Lecture Notes for Math 251: ODE and PDE. Lecture 15: 3.7 Electrical and Mechanical Vibrations

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1 Mechanical and Electrical Vibrations

Last Time: We studied the method of undetermined coefficients thoroughly, focusing mostly on determining guesses for particular solutions once we have solved for the complimentary solution.

1.1 Applications

The first application is mechanical vibrations. Consider an object of a given mass m hanging from a spring of natural length l, but there are a number of applications in engineering with the same general setup as this.

We will establish the convention that **always** the downward displacement and forces are **positive**, while upward displacements and forces are negative. BE CONSISTENT. We also measure all displacements from the equilibrium position. Thus if our displacement is $u(y)$, $u = 0$ corresponds to the center of gravity as it hangs at rest from a spring.

We need to develop a differential equation to model the displacement u of the object. Recall Newton's Second Law

$$F = ma \tag{1}$$

where m is the mass of the object. We want our equation to be for displacement, so we'll replace a by u'', and Newton's Second Law becomes

$$F(t, u, u') = mu''. \tag{2}$$

What are the various forces acting on the object? We will consider four different forces, some of which may or may not be present in a given situation.

(1) **Gravity**, F_g

The gravitational force always acts on an object. It is given by

$$F_g = mg \tag{3}$$

where g is the acceleration due to gravity. For simpler computations, you may take $g = 10$ m/s. Notice gravity is always positive since it acts downward.

(2) **Spring**, F_s

We attach an object to a spring, and the spring will exert a force on the object. Hooke's Law governs this force. The spring force is proportional to the displacement of the spring from its natural length. What is the displacement of the spring? When we attach an object to a spring, the spring gets stretched. The length of the stretched spring is L. Then the displacement from its natural length is $L + u$.

So the spring force is

$$F_s = -k(L + u) \tag{4}$$

where $k > 0$ is the **spring constant**. Why is it negative? It is to make sure the force is in the correct direction. If $u > -L$, i.e. the spring has been stretched beyond its natural length, then $u + L > 0$ and so $F_s < 0$, which is what we expect because the spring would pull upward on the object in this situation. If $u < -L$, so the spring is compressed, then the spring force would push the object back downwards and we expect to find $F_s > 0$.

(3) **Damping**, F_d

We will consider some situations where the system experiences damping. This will not always be present, but always notice if damping is involved. Dampers work to counteract motion (example: shocks on a car), so this will oppose the direction of the object's velocity.

In other words, if the object has downward velocity $u' > 0$, we would want the damping force to be acting in the upwards direction, so that $F_d < 0$. Similarly, if $u' < 0$, we want $F_d > 0$. Assume all damping is linear.

$$F_d = -\gamma u' \tag{5}$$

where $\gamma > 0$ is the **damping constant**.

(4) **External Force**, $F(t)$

This is encompasses all other forces present in a problem. An example is a spring hooked up to a piston that exerts an extra force upon it. We call $F(t)$ the **forcing function**, and it is just the sum of any of the external forces we have in a particular problem.

The most important part of any problem is identifying all the forces involved in the problem. Some may not be present. The forces will change depending on the particular situation. Let's consider the general form of our differential equation modeling a spring system. We have

$$F(t, u, u') = F_g + F_s + F_d + F(t) \tag{6}$$

so that Newton's Second Law becomes

$$mu'' = mg - k(L + u) - \gamma u' + F(t), \tag{7}$$

or upon reordering it becomes

$$mu'' + \gamma u' + ku = mg - kL + F(t). \tag{8}$$

What happens when the object is at rest. Equilibrium is $u = 0$, there are only two forces acting on the object: gravity and the spring force. Since the object is at rest, these two forces must balance to 0. So $F_g + F_s = 0$. In other words,

$$mg = kL. \tag{9}$$

So our equation simplifies to

$$mu'' + \gamma u' + ku = F(t), \tag{10}$$

and this is the most general form of our equation, with all forces present. We have the corresponding initial conditions

$$\begin{aligned} u(0) &= u_0 \quad \text{Initial displacement from equilibrium position} && (11) \\ u'(0) &= u'_0 \quad \text{Initial Velocity} && (12) \end{aligned}$$

Before we discuss individual examples, we need to touch on how we might figure out the constants k and γ if they are not explicitly given. Consider the spring constant k. We know if the spring is attached to some object with mass m, the object stretches the spring by some length L when it is at rest. We know at equilibrium $mg = kL$. Thus, if we know how much some object with a known mass stretches the spring when it is at rest, we can compute

$$k = \frac{mg}{L}. \tag{13}$$

How do we compute γ? If we do not know the damping coefficient from the beginning, we may know how much force a damper exerts to oppose motion of a given speed. Then set $|F_d| = \gamma|u'|$, where $|F_d|$ is the magnitude of the damping force and $|u'|$ is the speed of motion. So we have $\gamma = \frac{F_d}{u'}$. We will see how to compute in examples on damped motion. Let's consider specific spring mass systems.

1.2 Free, Undamped Motion

Start with free systems with no damping or external forces. This is the simplest situation since $\gamma = 0$. Our differential equation is

$$mu'' + ku = 0, \tag{14}$$

where $m, k > 0$. Solve by considering the characteristic equation

$$mr^2 + k = 0, \tag{15}$$

which has roots

$$r_{1,2} = \pm i\sqrt{\frac{k}{m}}. \tag{16}$$

We'll write

$$r_{1,2} = \pm i\omega_0, \tag{17}$$

where we've substituted

$$\omega_0 = \sqrt{\frac{k}{m}}. \tag{18}$$

ω_0 is called the **natural frequency** of the system, for reasons that will be clear shortly.

Since the roots of our characteristic equation are imaginary, the form of our general solution is

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \tag{19}$$

This is why we called ω_0 the natural frequency of the system: it is the frequency of motion when the spring-mass system has no interference from dampers or external forces.

Given initial conditions we can solve for c_1 and c_2. This is not the ideal form of the solution though since it is not easy to read off critical information. After we solve for the constants rewrite as

$$u(t) = R\cos(\omega_0 t - \delta), \tag{20}$$

where $R > 0$ is the **amplitude of displacement** and δ is the **phase angle of displacement**, sometimes called the **phase shift**.

Before determining how to rewrite the general solution in this desired form lets compare the two forms. When we keep it as the general solution is it easier to find the constants c_1 and c_2. But the new form is easier to work with since we can immediately see the amplitude making it much easier to graph. So ideally we will find the general solution, solve for c_1 and c_2, and then convert to the final form.

Assume we have c_1 and c_2 how do we find R and δ? Consider Equation (**??**) we can use a trig identity to write it as

$$u(t) = R\cos(\delta)\cos(\omega_0 t) + R\sin(\delta)\sin(\omega_0 t). \tag{21}$$

Comparing this to the general solution, we see that

$$c_1 = R\cos(\delta), \quad c_2 = R\sin(\delta). \tag{22}$$

Notice

$$c_1^2 + c_2^2 = R^2(\cos^2(\delta) + \sin^2(\delta)) = R^2, \tag{23}$$

so that, assuming $R > 0$,

$$R = \sqrt{c_1^2 + c_2^2}. \tag{24}$$

Also,

$$\frac{c_2}{c_1} = \frac{\sin(\delta)}{\cos(\delta)} = \tan(\delta). \tag{25}$$

to find δ.

Example 1. A 2kg object is attached to a spring, which it stretches by $\frac{5}{8}m$. The object is given an initial displacement of $1m$ upwards and given an initial downwards velocity of $4m/sec$. Assuming there are no other forces acting on the spring-mass system, find the displacement of the object at time t and express it as a single cosine.

The first step is to write down the initial value problem for this setup. We'll need to find an m and k. m is easy since we know the mass of the object is $2kg$. How about k? We know

$$k = \frac{mg}{L} = \frac{(2)(10)}{\frac{5}{8}} = 32. \tag{26}$$

So our differential equation is

$$2u'' + 32u = 0. \tag{27}$$

The initial conditions are given by

$$u(0) = -1, \quad u'(0) = 4. \tag{28}$$

The characteristic equation is

$$2r^2 + 32 = 0, \tag{29}$$

and this has roots $r_{1,2} = \pm 4i$. Hence $\omega_0 = 4$. Check: $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{32/2} = 4$. So our general solution is

$$u(t) = c_1 \cos(4t) + c_2 \sin(4t). \tag{30}$$

Using our initial conditions, we see

$$\begin{aligned} -1 &= u(0) = c_1 && (31) \\ 4 &= u'(0) = 4c_2 \Rightarrow c_2 = 1. && (32) \end{aligned}$$

So the solution is

$$u(t) = -\cos(4t) + \sin(4t). \tag{33}$$

We want to write this as a single cosine. Compute R

$$R = \sqrt{c_1^2 + c_2^2} = \sqrt{2}. \tag{34}$$

Now consider δ

$$\tan(\delta) = \frac{c_2}{c_1} = -1. \tag{35}$$

So δ is in Quadrants II or IV. To decide which look at the values of $\cos(\delta)$ and $\sin(\delta)$. We have

$$\begin{aligned} \sin(\delta) &= c_2 > 0 && (36) \\ \cos(\delta) &= c_1 < 0. && (37) \end{aligned}$$

So δ must be in Quadrant II, since there $\sin > 0$ and $\cos < 0$. If we take $\arctan(-1) = -\frac{\pi}{4}$, this has a value in Quadrant IV. Since $\tan$ is π-periodic, however, $-\frac{\pi}{4} + \pi = \frac{3\pi}{4}$ is in Quadrant II and also has a tangent of -1 Thus our desired phase angle is

$$\delta = \arctan(\frac{c_2}{c_1}) + \pi = \arctan(-1) + \pi = \frac{3\pi}{4} \tag{38}$$

and our solution has the final form

$$u(t) = \sqrt{2}\cos(4t - \frac{3\pi}{4}). \tag{39}$$

1.3 Free, Damped Motion

Now, let's consider what happens if we add a damper into the system with damping coefficient γ. We still consider free motion so $F(t) = 0$, and our differential equation becomes

$$mu'' + \gamma u' + ku = 0. \tag{40}$$

The characteristic equation is

$$mr^2 + \gamma r + k = 0, \tag{41}$$

and has solution

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}. \tag{42}$$

There are three different cases we need to consider, corresponding to the discriminant being positive, zero, or negative.

(1) $\gamma^2 - 4mk = 0$

This case gives a double root of $r = -\frac{\gamma}{2m}$, and so the general solution to our equation is

$$u(t) = c_1 e^{\frac{\gamma}{2m}} + c_2 t e^{-\frac{\gamma}{2m}} \tag{43}$$

Notice that $\lim_{t\to\infty} u(t) = 0$, which is good, since this signifies damping. This is called **critical damping** and occurs when

$$\begin{aligned} \gamma^2 - 4mk &= 0 && (44)\\ \gamma = \sqrt{4mk} &= 2\sqrt{mk} && (45) \end{aligned}$$

This value of $\gamma - 2\sqrt{mk}$ is denoted by γ_{CR} and is called the **critical damping coefficient**. Since this case separates the other two it is generally useful to be able to calculate this coefficient for a given spring-mass system, which we can do using this formula. Critically damped systems may cross $u = 0$ once but will never cross more than that. No oscillation

(2) $\gamma^2 - 4mk > 0$

In this case, the discriminant is positive and so we will get two distinct real roots r_1 and r_2. Hence our general solution is

$$u(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \tag{46}$$

But what is the behavior of this solution? The solution should die out since we have damping. We need to check $\lim_{t\to\infty} u(t) = 0$. Rewrite the roots

$$\begin{aligned} r_{1,2} &= \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} && (47)\\ &= \frac{-\gamma \pm \gamma(\sqrt{1 - \frac{4mk}{\gamma^2}})}{2m} && (48)\\ &= -\frac{\gamma}{2m}(1 \pm \sqrt{1 - \frac{4mk}{\gamma^2}}) && (49) \end{aligned}$$

By assumption, we have $\gamma^2 > 4mk$. Hence

$$1 - \frac{4mk}{\gamma^2} < 1 \tag{50}$$

and so

$$\sqrt{1 - \frac{4mk}{\gamma^2}} < 1. \tag{51}$$

so the quantity in parenthesis above is guaranteed to be positive, which means both of our roots are negative.

Thus the damping in this case has the desired effect, and the vibration will die out in the limit. This case, which occurs when $\gamma > \gamma_{CR}$, is called **overdamping**. The solution won't oscillate around equilibrium, but settles back into place. The overdamping kills all oscillation

(3) $\gamma^2 < 4mk$

The final case is when $\gamma < \gamma_{CR}$. In this case, the characteristic equation has complex roots

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = \alpha + i\beta. \tag{52}$$

The displacement is

$$\begin{aligned} u(t) &= c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t) && (53) \\ &= e^{\alpha t}(c_1\cos(\beta t) + c_2\sin(\beta t)). && (54) \end{aligned}$$

In analogy to the free undamped case we can rewrite as

$$u(t) = Re^{\alpha t}\cos(\beta t - \delta). \tag{55}$$

We know $\alpha < 0$. Hence the displacement will settle back to equilibrium. The difference is that solutions will oscillate even as the oscillations have smaller and smaller amplitude. This is called **overdamped**.

Notice that the solution $u(t)$ is not quite periodic. It has the form of a cosine, but the amplitude is not constant. A function $u(t)$ is called **quasi-periodic**, since it oscillates with a constant frequency but a varying amplitude. β is called the **quasi-frequency** of the oscillation.

So when we have free, damped vibrations we have one of these three cases. A good example to keep in mind when considering damping is car shocks. If the shocks are new its overdamping, when you hit a bump in the road the car settles back into place. As the shocks wear there is more of an initial bump but the car still settles does not bounce around. Eventually when your shocks where and you hit a bump, the car bounces up and down for a few minutes and then settles like underdamping. The critical point where the car goes from overdamped to underdamped is the critically damped case.

Another example is a washing machine. A new washing machine does not vibrate significantly due to the presence of good dampers. Old washing machines vibrate a lot.

In practice we want to avoid underdamping. We do not want cars to bounce around on the road or buildings to sway in the wind. With critical damping we have the right behavior, but its too hard to achieve this. If the dampers wear a little we are then underdamped. In practice we want to stay overdamped.

Example 2. A $2kg$ object stretches a spring by $\frac{5}{8}m$. A damper is attached that exerts a resistive force of $48N$ when the speed is $3m/sec$. If the initial displacement is $1m$ upwards and the initial velocity is $2m/sec$ downwards, find the displacement $u(t)$ at any time t.

This is actually the example from the last class with damping added and different initial conditions. We already know $k = 32$. What is the damping coefficient γ? We know $|F_d| = 48$ when the speed is $|u'| = 3$. So the damping coefficients is given by

$$\gamma = \frac{|F_d|}{|u'|} = \frac{48}{3} = 16. \tag{56}$$

Thus the initial value problem is

$$2u'' + 16u' + 32u = 0, \quad u(0) = -1, \quad u'(0) = 2. \tag{57}$$

Before we solve it, see which case we're in. To do so, let's calculate the critical damping coefficient.

$$\gamma_{CR} = 2\sqrt{mk} = 2\sqrt{64} = 16. \tag{58}$$

So we are critically damped, since $\gamma = \gamma_{CR}$. This means we will get a double root. Solving the characteristic equation we get $r_1 = r_2 = -4$ and the general solution is

$$u(t) = c_1 e^{-4t} + c_2 t e^{-4t}. \tag{59}$$

The initial conditions give coefficients $c_1 = -1$ and $c_2 = -2$. So the solution is

$$u(t) = -e^{-4t} - 2te^{-4t} \tag{60}$$

Notice there is no oscillations in this case.

Example 3. For the same spring-mass system as in the previous example, attach a damper that exerts a force of $40N$ when the speed is $2m/s$. Find the displacement at any time t.

the only difference from the previous example is the damping force. Lets compute γ

$$\gamma = \frac{|F_d|}{|u'|} = \frac{40}{2} = 20. \tag{61}$$

Since we computed $\gamma_{CR} = 16$, this means we are overdamped and the characteristic equation should give us distinct real roots. The IVP is

$$2u'' + 20u' + 32u = 0, \quad u(0) = -1, \quad u(0) = 2. \tag{62}$$

The characteristic equation has roots $r_1 = -8$ and $r_2 = -2$. So the general solution is

$$u(t) = c_1 e^{-8t} + c_2 e^{-2t} \tag{63}$$

The initial conditions give $c_1 = 0$ and $c_2 = -1$, so the displacement is

$$u(t) = -e^{-2t} \tag{64}$$

Notice here we do not actually have a "vibration" as we normally think of them. The damper is strong enough to force the vibrations to die out so quickly that we do not notice much if any of them.

Example 4. For the same spring-mass system as in the previous two examples, add a damper that exerts a force of $16N$ when the speed is $2m/s$.

In this case, the damping coefficient is

$$\gamma = \frac{16}{2} = 8, \tag{65}$$

which tells us that this case is underdamped as $\gamma < \gamma_{CR} = 16$. We should expect complex roots of the characteristic equation. The IVP is

$$2u'' + 8u' + 32u = 0, \quad u(0) = -1, \quad u'(0) = 3. \tag{66}$$

The characteristic equation has roots

$$r_{1,2} = \frac{-8 \pm \sqrt{192}}{4} = -2 \pm i\sqrt{12}. \tag{67}$$

Thus our general solution is

$$u(t) = c_1 e^{-2t}\cos(\sqrt{12}t) + c_2 e^{2t}\sin(\sqrt{12}t) \tag{68}$$

The initial conditions give the constants $c_1 = 1$ and $c_2 = \frac{1}{\sqrt{12}}$, so we have

$$u(t) = -e^{-2t}\cos(\sqrt{12}t) + \frac{1}{\sqrt{12}}e^{2t}\sin(\sqrt{12}t). \tag{69}$$

Let's write this as a single cosine

$$R = \sqrt{(-1)^2 + (\frac{1}{\sqrt{12}})^2} = \sqrt{\frac{13}{12}} \tag{70}$$

$$\tan(\delta) = -\frac{1}{\sqrt{12}} \tag{71}$$

As in the undamped case, we look at the signs of c_1 and c_2 to figure out what quadrant δ is in. By doing so, we see that δ has negative cosine and positive sine, so it is in Quadrant II. Hence we need to take the arctangent and add π to it

$$\delta = \arctan(-\frac{1}{\sqrt{12}}) + \pi. \tag{72}$$

Thus our displacement is

$$u(t) = \sqrt{\frac{13}{12}} e^{-2t} \cos(\sqrt{12}t - \arctan(-\frac{1}{\sqrt{12}} - \pi). \tag{73}$$

In this case, we actually get a vibration, even though its amplitude steadily decreases until it is negligible. The vibration has quasi-frequency $\sqrt{12}$.

HW 3.7 # 1, 6, 11, 13, 14, 26ab 3 additional problems Hint: Period $T = \frac{2\pi}{frequency}$.

HW 1: A $2kg$ object stretches a spring by $\frac{1}{2}m$. A damper is attached that exerts a resistive force of $24N$ when the speed is $3m/sec$. If the initial displacement is $1m$ upwards and the initial velocity is $2m/sec$ downwards, find the displacement $u(t)$ at any time t. Find quasi-frequency μ, phase angle δ, and amplitude R.

HW 2: Same problem with mass $m = .5kg$. Just find the displacement. Don't need to find μ, δ, or R.

HW 3: Determine the critical value γ_{CR} that determines when the system is critically damped for HW1 and HW2.