Lecture Notes for Math 251: ODE and PDE. Lecture 14: 3.5 Nonhomogeneous Equations; Method of Undetermined Coefficients

Shawn D. Ryan

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1 Nonhomogeneous Equations with Constant Coefficients

Last Time: We considered cases of homogeneous second order equations where the roots of the characteristic equation were repeated real roots. Then we looked at the method of reduction of order to produce a second solution to an equation given the first solution.

1.1 Nonhomogeneous Equations

A second order nonhomogeneous equation has the form

$$p(t)y'' + q(t)y' + r(t)y = g(t) \tag{1}$$

where $g(t) \neq 0$. How do we get the general solution to these?

Suppose we have two solutions $Y_1(t)$ and $Y_2(t)$. The Principle of Superposition no longer holds for nonhomogeneous equations. We cannot just take a linear combination of the two to get another solution. Consider the equation

$$p(t)y'' + q(t)y' + r(t)y = 0 \tag{2}$$

which we will call the **associated homogeneous equation**.

Theorem 1. *Suppose that $Y_1(t)$ and $Y_2(t)$ are two solutions to equation* (1) *and that $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions to* (2)*. Then $Y_1(t) - Y_2(t)$ is a solution to Equation* (2) *and has the form*

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t) \tag{3}$$

Notice the notation used, it will be standard. Uppercase letters are solutions to the nonhomogeneous equation and lower case letters to denote solutions to the homogeneous equation.

Let's verify the theorem by plugging in $Y_1 - Y_2$ to (2)

$$\begin{aligned} p(t)(Y_1 - Y_2)'' + q(t)(Y_1 - Y_2)' + r(t)(Y_1 - Y_2) &= 0 && (4) \\ \big(p(t)Y_1'' + q(t)Y_1' + r(t)Y_1\big) - \big(p(t)Y_2'' + q(t)Y_2' + r(t)Y_2\big) &= 0 && (5) \\ g(t) - g(t) &= 0 && (6) \\ 0 &= 0 && (7) \end{aligned}$$

So we have that $Y_1(t) - Y_2(t)$ solves equation (2). We know that $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions to equation (2) and so any solution can be written as a linear combination of them. Thus for constants c_1 and c_2

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t) \tag{8}$$

So the difference of any two solutions of (1) is a solution to (2). Suppose we have a solution to (1), which we denote by $Y_p(t)$. Let $Y(t)$ denote the general solution. We have seen

$$Y(t) - Y_p(t) = c_1 y_1(t) + c_2 y_2(t) \tag{9}$$

or

$$Y(t) = c_1 y_1(t) + c_2 y_2(t) + Y_p(t) \tag{10}$$

where y_1 and y_2 are a fundamental set of solutions to $Y(t)$. We will call

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t) \tag{11}$$

the **complimentary solution** and $Y_p(t)$ a **particular solution**. So, the general solution can be expressed as

$$Y(t) = y_c(t) + Y_p(t). \tag{12}$$

Thus, to find the general solution of (1), we'll need to find the general solution to (2) and then find some solution to (1). Adding these two pieces together give the general solution to (1).

If we vary a solution to (1) by just adding in some solution to Equation (2), it will still solve Equation (1). Now the goal of this section is to find some particular solution $Y_p(t)$ to Equation (1). We have two methods. The first is the method of **Undetermined Coefficients**, which reduces the problem to an algebraic problem, but only works in a few situations. The other called Variation of Parameters is a much more general method that always works but requires integration which may or may not be tedious.

1.2 Undetermined Coefficients

The major disadvantage of this solution method is that it is only useful for constant coefficient differential equations, so we will focus on

$$ay'' + by' + cy = g(t) \tag{13}$$

for $g(t) \neq 0$. The other disadvantage is it only works for a small class of $g(t)$'s.

Recall that we are trying to find some particular solution $Y_p(t)$ to Equation (13). The idea behind the method is that for certain classes of nonhomogeneous terms, we're able to make a good educated guess as to how $Y_p(t)$ should look, up to some unknown coefficients. Then we plug our guess into the differential equation and try to solve for the coefficients. If we can, our guess was correct and we have determined $Y_p(t)$. If we cannot solve for the coefficients, then we guessed incorrectly and we will need to try again.

1.3 The Basic Functions

There are three types of basic types of nonhomogeneous terms $g(t)$ that can be used for this method: exponentials, trig functions ($\sin$ and $\cos$), and polynomials. One we know how they work individually and combination will be similar.

1.3.1 Exponentials

Let's walk through an example where $g(t)$ is an exponential and see how to proceed.

Example 2. Determine a particular solution to

$$y'' - 4y' - 12y = 2e^{4t}. \tag{14}$$

How can we guess the form of $Y_p(t)$? When we plug $Y_p(t)$ into the equation, we should get $g(t) = 2e^{4t}$. We know that exponentials never appear or disappear during differentiation, so try

$$Y_p(t) = Ae^{4t} \tag{15}$$

for some coefficient A. Differentiate, plug in, and see if we can determine A. Plugging in we get

$$16Ae^{4t} - 4(4Ae^{4t}) - 12Ae^{4t} = 2e^{4t} \tag{16}$$

$$-12Ae^{4t} = 2e^{4t} \tag{17}$$

For these to be equal we need A to satisfy

$$-12A = 2 \Rightarrow A = -\frac{1}{6}. \tag{18}$$

So with this choice of A, our guess works, and the particular solution is

$$Y_p(t) = -\frac{1}{6}e^{4t}. \tag{19}$$

Consider the following full problem:

Example 3. Solve the IVP

$$y'' - 4y' - 12y = 2e^{4t}, \quad y(0) = -\frac{13}{6}, \quad y'(0) = \frac{7}{3}. \tag{20}$$

We know the general solution has the form

$$y(t) = y_c(t) + Y_p(t) \tag{21}$$

where the complimentary solution $y_c(t)$ is the general solution to the associated homogeneous equation

$$y'' - 4y' - 12y = 0 \tag{22}$$

and $Y_p(t)$ is the particular solution to the original differential equation. From the previous example we know

$$Y_p(t) = -\frac{1}{6}e^{4t}. \tag{23}$$

What is the complimentary solution? Our associated homogeneous equation has constant coefficients, so we need to find roots of the characteristic equation.

$$r^2 - 4r - 12 = 0 \tag{24}$$
$$(r-6)(r+2) = 0 \tag{25}$$

So we conclude that $r_1 = 6$ and $r_2 = -2$. These are distinct roots, so the complimentary solution will be

$$y_c(t) = c_1 e^{6t} + c_2 e^{-2t} \tag{26}$$

We must be careful to remember the initial conditions are for the non homogeneous equation, not the associated homogeneous equation. Do not apply them at this stage to y_c, since that is not a solution to the original equation.

So our general solution is the sum of $y_c(t)$ and $Y_p(t)$. We'll need it and its derivative to apply the initial conditions

$$y(t) = c_1 e^{6t} + c_2 e^{-2t} - \frac{1}{6}e^{4t} \tag{27}$$
$$y'(t) = 6c_1 e^{6t} - 2c_2 e^{2t} - \frac{2}{3}e^{4t} \tag{28}$$

Now apply the initial conditions

$$-\frac{13}{6} = y(0) = c_1 + c_2 - \frac{1}{6} \tag{29}$$
$$\frac{7}{3} = y'(0) = 6c_1 - 2c_2 - \frac{2}{3} \tag{30}$$

This system is solved by $c_1 = -\frac{1}{8}$ and $c_2 = -\frac{15}{8}$, so our solution is

$$y(t) = -\frac{1}{8}e^{6t} - \frac{15}{8}e^{-2t} - \frac{1}{6}e^{4t}. \tag{31}$$

1.3.2 Trig Functions

The second class of nonhomogeneous terms for which we can use this method are trig functions, specifically $\sin$ and $\cos$.

Example 4. Find a particular solution for the following IVP

$$y'' - 4y' - 12y = 6\cos(4t). \tag{32}$$

In the first example the nonhomogeneous term was exponential, and we know when we differentiate exponentials they persist. In this case, we've got a cosine function. When we differentiate a cosine, we get sine. So we expect an initial guess to require a sine term in addition to cosine. Try

$$Y_p(t) = A\cos(4t) + B\sin(4t). \tag{33}$$

Now differentiate and plug in

$$-16A\cos(4t) - 16B\sin(4t) - 4\big(-4A\sin(4t) + 4B\cos(4t)\big) - 12\big(A\cos(4t) + B\sin(4t)\big) = 13\cos(4t) \tag{34}$$

$$(-16A - 16B - 12A)\cos(4t) + (-16B + 16A - 12B)\sin(4t) = 13\cos(4t) \tag{35}$$

$$(-28A - 16B)\cos(4t) + (16A - 28B)\sin(4t) = 13\cos(4t) \tag{36}$$

To solve for A and B set the coefficients equal. Note that the coefficient for $\sin(4t)$ on the right hand side is 0. So we get the system of equations

$$\cos(4t) \quad : \qquad -28A - 16B = 13 \tag{37}$$

$$\sin(4t) \quad : \qquad 16A - 28B = 0. \tag{38}$$

This system is solved by $A = -\frac{7}{20}$ and $B = -\frac{1}{5}$. So a particular solution is

$$Y_p(t) = -\frac{7}{20}\cos(4t) - \frac{1}{5}\sin(4t) \tag{39}$$

Note that the guess would have been the same if $g(t)$ had been sine instead of cosine.

1.3.3 Polynomials

The third and final class of nonhomogeneous term we can use with this method are polynomials.

Example 5. Find a particular solution to

$$y'' - 4y' - 12y = 3t^3 - 5t + 2. \tag{40}$$

In this case, $g(t)$ is a cubic polynomial. When differentiating polynomials the order decreases. So if our initial guess is a cubic, we should capture all terms that will arise. Our guess

$$Y_p(t) = At^3 + Bt^2 + Ct + D. \tag{41}$$

Note that we have a t^2 term in our equation even though one does not appear in $g(t)$! Now differentiate and plug in

$$\begin{aligned} 6At + 2B - 4(3At^2 + 2Bt + C) - 12(At^3 + Bt^2 + Ct + D) &= 3t^2 - 5t + 2 \quad (42) \\ -12At^3 + (12A - 12B)t^2 + (6A - 8B - 12C)t + (2B - 4C - 12D) &= 3t^2 - 5t + 2 \quad (43) \end{aligned}$$

We obtain a system of equations by setting coefficients equal

$$t^3 \ : \ -12A = 3 \Rightarrow A = -\frac{1}{4} \tag{44}$$

$$t^2 \ : \ -12A - 12B = 0 \Rightarrow B = \frac{1}{4} \tag{45}$$

$$t \ : \ 6A - 8B - 12C = -5 \Rightarrow C = \frac{1}{8} \tag{46}$$

$$1 \ : \ 2B - 4C - 12D = 2 \Rightarrow D = -\frac{1}{6} \tag{47}$$

So a particular solution is

$$Y_p(t) = -\frac{1}{4}t^3 + \frac{1}{4}t^2 + \frac{1}{8}t - \frac{1}{6} \tag{48}$$

1.3.4 Summary

Given each of the basic types, we make the following guess

$$ae^{\alpha t} \Rightarrow Ae^{\alpha t} \tag{49}$$

$$a\cos(\alpha t) \Rightarrow A\cos(\alpha t) + B\sin(\alpha t) \tag{50}$$

$$a\sin(\alpha t) \Rightarrow A\cos(\alpha t) + B\sin(\alpha t) \tag{51}$$

$$a_n t^n + a_{n-1}t^{n-1} + ... + a_1 t + a_0 \Rightarrow A_n t^n + A_{n-1}t^{n-1} + ... + A_1 t + A_0 \tag{52}$$

1.4 Products

The idea for products is to take products of our forms above.

Example 6. Find a particular solution to

$$y'' - 4y' - 12y = te^{4t} \tag{53}$$

Start by writing the guess for the individual pieces. $g(t)$ is the product of a polynomial and an exponential. Thus guess for the polynomial is $At + B$ while the guess for the exponential is Ce^{4t}. So the guess for the product should be

$$Ce^{4t}(At + B) \tag{54}$$

We want to minimize the number of constants, so

$$Ce^{4t}(At + B) = e^{4t}(ACt + BC). \tag{55}$$

Rewrite with two constants

$$Y_p(t) = e^{4t}(At + B) \tag{56}$$

Notice this is the guess as if it was just t with the exponential multiplied to it. Differentiate and plug in

$$\begin{aligned}
16e^{4t}(At+B) + 8Ae^{4t} - 4(4e^{4t}(At+B) + Ae^{4t}) - 12e^{4t}(At+B) &= te^{4t} & (57)\\
(16A - 16A - 12A)t^{4t} + (16B + 8A - 16B - 4A - 12B)e^{4t} &= te^{4t} & (58)\\
-12Ate^{4t} + (4A - 12B)e^{4t} &= te^{4t} & (59)
\end{aligned}$$

Then we set the coefficients equal

$$\begin{aligned}
te^{4t}: \quad -12A = 1 \quad \Rightarrow \quad A &= -\frac{1}{12} & (60)\\
e^{4t}: \quad (4A - 12B) = 0 \quad \Rightarrow \quad B &= -\frac{1}{36} & (61)
\end{aligned}$$

So, a particular solution for this differential equation is

$$Y_p(t) = e^{4t}(-\frac{1}{12}t - \frac{1}{36}) = -\frac{e^{4t}}{36}(3t+1). \tag{62}$$

Basic Rule: If we have a product with an exponential write down the guess for the other piece and multiply by an exponential without any leading coefficient.

Example 7. Find a particular solution to

$$y'' - 4y' - 12y = 29e^{5t}\sin(3t). \tag{63}$$

We try the following guess

$$Y_p(t) = e^{5t}(A\cos(3t) + B\sin(3t)). \tag{64}$$

So differentiate and plug in

$$\begin{aligned}
&25e^{5t}(A\cos(3t) + B\sin(3t)) + 30e^{5t}(-A\sin(3t) + B\cos(3t)+\\
&9e^{5t}(-A\cos(3t) - B\sin(3t)) - 4(5e^{5t}(A\cos(3t) + B\sin(3t))+\\
&3e^{5t}(-A\sin(3t) + B\cos(3t))) - 12e^{5t}(A\cos(3t) + B\sin(3t)) = 29e^{5t}\sin(3t)
\end{aligned} \tag{65}$$

Gather like terms

$$(-16A + 18B)e^{5t}\cos(3t) + (-18A - 16B)e^{5t}\sin(3t) = 29e^{5t}\sin(3t) \tag{66}$$

Set the coefficients equal

$$\begin{aligned}
e^{5t}\cos(3t): \quad -16A + 18B &= 0 & (67)\\
e^{5t}\sin(3t): \quad -18A - 16B &= 29 & (68)
\end{aligned}$$

This is solved by $A = -\frac{9}{10}$ and $B = -\frac{4}{5}$. So a particular solution to this differential equation is

$$Y_p(t) = e^{5t}\big(-\frac{9}{10}t - \frac{4}{5}\big) = -\frac{e^{5t}}{10}(9t+8). \tag{69}$$

Example 8. Write down the form of the particular solution to

$$y'' - 4y' - 12y = g(t) \tag{70}$$

for the following $g(t)$:
(1) $g(t) = (9t^2 - 103t)\cos(t)$

Here we have a product of a quadratic and a cosine. The guess for the quadratic is

$$At^2 + Bt + C \tag{71}$$

and the guess for the cosine is

$$D\cos(t) + E\sin(t). \tag{72}$$

Multiplying the two guesses gives

$$(At^2 + Bt + C)(D\cos(t)) \quad + \quad (At^2 + Bt + C)(E\sin(t)) \tag{73}$$

$$(ADt^2 + BDt + CD)\cos(t) \quad + \quad (AEt^2 + BEt + CE)\sin(t). \tag{74}$$

Each of the coefficients is a product of two constants, which is another constant. Simply to get our final guess

$$Y_p(t) = (At^2 + Bt + C)\cos(t) + (Dt^2 + Et + F)\sin(t) \tag{75}$$

This is indicative of the general rule for a product of a polynomial and a trig function. Write down the guess for the polynomial, multiply by cosine, then add to that the guess for the polynomial multiplied by a sine.

(2) $g(t) = e^{-2t}(3 - 5t)\cos(9t)$

This homogeneous term has all three types of special functions. So combining the two general rules above, we get

$$Y_p(t) = e^{-2t}(At + B)\cos(9t) + e^{-2t}(Ct + D)\sin(9t). \tag{76}$$

1.5 Sums

We have the following important fact. If Y_1 satisfies

$$p(t)y'' + q(t)y' + r(t)y = g_1(t) \tag{77}$$

and Y_2 satisfies

$$p(t)y'' + q(t)y' + r(t)y = g_2(t) \tag{78}$$

then $Y_1 + Y_2$ satisfies

$$p(t)y'' + q(t)y' + r(t)y = g_1(t) + g_2(t) \tag{79}$$

This means that if our nonhomogeneous term $g(t)$ is a sum of terms we can write down the guesses for each of those terms and add them together for our guess.

Example 9. Find a particular solution to

$$y'' - 4y' - 12y = e^{7t} + 12. \tag{80}$$

Our nonhomogeneous term $g(t) = e^{7t} + 12$ is the sum of an exponential $g_1(t) = e^{7t}$ and a 0 degree polynomial $g_2(t) = 12$. The guess is

$$Y_p(t) = Ae^{7t} + B \tag{81}$$

This cannot be simplified, so this is our guess. Differentiate and plug in

$$49Ae^{7t} - 28Ae^{7t} - 12Ae^{7t} - 12B = e^{7t} + 12 \tag{82}$$
$$9Ae^{7t} - 12B = e^{7t} + 12. \tag{83}$$

Setting the coefficients equal gives $A = \frac{1}{9}$ and $B = -1$, so our particular solution is

$$Y_p(t) = \frac{1}{9}e^{7t} - 1. \tag{84}$$

Example 10. Write down the form of a particular solution to

$$y'' - 4y' - 12y = g(t) \tag{85}$$

for each of the following $g(t)$:
(1) $g(t) = 2\cos(t) - 9\sin(3t)$

Our guess for the cosine is

$$A\cos(3t) + B\sin(3t) \tag{86}$$

Additionally, our guess for the sine is

$$C\cos(3t) + D\sin(3t) \tag{87}$$

So if we add the two of them together, we obtain

$$A\cos(3t) + B\sin(3t) + C\cos(3t) + D\sin(3t) = (A + C)\cos(3t) + (B + D)\sin(3t) \tag{88}$$

But $A + C$ and $B + D$ are just some constants, so we can replace them with the guess

$$Y_p(t) = A\cos(3t) + B\sin(3t). \tag{89}$$

(2) $g(t) = \sin(t) - 2\sin(14t) - 5\cos(14t)$

Start with a guess for the $\sin(t)$

$$A\cos(t) + B\sin(t). \tag{90}$$

Since they have the same argument, the previous example showed we can combine the guesses for $\cos(14t)$ and $\sin(14t)$ into

$$C\cos(14t) + D\sin(14t) \tag{91}$$

So the final guess is

$$Y_p(t) = A\cos(t) + B\sin(t) + C\cos(14t) + D\cos(14t) \tag{92}$$

(3) $g(t) = 7\sin(10t) - 5t^2 + 4t$
Here we have the sum of a trig function and a quadratic so the guess will be

$$Y_p(t) = A\cos(10t) + B\sin(10t) + Ct^2 + Dt + E. \tag{93}$$

(4) $g(t) = 9e^t + 3te^{-5t} - 5e^{-5t}$
This can be rewritten as $9e^t + (3t-5)e^{-5t}$. So our guess will be

$$Y_p(t) = Ae^t + (Bt + C)e^{-5t} \tag{94}$$

(5) $g(t) = t^2\sin(t) + 4\cos(t)$
So our guess will be

$$Y_p(t) = (At^2 + Bt + C)\cos(t) + (Dt^2 + Et + F)\sin(t). \tag{95}$$

(6) $g(t) = 3e^{-3t} + e^{-3t}\sin(3t) + \cos(3t)$
Our guess

$$Y_p(t) = Ae^{-3t} + e^{-3t}(B\cos(3t) + C\sin(3t)) + D\cos(3t) + E\sin(3t). \tag{96}$$

This seems simple, right? There is one problem which can arise you need to be aware of

Example 11. Find a particular solution to

$$y'' - 4y' - 12y = e^{6t} \tag{97}$$

This seems straightforward, so try $Y_p(t) = Ae^{6t}$. If we differentiate and plug in

$$36Ae^{6t} - 24Ae^{6t} - 12Ae^{6t} = e^{6t} \tag{98}$$
$$0 = e^{6t} \tag{99}$$

Exponentials are never zero. So this cannot be possible. Did we make a mistake on our original guess? Yes, if we went through the normal process and found the complimentary solution in this case

$$y_c(t) = c_1e^{6t} + c_2e^{-2t}. \tag{100}$$

So our guess for the particular solution was actually part of the complimentary solution. So we need to find a different guess. Think back to repeated root solutions and try $Y_p(t) = Ate^{6t}$. Try it

$$(36Ate^{6t} + 12Ae^{6t}) - 4(6Ate^{6t} + Ae^{6t}) - 12Ate^{6t} = e^{6t} \tag{101}$$
$$(36A - 24A - 12A)te^{6t} + (12A - 4A)e^{6t} = e^{6t} \tag{102}$$
$$8Ae^{6t} = e^{6t} \tag{103}$$

Setting the coefficients equal, we conclude that $A = \frac{1}{8}$, so

$$Y_p(t) = \frac{1}{8} t e^{6t}. \tag{104}$$

NOTE: If this situation arises when the complimentary solution has a repeated root and has the form

$$y_c(t) = c_1 e^{rt} + c_2 t e^{rt} \tag{105}$$

then our guess for the particular solution should be

$$Y_p(t) = A t^2 e^{rt}. \tag{106}$$

1.6 Method of Undetermined Coefficients

Then we want to construct the general solution $y(t) = y_c(t) + Y_p(t)$ by following these steps:
(1) Find the general solution of the corresponding homogeneous equation.
(2) Make sure $g(t)$ belongs to a special set of basic functions we will define shortly.
(3) If $g(t) = g_1(t) + ... + g_n(t)$ is the sum of n terms, then form n subproblems each of which contains only one $g_i(t)$. Where the ith subproblem is

$$ay'' + by' + cy = g_i(t) \tag{107}$$

(4) For the ith subproblem assume a particular solution of the appropriate functions (exponential, sine, cosine, polynomial). If there is a duplication in $Y_i(t)$ with a solution to the homogeneous problem then multiply $Y_i(t)$ by t (or if necessary t^2).
(5) Find the particular solution $Y_i(t)$ for each subproblem. Then the sum of the Y_i is a particular solution for the full nonhomogeneous problem.
(6) Form the general solution by summing all the complimentary solutions from the homogeneous equation and the n particular solutions.
(7) Use the initial conditions to determine the values of the arbitrary constants remaining in the general solution.

Now for more examples, write down the guess for the particular solution:
(1) $y'' - 3y' - 28y = 6t + e^{-4t} - 2$

First we find the complimentary solution using the characteristic equation

$$y_c(t) = c_1 e^{7t} + c_2 e^{-4t} \tag{108}$$

Now look at the nonhomogeneous term which is a polynomial and exponential, $6t - 2 + e^{-4t}$. So our initial guess should be

$$At + B + Ce^{-4t} \tag{109}$$

The first two terms are fine, but the last term is in the complimentary solution. Since Cte^{-4t} does not show up in the complimentary solution our guess should be

$$Y_p(t) = At + B + Cte^{-4t}. \tag{110}$$

(2) $y'' - 64y = t^2 e^{8t} + \cos(t)$
The complimentary solution is

$$y_c(t) = c_1 e^{8t} + c_2 e^{-8t}. \tag{111}$$

Our initial guess for a particular solution is

$$(At^2 + Bt + C)e^{8t} + D\cos(t) + E\sin(t) \tag{112}$$

Again we have a Ce^{8t} term which is also in the complimentary solution. So we need to multiply the entire first term by t, so our final guess is

$$Y_p(t) = (At^3 + Bt^2 + Ct)e^{8t} + D\cos(t) + E\sin(t). \tag{113}$$

(3) $y'' + 4y' = e^{-t}\cos(2t) + t\sin(2t)$
The complimentary solution is

$$y_c(t) = c_1\cos(2t) + c_2\sin(2t) \tag{114}$$

Our first guess for a particular solution would be

$$e^{-t}(A\cos(2t) + B\sin(2t)) + (Ct + D)\cos(2t) + (Et + F)\sin(2t) \tag{115}$$

We notice the second and third terms contain parts of the complimentary solution so we need to multiply by t, so we have a our final guess

$$Y_p(t) = e^{-t}(A\cos(2t) + B\sin(2t)) + (Ct^2 + Dt)\cos(2t) + (Et^2 + Ft)\sin(2t). \tag{116}$$

(4) $y'' + 2y' + 5 = e^{-t}\cos(2t) + t\sin(2t)$
Notice the nonhomogeneous term in this example is the same as in the previous one, but the equation has changed. Now the complimentary solution is

$$y_c(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) \tag{117}$$

So our initial guess for the particular solution is the same as the last example

$$e^{-t}(A\cos(2t) + B\sin(2t)) + (Ct + D)\cos(2t) + (Et + F)\sin(2t) \tag{118}$$

This time the first term causes the problem, so multiply the first term by t to get the final guess

$$Y_p(t) = te^{-t}(A\cos(2t) + B\sin(2t)) + (Ct + D)\cos(2t) + (Et + F)\sin(2t) \tag{119}$$

So even though the nonhomogeneous parts are the same the guess also depends critically on the complimentary solution and the differential equation itself.
(5) $y'' + 4y' + 4y = t^2 e^{-2t} + 2e^{-2t}$
The complimentary solution is

$$y_c(t) = c_1 e^{-2t} + c_2 t e^{-2t} \tag{120}$$

Notice that we can factor out a e^{-2t} from out nonhomogeneous term, which becomes $(t^2 + 2)e^{-2t}$. This is the product of a polynomial and an exponential, so our initial guess is

$$(At^2 + Bt + C)e^{-2t} \tag{121}$$

But the Ce^{-2t} term is in $y_c(t)$. Also, Cte^{-2t} is in $y_c(t)$. So we must multiply by t^2 to get our final guess

$$Y_p(t) = (At^4 + Bt^3 + C)e^{-2t}. \tag{122}$$

HW 3.5 # 1, 2, 4, 9, 11, 14, 18, 23a, 25a, 28