -2c_1 \sin(2t) + 2c_2 \cos(2t). \tag{46}$$

The initial conditions give the system

$$-10 = c_2 \tag{47}$$

$$4 = -2c_1 \tag{48}$$

so we conclude that $c_1 = -2$ and $c_2 = -10$ and the particular solution is

$$y(t) = -2\cos(2t) - 10\sin(2t). \tag{49}$$

HW 3.3 # 1, 4, 14, 15, 18, 19