Lecture Notes for Math 251: ODE and PDE. Lecture 16: 3.8 Forced Vibrations Without Damping

Shawn D. Ryan

Spring 2012

1 Forced Vibrations

Last Time: We studied non-forced vibrations with and without damping. We studied the four forces acting on an object gravity, spring force, damping, and external forces.

1.1 Forced, Undamped Motion

What happens when the external force $F(t)$ is allowed to act on our system. The function $F(t)$ is called the **forcing function**. We will consider the undamped case

$$mu'' + ku = F(t). \tag{1}$$

This is a nonhomogeneous equation, so we will need to find both the complimentary and particular solution.

$$u(t) = u_c(t) + U_p(t), \tag{2}$$

Recall that $u_c(t)$ is the solution to the associated homogeneous equation. We will use undetermined coefficients to find the particular solution $U_p(t)$ (if $F(t)$ has an appropriate form) or variation of parameters.

We restrict our attention to the case which appears frequently in applications

$$F(t) = F_0 \cos(\omega t) \quad \text{or} \quad F(t) = F_0 \sin(\omega t) \tag{3}$$

The force we are applying to our spring-mass system is a simple periodic function with frequency ω. For now we assume $F(t) = F_0 \cos(\omega t)$, but everything is analogous if it is a sine function. So consider

$$mu'' + ku' = F_0 \cos(\omega t). \tag{4}$$

Where the complimentary solution to the analogous free undamped equation is

$$u_c(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t), \tag{5}$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural frequency.

We can use the method of undetermined coefficients for this nonhomogeneous term $F(t)$. The initial guess for the particular solution is

$$U_p(t) = A\cos(\omega t) + B\sin(\omega t). \tag{6}$$

We need to be careful, note that we are okay since $\omega_0 \neq \omega$, but if the frequency of the forcing function is the same as the natural frequency, then this guess is the complimentary solution $u_c(t)$. Thus, if $\omega_0 = \omega$, we need to multiply by a factor of t. So there are two cases.

(1) $\omega \neq \omega_0$

In this case, our initial guess is not the complimentary solution, so the particular solution will be

$$U_p(t) = A\cos(\omega t) + B\sin(\omega t). \tag{7}$$

Differentiating and plugging in we get

$$\begin{aligned} m\omega^2(-A\cos(\omega t) - B\sin(\omega t)) + k(A\cos(\omega t) + B\sin(\omega t)) &= F_0\cos(\omega t) && (8)\\ (-m\omega^2 A + kA)\cos(\omega t) + (-m\omega^2 B + kB)\sin(\omega t) &= F_0\cos(\omega t). && (9) \end{aligned}$$

Setting the coefficients equal, we get

$$\begin{aligned} \cos(\omega t): \quad (-m\omega^2 + k)A &= F_0 \quad \Rightarrow \quad A = \frac{F_0}{k - m\omega^2} && (10)\\ \sin(\omega t): \quad (-m\omega^2 + k)B &= 0 \quad \Rightarrow \quad B = 0. && (11) \end{aligned}$$

So our particular solution is

$$\begin{aligned} U_p(t) &= \frac{F_0}{k - m\omega^2}\cos(\omega t) && (12)\\ &= \frac{F_0}{m(\frac{k}{m} - \omega^2)}\cos(\omega t) && (13)\\ &= \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t). && (14) \end{aligned}$$

Notice that the amplitude of the particular solution is dependent on the amplitude of the forcing function F_0 and the difference between the natural frequency and the forcing frequency.

We can write our displacement function in two forms, depending on which form we use for complimentary solution.

$$\begin{aligned} u(t) &= c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t) && (15)\\ u(t) &= R\cos(\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t) && (16) \end{aligned}$$

Again, we get an analagous solution if the forcing function were $F(t) = F_0 \sin(\omega t)$.

The key feature of this case can be seen in the second form. We have two cosine functions with different frequencies. These will interfere with each other causing the net oscillation to vary between great and small amplitude. This phenomena has a name ”beats” derived from musical terminology. Thing of hitting a tuning fork after it has already been struck, the volume will increase and decrease randomly. One hears the waves created here in the exact form of our solution.

(2) $\omega = \omega_0$

If the frequency of the forcing function is the same as the natural frequency, so the guess for the particular solution is

$$U_p(t) = At\cos(\omega_0 t) + Bt\sin(\omega_0 t) \tag{17}$$

Differentiate and plug in

$$\begin{aligned}(-m\omega_0^2 + k)At\cos(\omega_0 t) + (-m\omega_0^2 + k)Bt\sin(\omega_0 t)\\ + 2m\omega_0 B\cos(\omega_0 t) - 2m\omega_0 A\sin(\omega_0 t) = F_0\cos(\omega_0 t).\end{aligned} \tag{18}$$

To begin simplification recall that $\omega_0^2 = \frac{k}{m}$, so $m\omega_0^2 = k$. this means the first two terms will vanish (expected since no analogous terms on right side), and we get

$$2m\omega_0 B\cos(\omega_0 t) - 2m\omega_0 A\sin(\omega_0 t) = F_0\cos(\omega_0 t). \tag{19}$$

Now set the coefficients equal

$$\cos(\omega_0 t): \quad 2m\omega_0 B = F_0 \quad B = \frac{F_0}{2m\omega_0} \tag{20}$$

$$\sin(\omega_0 t): \quad -2m\omega_0 A = 0 \quad A = 0 \tag{21}$$

Thus the particular solution is

$$U_p(t) = \frac{F_0}{2m\omega_0} t\sin(\omega_0 t) \tag{22}$$

and the displacement is

$$u(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{2m\omega_0} t\sin(\omega_0 t) \tag{23}$$

or

$$u(t) = R\cos(\omega_0 t - \delta) + \frac{F_0}{2m\omega_0} t\sin(\omega_0 t). \tag{24}$$

What stands out most about this equation? Notice that as $t \to \infty$, $u(t) \to \infty$ due to the form of the particular solution. Thus, in the case where the forcing frequency is the same as the natural frequency, the oscillation will have an amplitude that continues to increase for all time since the external force adds energy to the system in a way that reinforces the natural motion of the system.

This phenomenon is called **resonance**. Resonance is the phenomenon behind microwave ovens. The microwave radiation strikes the water molecules in what’s being heated at their natural frequency, causing them to vibrate faster and faster, which generates heat. A similar trait is noticed in the Bay of Fundy, where tidal forces cause the ocean to resonate, yielding larger and larger tides. Resonance in the ear causes us to be able to distinguish between tones in sound.

A common example is the Tacoma Narrows Bridge. This is incorrect because the oscillation that led to the collapse of the bridge was from a far more complicated phenomenon than the simple resonance we're considering now. In general, for engineering purposes, resonance is something we would like to avoid unless we understand the situation and the effect on the system.

In summary when we drive our system at a different frequency than the natural frequency, the two frequencies interfere and we observe beats in motion. When the system is driven at a natural frequency, the natural motion of the system is reinforced, causing the amplitude of the motion to increase to infinity.

Example 1. A $3kg$ object is attached to a spring, which it stretches by $40cm$. There is no damping, but the system is forced with the forcing function

$$F(t) = 10\cos(\omega t) \tag{25}$$

such that the system will experience resonance. If the object is initially displaced $20cm$ downward and given an initial upward velocity of $10cm/s$, find the displacement at any time t.

We need to be aware of the units, convert all lengths to meters. Find k

$$k = \frac{mg}{L} = \frac{(3)(10)}{.4} = 75 \tag{26}$$

Next, we are told the system experiences resonance. Thus the forcing frequency ω must be the natural frequency ω_0.

$$\omega = \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{75}{3}} = 5 \tag{27}$$

Thus our initial value problem is

$$3u'' + 75u = 10\cos(5t) \quad u(0) = .2, \quad u'(0) = -.1 \tag{28}$$

The complimentary solution is the general solution of the associated free, undamped case. Since we have computed the natural frequency already, the complimentary solution is just

$$u_c(t) = c_1\cos(5t) + c_2\sin(5t). \tag{29}$$

The particular solution (using formula derived above) is

$$\frac{1}{3}t\sin(5t), \tag{30}$$

and so the general solution is

$$u(t) = c_1\cos(5t) + c_2\sin(5t) + \frac{1}{3}t\sin(5t). \tag{31}$$

The initial conditions give $c_1 = \frac{1}{5}$ and $c_2 = -\frac{1}{50}$, so the displacement can be given as

$$u(t) = \frac{1}{5}\cos(5t) - \frac{1}{50}\sin(5t) + \frac{1}{3}t\sin(5t) \tag{32}$$

Let's convert the first two terms to a single cosine.

$$R = \sqrt{(\frac{1}{5})^2 + (-\frac{1}{50})^2} = \sqrt{\frac{101}{2500}} \tag{33}$$

$$\tan(\delta) = \frac{-\frac{1}{50}}{\frac{1}{5}} = -\frac{1}{10} \tag{34}$$

Looking at the signs of c_1 and c_2, we see that $\cos(\delta) > 0$ and $\sin(\delta) < 0$. Thus δ is in Quadrant IV, and so we can just take the arctangent.

$$\delta = \arctan(-\frac{1}{10}) \tag{35}$$

The displacement is then

$$u(t) = \sqrt{\frac{101}{2500}}\cos\left(5t - \arctan(-\frac{1}{10})\right) + \frac{1}{3}t\sin(5t) \tag{36}$$

HW 3.8 # 2, 5, 6, 9, 18a, 19a

Hint: For # 2 recall identities for $\cos(\alpha \pm \beta)$ and $\sin(\alpha \pm \beta)$. Just set up 6 even though it has damping. If given in pounds divide weight by 32 to get mass.