Lecture Notes for Math 251: ODE and PDE. Lecture 18: 4.2 Homogeneous Equations With Constant Coefficients

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1 Homogeneous Equations with Constant Coefficients

Last Time: We studied the general theory of nth order linear equations. Now we want to find the general solution of an nth order equation using the methods developed for homogeneous equations with constant coefficients.

Consider the nth order linear homogeneous equation with constant coefficients

$$L[y] = a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_n y = 0 \tag{1}$$

Just like in the second order case we have the characteristic equation

$$a_n r^n + ... + a_1 r + a_n = 0 \tag{2}$$

Example 1. (Distinct Real Roots) Find the general solution of

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0, \quad y(0) = 1, y'(0) = 0, y''(0) = -2, y'''(0) = -1 \tag{3}$$

So the characteristic equation is

$$r^4 + r^3 - 7r^2 - r + 6 = 0. \tag{4}$$

Thus the roots are $1, -1, 2, -3$. So the general solution is

$$y = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-3t} \tag{5}$$

Using the initial conditions we find $c_1 = \frac{11}{8}, c_2 = \frac{5}{12}, c_3 = -\frac{2}{3}, c_4 = -\frac{1}{8}$. Therefore the solution to the IVP is

$$y = \frac{11}{8} e^t + \frac{5}{12} e^{-t} - \frac{2}{3} e^{2t} - \frac{1}{8} e^{-3t} \tag{6}$$

Example 2. (Complex Roots) Find the general solution of

$$y^{(4)} - y = 0, \quad y(0) = 7/2, y'(0) = -4, y''(0) = 5/2, y'''(0) = -2 \tag{7}$$

The characteristic equation is

$$r^4 - 1 = (r^2 + 1)(r^2 - 1) = 0 \tag{8}$$

Therefore the roots are $1, -1, i, -i$. Thus the general solution is

$$y = c_1 e^t + c_2 e^{-t} + c_3 \cos(t) + c_4 \sin(t) \tag{9}$$

By imposing the initial conditions we get

$$c_1 = 0, c_2 = 3, c_3 = 1/2, c_4 = -1 \tag{10}$$

Thus the solution of the IVP is

$$y = 3e^{-t} + \frac{1}{2}\cos(t) - \sin(t) \tag{11}$$

Example 3. (Repeated Roots) Find the general solution of

$$y^{(4)} + 2y'' + y = 0. \tag{12}$$

The characteristic equation is

$$r^4 + 2r^2 + 1 = (r^2 + 1)(r^2 + 1) = 0 \tag{13}$$

The roots are $i, i, -i, -i$. So the general solution is

$$y = c_1 \cos(t) + c_2 \sin(t) + c_3 t \cos(t) + c_4 t \sin(t) \tag{14}$$

HW 4.2 # 11, 13, 14, 16, 17, 18