Lecture Notes for Math 251: ODE and PDE. Lecture 13: 3.4 Repeated Roots and Reduction Of Order

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Spring 2012

1 Repeated Roots of the Characteristic Equation and Reduction of Order

Last Time: We considered cases of homogeneous second order equations where the roots of the characteristic equation were complex.

1.1 Repeated Roots

The last case of the characteristic equation to consider is when the characteristic equation has repeated roots $r_1 = r_2 = r$. This is a problem since our usual solution method produces the same solution twice

$$y_1(t) = e^{r_1 t} = e^{r_2 t} = y_2(t) \tag{1}$$

But these are the same and are not linearly independent. So we will need to find a second solution which is "different" from $y_1(t) = e^{rt}$. What should we do?

Start by recalling that if the quadratic equation $ar^2 + br + c = 0$ has a repeated root r, it must be $r = -\frac{b}{2a}$. Thus our solution is $y_1(t) = e^{-\frac{b}{2a}}$. We know any constant multiple of $y_1(t)$ is also a solution. These will still be linearly dependent to $y_1(t)$. Can we find a solution of the form

$$y_2(t) = v(t)y_1(t) = v(t)e^{-\frac{b}{2a}t} \tag{2}$$

i.e. y_2 is the product of a function of t and y_1.

Differentiate $y_2(t)$:

$$\begin{aligned}
y_2'(t) &= v'(t)e^{-\frac{b}{2a}t} - \frac{b}{2a}v(t)e^{-\frac{b}{2a}t} && (3)\\
y_2''(t) &= v''(t)e^{-\frac{b}{2a}} - \frac{b}{2a}v'(t)e^{-\frac{b}{2a}t} - \frac{b}{2a}v'(t)e^{-\frac{b}{2a}t} + \frac{b^2}{4a^2}v(t)e^{-\frac{b}{2a}t} && (4)\\
&= v''(t)e^{-\frac{b}{2a}t} - \frac{b}{a}v'(t)e^{-\frac{b}{2a}t} + \frac{b^2}{4a^2}v(t)e^{-\frac{b}{2a}t}. && (5)
\end{aligned}$$

Plug in differential equation:

$$\begin{aligned} a(v''e^{-\frac{b}{2a}t} - \frac{b}{a}v'e^{-\frac{b}{2a}t} + \frac{b^2}{4a^2}ve^{-\frac{b}{2a}t}) + b(v'e^{-\frac{b}{2a}t} - \frac{b}{2a}ve^{-\frac{b}{2a}t}) + c(ve^{-\frac{b}{2a}t}) &= 0 \quad (6) \\ e^{-\frac{b}{2a}t}\big(av'' + (-b+b)v' + (\frac{b^2}{4a} - \frac{b^2}{2a} + c)v\big) &= 0 \quad (7) \\ e^{-\frac{b}{2a}t}\big(av'' - \frac{1}{4a}(b^2 - 4ac)v\big) &= 0 \quad (8) \end{aligned}$$

Since we are in the repeated root case, we know the discriminant $b^2 - 4ac = 0$. Since exponentials are never zero, we have

$$av'' = 0 \Rightarrow v'' = 0 \quad (9)$$

We can drop the a since it cannot be zero, if a were zero it would be a first order equation! So what does v look like

$$v(t) = c_1 t + c_2 \quad (10)$$

for constants c_1 and c_2. Thus for any such $v(t)$, $y_2(t) = v(t)e^{-\frac{b}{2a}t}$ will be a solution. The most general possible $v(t)$ that will work for us is $c_1 t + c_2$. Take $c_1 = 1$ and $c_2 = 0$ to get a specific $v(t)$ and our second solution is

$$y_2(t) = te^{-\frac{b}{2a}t} \quad (11)$$

and the general solution is

$$y(t) = c_1 e^{-\frac{b}{2a}t} + c_2 te^{-\frac{b}{2a}t} \quad (12)$$

REMARK: Here's another way of looking at the choice of constants. Suppose we do not make a choice. Then we have the general solution

$$\begin{aligned} y(t) &= c_1 e^{-\frac{b}{2a}t} + c_2(ct+k)e^{-\frac{b}{2a}t} \quad (13) \\ &= c_1 e^{-\frac{b}{2a}t} + c_2 cte^{-\frac{b}{2a}t} + c_2 ke^{-\frac{b}{2a}t} \quad (14) \\ &= (c_1 + c_2 k)e^{-\frac{b}{2a}t} + c_2 cte^{-\frac{b}{2a}t} \quad (15) \end{aligned}$$

since they are all constants we just get

$$y(t) = c_1 e^{-\frac{b}{2a}t} + c_2 te^{-\frac{b}{2a}t} \quad (16)$$

To summarize: if the characteristic equation has repeated roots $r_1 = r_2 = r$, the general solution is

$$y(t) = c_1 e^{rt} + c_2 te^{rt} \quad (17)$$

Now for examples:

Example 1. Solve the IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = -1, \quad y'(0) = 6 \quad (18)$$

The characteristic equation is

$$
\begin{aligned}
r^2 - 4r + 4 &= 0 && (19)\\
(r-2)^2 = 0 & && (20)
\end{aligned}
$$

so we see that we have a repeated root $r = 2$. The general solution and its derivative are

$$
\begin{aligned}
y(t) &= c_1 e^{2t} + c_2 t e^{2t} && (21)\\
y'(t) &= 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t e^{2t} && (22)
\end{aligned}
$$

and plugging in initial conditions yields

$$
\begin{aligned}
-1 &= c_1 && (23)\\
6 &= 2c_1 + c_2 && (24)
\end{aligned}
$$

so we have $c_1 = -1$ and $c_2 = 8$. The particular solution is

$$y(t) = -e^{2t} + 6te^{2t} \qquad (25)$$

Example 2. Solve the IVP

$$16y'' + 40y' + 25y = 0, \quad y(0) = -1, \quad y'(0) = 2. \qquad (26)$$

The characteristic equation is

$$
\begin{aligned}
16r^2 + 40r + 25 &= 0 && (27)\\
(4r+5)^2 &= 0 && (28)
\end{aligned}
$$

and so we conclude that we have a repeated root $r = -\frac{5}{4}$ and the general solution and its derivative are

$$
\begin{aligned}
y(t) &= c_1 e^{-\frac{5}{4}t} + c_2 t e^{-\frac{5}{4}t} && (29)\\
y'(t) &= -\frac{5}{4}c_1 e^{-\frac{5}{4}t} + c_2 e^{-\frac{5}{4}t} - \frac{5}{4}c_2 t e^{-\frac{5}{4}t} && (30)
\end{aligned}
$$

Plugging in the initial conditions yields

$$
\begin{aligned}
-1 &= c_1 && (31)\\
2 &= -\frac{5}{4}c_1 + c_2 && (32)
\end{aligned}
$$

so $c_1 = -1$ and $c_2 = \frac{5}{4}$. The particular solution is

$$y(t) = -e^{-\frac{5}{4}t} + \frac{3}{4}te^{-\frac{5}{4}t} \qquad (33)$$

1.2 Reduction of Order

We have spent the last few lectures analyzing second order linear homogeneous equations with constant coefficients, i.e. equations of the form

$$ay'' + by' + cy = 0 \tag{34}$$

Let's now consider the case when the coefficients are not constants

$$p(t)y'' + q(t)y' + r(t)y = 0 \tag{35}$$

In general this is not easy, but if we can guess a solution, we can use the techniques developed in the repeated roots section to find another solution. This method will be called **Reduction Of Order**. Consider a few examples

Example 3. Find the general solution to

$$2t^2y'' + ty' - 3y = 0 \tag{36}$$

given that $y_1(t) = t^{-1}$ is a solution.

ANS: Think back to repeated roots. We know we had a solution $y_1(t)$ and needed to find a distinct solution. What did we do? We asked which nonconstant function $v(t)$ make $y_2(t) = v(t)y_1(t)$ is also a solution. The y_2 derivatives are

$$\begin{aligned} y_2 &= vt^{-1} && (37)\\ y_2' &= v't^{-1} - vt^{-2} && (38)\\ y_2'' &= v''t^{-1} - v't^{-2} + 2vt^{-3} = v''t^{-1} - 2v't^{-2} + 2vt^{-3} && (39) \end{aligned}$$

The next step is to plug into the original equation so we can solve for v:

$$\begin{aligned} 2t^2(v''t^{-1} - 2v't^{-2} + 2vt^{-3}) + t(v't^{-1} - vt^{-2}) - 3vt^{-1} &= 0 && (40)\\ 2v''t - 4v' + 4vt^{-1} + v' - vt^{-1} - 3vt^{-1} &= 0 && (41)\\ 2tv'' - 3v' &= 0 && (42) \end{aligned}$$

Notice that the only terms left involve v'' and v', not v. This also happened in the repeated root case. The v term should always disappear at this point, so we have a check on our work. If there is a v term left we have done something wrong.

Now we know that if y_2 is a solution, the function v must satisfy

$$2tv'' - 3v' = 0 \tag{43}$$

But this is a second order linear homogeneous equation with nonconstant coefficients. Let $w(t) = v'(t)$. By changing variables our equation becomes

$$w' - \frac{3}{2t}w = 0. \tag{44}$$

So by Integrating Factor

$$\begin{aligned}
\mu(t) &= e^{\int -\frac{3}{2t}dt} = e^{-\frac{3}{2}\ln(t)} = t^{-\frac{3}{2}} &(45)\\
(t^{-\frac{3}{2}}w)' &= 0 &(46)\\
t^{-\frac{3}{2}}w &= c &(47)\\
w(t) &= ct^{\frac{3}{2}} &(48)
\end{aligned}$$

So we know what $w(t)$ must solve the equation. But to solve our original differential equation, we do not need $w(t)$, we need $v(t)$. Since $v'(t) = w(t)$, integrating w will give our v

$$\begin{aligned}
v(t) &= \int w(t)dt &(49)\\
&= \int ct^{\frac{3}{2}t}dt &(50)\\
&= \frac{2}{5}ct^{\frac{5}{2}} + k &(51)
\end{aligned}$$

Now this is the general form of $v(t)$. Pick $c = 5/2$ and $k = 0$. Then $v(t) = t^{\frac{5}{2}}$, so $y_2(t) = v(t)y_1(t) = t^{\frac{3}{2}}$, and the general solution is

$$y(t) = c_1t^{-1} + c_2t^{\frac{3}{2}} \tag{52}$$

Reduction of Order is a powerful method for finding a second solution to a differential equation when we do not have any other method, but we need to have a solution to begin with. Sometimes even finding the first solution is difficult.

We have to be careful with these problems sometimes the algebra is tedious and one can make sloppy mistakes. Make sure the v terms disappears when we plug in the derivatives for y_2 and check the solution we obtain in the end in case there was an algebra mistake made in the solution process.

Example 4. Find the general solution to

$$t^2y'' + 2ty' - 2y = 0 \tag{53}$$

given that

$$y_1(t) = t \tag{54}$$

is a solution.

Start by setting $y_2(t) = v(t)y_1(t)$. So we have

$$\begin{aligned}
y_2 &= tv &(55)\\
y_2' &= tv' + v &(56)\\
y_2'' &= tv'' + v' + v' = tv'' + 2v'. &(57)
\end{aligned}$$

Next, we plug in and arrange terms

$$\begin{aligned} t^2(tv'' + 2v') + 2t(tv' + v) - 2tv &= 0 && (58) \\ t^3v'' + 2t^2v' + 2t^2v' + 2tv - 2tv &= 0 && (59) \\ t^3v'' + 4t^2v' &= 0. && (60) \end{aligned}$$

Notice the v drops out as desired. We make the change of variables $w(t) = v'(t)$ to obtain

$$t^3w' + 4t^2w = 0 \quad (61)$$

which has integrating factor $\mu(t) = t^4$.

$$\begin{aligned} (t^4w)' &= 0 && (62) \\ t^4w &= c && (63) \\ w(t) &= ct^{-4} && (64) \end{aligned}$$

So we have

$$\begin{aligned} v(t) &= \int w(t)dt && (65) \\ &= \int ct^{-4}dt && (66) \\ &= -\frac{c}{3}t^{-3} + k. && (67) \end{aligned}$$

A nice choice for the constants is $c = -3$ and $k = 0$, so $v(t) = t^{-3}$, which gives a second solution of $y_2(t) = v(t)y_1(t) = t^{-2}$. So our general solution is

$$y(t) = c_1t + c_2t^{-2} \quad (68)$$

HW 3.4 # 7, 13, 18, 20, 23