Lecture Notes for Math 251: ODE and PDE. Lecture 17: 4.1 General Theory for nth Order Linear Equations

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1 Chapter 4: Higher Order Linear Equations

Last Time: We studied forced vibrations by a sine or a cosine function without damping. That concluded Chapter 3. Now we start Chapter 4.

1.1 General Theory for nth Order Linear Equations

Recall that an nth order linear differential equation is an equation of the form

$$P_n(t)\frac{d^ny}{dt^n} + P_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + ... + P_1(t)\frac{dy}{dt} + P_0(t)y = g(t). \tag{1}$$

We will assume $P_i(t)$ and $g(t)$ are continuous on some interval $\alpha < t < \beta$ and $P_n(t)$ is nonzero on this interval. We can divide by $P_n(t)$ and consider the equation as a linear operator

$$L[y] = \frac{d^ny}{dt^n} + p_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + ... + p_1(t)\frac{dy}{dt} + p_0(t)y = g(t). \tag{2}$$

where $p_i(t) = \frac{P_i(t)}{P_n(t)}$. The theory involved with an nth order equation is analogous to the theory for second order linear equations. To obtain a unique solution we need n initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad ..., \quad y^{(n-1)}(t_0) = y_0^{(n-1)} \tag{3}$$

we have the following theorem regarding uniqueness of a solution

Theorem 1. *If $p_0, p_1, ...p_{n-1}$ and g are continuous on the open interval I, then there exists exactly one solution $y = \phi(t)$ of the differential equation that also satisfies the initial conditions. This solution exists on all of I. Note that the leading derivative (nth order) must have the coefficient 1.*

Example 2. Determine the intervals in which a solution is sure to exist

$$(t-2)y''' + \sqrt{t+3}y'' + \ln(t)y' + e^ty = \sin(t) \tag{4}$$

Ans: First put the equation in standard form

$$y''' + \frac{\sqrt{t+3}}{t-2}y'' + \frac{\ln(t)}{t-2}y' + \frac{e^t}{t-2}y = \frac{\sin(t)}{t-2} \tag{5}$$

By Theorem 1 we are looking for the intervals where all the coefficients are continuous. The coefficient for y'' is continuous on $[-3,2) \cup (2,\infty)$, The coefficient for y' is continuous on $(0,2) \cup (2,\infty)$, the coefficient of y is continuous on $(-\infty,2) \cup (2,\infty)$, and $g(t)$ is continuous on $(-\infty,2) \cup (2,\infty)$. So the intervals where a solution exists are

$$(0,2) \cup (2,\infty) \tag{6}$$

Theorem 3. *If the functions $p_1, p_2, ..., p_n$ are continuous on the open interval I, the functions $y_1, y_2, ..., y_n$ are solutions of the homogeneous differential equation, and if $W(y_1, y_2, ..., y_n)(t) \neq 0$ for at least one point in I, then every solution on the differential equation can be expressed as a linear combination of the solutions $y_1, y_2, ..., y_n$ (Linear Superposition).*

Theorem 4. *If $y_1(t), y_2(t), ..., y_n(t)$ form a fundamental set of solutions of the homogeneous differential equation on an interval I, then $y_1(t), ..., y_n(t)$ are linearly independent solutions of the equation on I, then they form a fundamental set of solutions for the inhomogeneous equation on I.*

Example 5. Determine whether the given set of functions is linearly dependent or linearly independent.

$$t^2 - 1, t - 1, 2t + 1 \tag{7}$$

Consider

$$c_1(t^2 - 1) + c_2(t - 1) + c_3(2t + 1) = 0 \tag{8}$$

So we need to match coefficients

$$\begin{aligned} c_1 &= 0 \quad (9) \\ c_2 + 2c_3 &= 0 \quad (10) \\ -c_2 + c_3 &= 0 \quad (11) \end{aligned}$$

Rearrange the third equation $c_2 = c_3$ substitute into the second equation $3c_2 = 0$. Thus $c_1 = c_2 = c_3 = 0$. So the set of functions is linearly independent.

NO HOMEWORK from 4.1