Lecture Notes for Math 251: ODE and PDE. Lecture 19: 6.1 Definition of the Laplace Transform

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1 Definition of the Laplace Transform

Last Time: We studied nth order linear differential equations and used the method of characteristics to solve them.

We have spent most of the course solving differential equations directly, but sometimes a transformation of the problem can make it much easier. One such example is the Laplace Transform.

1.1 The Definition

Definition 1. A function f is called **piecewise continuous** on an interval $[a, b]$, if $[a, b]$ can be broken into a finite number of subintervals $[a_n, b_n]$ such that f is continuous on each open subinterval (a_n, b_n) and has a finite limit at each endpoint a_n, b_n.

So a piecewise continuous function has only finitely many jumps and does not have any asymptotes where it blows up to infinity or minus infinity.

Definition 2. *(Laplace Transformation)* Suppose that $f(t)$ is a piecewise continuous function. The Laplace Transform of $f(t)$, denoted by $\mathcal{L}\{f(t)\}$, is given by

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt. \tag{1}$$

REMARK: There is an alternate notation for the Laplace Transform that we will commonly use. Notice that the definition of $\mathcal{L}\{f(t)\}$ introduces a new variable, s, in the definite integral with respect to t. As a result, computing the transform yields a function which depends on s. Thus

$$\mathcal{L}\{f(t)\} = F(s) \tag{2}$$

It should also be noted that the integral definition of $\mathcal{L}\{f(t)\}$ is an improper integral. In our first examples of computing Laplace Transforms, we will review how to handle them.

Example 3. Compute $\mathcal{L}\{1\}$. Plugging $f(t) = 1$ into the definition we have

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}dt. \tag{3}$$

Recall we need to convert the improper integral into a limit

$$\begin{aligned} &= \lim_{N\to\infty} \int_0^N e^{-st}dt && (4)\\ &= \lim_{N\to\infty} \big[-\frac{1}{s}e^{-st}\big]_0^N && (5)\\ &= \lim_{N\to\infty} \big(-\frac{1}{s}e^{-Ns} + \frac{1}{s}\big) && (6)\end{aligned}$$

Note the value of s will affect our answer. If $s < 0$, the exponent of our exponential is positive, so the limit in question will diverge as the exponential goes to infinity. On the other hand, if $s > 0$, the exponential will go to 0 and the limit will converge.

Thus, we restrict our attention to the case where $s > 0$ and conclude that

$$\mathcal{L}\{1\} = \frac{1}{s} \quad \text{for} \quad s > 0 \tag{7}$$

Notice that we had to put a restriction on the domain of our Laplace Transform. This will always be the case: these integrals will not always converge for any s.

Example 4. Compute $\mathcal{L}\{e^{at}\}$ for $a \neq 0$.

By definition

$$\begin{aligned} \mathcal{L}\{e^{at}\} &= \int_0^\infty e^{-st}e^{at}dt && (8)\\ &= \int_0^\infty e^{(a-s)t}dt && (9)\\ &= \lim_{N\to\infty} \big[\frac{1}{a-s}e^{(a-s)t}\big]_0^N && (10)\\ &= \lim_{N\to\infty} \big(\frac{1}{a-s}e^{(a-s)N} - \frac{1}{a-s}\big) && (11)\\ &= \frac{1}{s-a} \quad \text{for} \quad s > a. && (12)\end{aligned}$$

Example 5. Compute $\mathcal{L}\{\sin(at)\}$.

$$\begin{aligned} \mathcal{L}\{\sin(at)\} &= \int_0^\infty e^{-st}\sin(at)dt && (13)\\ &= \lim_{N\to\infty} \int_0^N e^{-st}\sin(at)dt && (14)\end{aligned}$$

Integration By Parts (twice) yields

$$= \lim_{N\to\infty}\left(\frac{1}{a}(1-e^{-sN}\cos(aN)\right)-\frac{s}{a}\left(\frac{1}{a}e^{-sN}\sin(aN)+\frac{s}{a}\int_0^N e^{-st}\sin(at)dt\right). \tag{15}$$

After rewriting, we get

$$F(s) = \frac{1}{a}-\frac{s^2}{a^2}F(s) \tag{16}$$

$$\mathcal{L}\{\sin(at)\} = F(s) = \frac{a}{s^2+a^2} \quad \text{provided} \quad s>0. \tag{17}$$

Example 6. If $f(t)$ is a piecewise continuous function with piecewise continuous derivative $f'(t)$, express $\mathcal{L}\{f'(t)\}$ in terms of $\mathcal{L}\{f(t)\}$.

We plug f' into the definition for the Laplace transform

$$\mathcal{L}\{f'\} = \int_0^\infty e^{-st}f'dt \tag{18}$$

$$= \lim_{N\to\infty}\int_0^N e^{-st}f'dt \tag{19}$$

The next step is to integrate by parts

$$= \lim_{N\to\infty}\left(e^{-st}f|_0^N + s\int_0^N e^{-st}fdt\right) \tag{20}$$

$$= \lim_{N\to\infty}e^{-sN}f(N)-f(0)+s\int_0^\infty e^{-st}fdt \tag{21}$$

$$= s\mathcal{L}\{f(t)\}-f(0) \quad \text{provided} \quad s>0 \tag{22}$$

Doing this repeatedly one finds

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\}-s^{n-1}f(0)-s^{n-2}f'(0)-...-sf^{(n-2)}(0)-f^{(n-1)}(0). \tag{23}$$

Example 7. If $f(t)$ is a piecewise continuous function, express $\mathcal{L}\{e^{at}f(t)\}$ in terms of $\mathcal{L}\{e^{at}f(t)\}$

We begin by plugging into the definition

$$\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}dt \tag{24}$$

$$= \int_0^\infty e^{(a-s)t}f(t)dt \tag{25}$$

This looks like the definition of $F(s)$, but its not the same, since the exponent is $a-s$. However, if we substitute $u=s-a$, we get

$$= \int_0^\infty e^{-ut}f(t)dt \tag{26}$$

$$= F(u) \tag{27}$$

$$= F(s-a) \tag{28}$$

Thus if we take the Laplace transform of a function multiplied by e^{at}, we'll get the Laplace Transform of the original function shifted by a.

1.2 Laplace Transforms

In general, we won't be using the definition we will be using a table of Laplace Transforms. I would make an effort to know all the common transforms above as well as the definition. From now on we will use a table, but be prepared on an exam to do a basic transform using the definition.

Note the Laplace Transform is linear

Theorem 8. *Given piecewise continuous functions $f(t)$ and $g(t)$,*

$$\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} \tag{29}$$

for any constants a, b.

This follows from the linearity of integration. From a practical perspective we will not have to worry about constants or sums. We can decompose our function into individual pieces, transform them, and then put everything back together.

Example 9. Find the Laplace Transforms of the following functions
(i) $f(t) = 6e^{-5t} + e^{3t} + 5t^3 - 9$

$$\begin{aligned}
F(s) = \mathcal{L}\{f(t)\} &= 6\mathcal{L}\{e^{-5t}\} + \mathcal{L}\{e^{3t}\} + 5\mathcal{L}\{t^3\} - 9\mathcal{L}\{1\} && (30)\\
&= 6\frac{1}{s-(-5)} + \frac{1}{s-3} + 5\frac{3!}{s^{3+1}} - 9\frac{1}{s} && (31)\\
&= \frac{6}{s+5} + \frac{1}{s-3} + \frac{30}{s^4} - \frac{9}{s} && (32)
\end{aligned}$$

(ii) $g(t) = 4\cos(4t) - 2\sin(4t) - 3\cos(8t)$

$$\begin{aligned}
G(s) = \mathcal{L}\{g(t)\} &= 4\mathcal{L}\{\cos(4t)\} - 2\mathcal{L}\{\sin(4t)\} - 3\mathcal{L}\{\cos(10t)\} && (33)\\
&= 4\frac{s}{s^2+4^2} - 2\frac{4}{s^2+4^2} - 3\frac{s}{s^2+10^2} && (34)\\
&= \frac{4s-8}{s^2+16} - \frac{3s}{s^2+100} && (35)
\end{aligned}$$

(iii) $h(t) = e^{2t} + \cos(3t) + e^{2t}\cos(3t)$

$$\begin{aligned}
H(t) = \mathcal{L}\{h(t)\} &= \mathcal{L}\{e^{2t}\} + \mathcal{L}\{\cos(3t)\} - \mathcal{L}\{e^{2t}\cos(3t)\} && (36)\\
&= \frac{1}{s-2} + \frac{s}{s^2+3^2} - \frac{s-2}{(s-2)^2+3^2} && (37)\\
&= \frac{1}{s-2} + \frac{2}{s^2+9} - \frac{s-2}{(s-2)^2+9} && (38)
\end{aligned}$$

1.3 Initial Value Problems

We study Laplace Transforms to solve Initial Value Problems.

Example 10. Solve the following initial value problem using Laplace Transforms.

$$y'' - 6y' + 5y = 7t, \quad y(0) = -1, \quad y'(0) = 2. \tag{39}$$

The first step is using the Laplace Transform to solve an IVP is to transform both sides of the equation.

$$\begin{aligned}
\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} &= 7\mathcal{L}\{t\} && (40)\\
s^2Y(s) - sy(0) - y'(0) - 6\big(sY(s) - y(0)\big) + 5Y(s) &= \frac{7}{s^2} && (41)\\
s^2Y(s) + s - 2 - 6\big(sY(s) + 1\big) + 5Y(s) &= \frac{7}{s^2} && (42)
\end{aligned}$$

Now solve for $Y(s)$.

$$\begin{aligned}
(s^2 - 6s + 5)Y(s) + s - 8 &= \frac{7}{s^2} && (43)\\
Y(s) &= \frac{7}{s^2(s^2 - 6s + 5)} + \frac{8 - s}{s^2 - 6s + 5} && (44)
\end{aligned}$$

Now we want to solve for $y(t)$, but we have an expression for $Y(s) = \mathcal{L}\{y(t)\}$. Thus to finish solving this problem, we need the inverse Laplace Transform

1.4 Inverse Laplace Transform

In this section we have $F(s)$ and want to find $f(t)$. $f(t)$ is an inverse Laplace Transform of $F(s)$ with notation

$$f(t) = \mathcal{L}^{-1}\{F(s)\}. \tag{45}$$

Our starting point is that the inverse Laplace Transform is linear.

Theorem 11. *Given two Laplace Transforms $F(s)$ and $G(s)$,*

$$\mathcal{L}^{-1}\{aF(s) + bG(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\} \tag{46}$$

for any constants a, b.

So we decompose our original transformed function into pieces, inverse transform, and then put everything back together. Using the table we want to look at the denominator, which will tell us what the original function will have to be, but sometimes we have to look at the numerator to distinguish between two potential inverses (i.e. $\sin(at)$ and $\cos(at)$).

Example 12. Find the inverse transforms of the following
(i) $F(s) = \frac{6}{s} - \frac{1}{s-8} + \frac{4}{s-3}$

The denominator of the first term is s indicating that this will be the Laplace Transform of 1. Since $\mathcal{L}\{1\} = \frac{1}{s}$, we will factor out the 6 before taking the inverse transform. For the second term, this is just the Laplace Transform of e^{8t}, and there's nothing else to do with it. The third term is also an exponential, e^{3t}, and we'll need to factor out the 4 in the numerator before we take the inverse transform.

So we have

$$\begin{aligned}
\mathcal{L}^{-1}\{F(s)\} &= 6\mathcal{L}^{-1}\{\frac{1}{s}\} - \mathcal{L}^{-1}\{\frac{1}{s-8}\} + 4\mathcal{L}^{-1}\frac{1}{s-3} && (47)\\
f(t) &= 6(1) - e^{8t} + 4(e^{3t}) && (48)\\
&= 6 - e^{8t} + 4e^{3t} && (49)
\end{aligned}$$

(ii) $G(s) = \frac{12}{s+3} - \frac{1}{2s-4} + \frac{2}{s^4}$

The first term is just the transform of e^{-3t} multiplied by 12, which we will factor out before applying the inverse transform. The second term looks like it should be exponential, but it has a $2s$ instead of an s in the denominator, and transforms of exponentials should just have s. Fix this by factoring out the 2 in the denominator and then taking the inverse transform. The third term has s^4 as its denominator. This indicates that it will be related to the transform of t^3. The numerator is not correct since $\mathcal{L}\{t^3\} = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$. So we would need the numerator to be 6, and right now is 2. How do we fix this? We'll multiply by $\frac{3}{3}$, absorb the top 3 into the transform, with these fixes incorporated we have

$$\begin{aligned}
G(s) &= 12\frac{1}{s-(-3)} - \frac{1}{2(s-2)} + \frac{3}{3}\frac{2}{s^4} && (50)\\
&= 12\frac{1}{s-(-3)} - \frac{1}{2}\frac{1}{(s-2)} + \frac{1}{3}\frac{6}{s^4} && (51)
\end{aligned}$$

Now we can take the inverse transform.

$$g(t) = 12e^{-3t} - \frac{1}{2}e^{2t} + \frac{1}{3}t^3 \tag{52}$$

(iv) $H(s) = \frac{4s}{s^2+25} + \frac{3}{s^2+16}$

The denominator of the first term is, $s^2 + 25$, indicates that this should be the transform of either $\sin(5t)$ or $\cos(5t)$. The numerator is $4s$, which tells us that once we factor out the 4, it will be the transform of $\cos(5t)$. The second term's denominator is $s^2 + 16$, so it will be the transform of either $\sin(4t)$ or $\cos(4t)$. The numerator is a constant, 3, so it will be the transform of $\sin(4t)$. The only problem is that the numerator of $\mathcal{L}\{\sin(4t)\}$ should be 4, while here it is 3. We fix this by multiplying by $\frac{4}{4}$. Rewrite the transform

$$\begin{aligned}
H(s) &= 4\frac{1}{s^2+5^2} + \frac{4}{4}\frac{3}{s^2+4^2} && (53)\\
&= 4\frac{1}{s^2+5^2} + \frac{3}{4}\frac{4}{s^2+4^2} && (54)
\end{aligned}$$

Then take the inverse

$$h(t) = 4\cos(5t) + \frac{3}{4}\sin(4t) \tag{55}$$

Let's do some examples which require more work.

Example 13. Find the inverse Laplace Transforms for each of the following.
(i) $F(s) = \frac{3s-7}{s^2+16}$

Looking at the denominator, we recognize it will be a sine or cosine, since it has the form $s^2 + a^2$. It is not either because it has both s and a constant in the numerator, while a cosine just has s and the sine just has the constant. This is easy to compensate for, we split the fraction into the difference of two fractions, then fix them up as we did in the previous example, we will be able to then take the Inverse Laplace Transform

$$\begin{aligned} F(s) &= \frac{3s-7}{s^2+16} && (56)\\ &= \frac{3s}{s^2+16} - \frac{7}{s^2+16} && (57)\\ &= 3\frac{s}{s^2+16} - \frac{4}{4}\frac{7}{s^2+16} && (58)\\ &= 3\frac{s}{s^2+16} - \frac{7}{4}\frac{4}{s^2+16} && (59) \end{aligned}$$

Now each term is the correct form, and we can take the inverse transform.

$$f(t) = 3\cos(4t) - \frac{7}{4}\sin(4t) \tag{60}$$

(ii) $G(s) = \frac{1-3s}{s^2+2s+10}$

If we look at the table of Laplace Transforms, we might see that there are no denominators that look like a quadratic polynomial. Also, this polynomial does not factor nicely into linear terms. There are, however, some terms in the table with denominator of the form $(s-a)^2 + b^2$. Those for $e^{at}\cos(bt)$ and $e^{at}\sin(bt)$. Put the denominator in this form by completing the square.

$$\begin{aligned} s^2 + 2s + 10 &= s^2 + 2s + 1 - 1 + 10 && (61)\\ &= s^2 + 2s + 1 + 9 && (62)\\ &= (s+1)^2 + 9 && (63) \end{aligned}$$

Thus, our transformed function can be written as

$$G(s) = \frac{1-3s}{(s+1)^2+9} \tag{64}$$

We will not split this into two pieces yet. First, we need the s in the numerator to be $s+1$ so we can have the numerator of $e^{-t}\cos(3t)$. We do this by adding and subtracting 1 from the s. This will produce some other constant term, which we will combine with the already present constant and try to have the remaining terms look like the numerator for $e^{-t}\sin(3t)$.

$$\begin{aligned} G(s) &= \frac{1-3(s+1-1)}{(s+1)^2+9} && (65) \\ &= \frac{1-3(s+1)+3}{(s+1)^2+9} && (66) \\ &= \frac{-3(s+1)+4}{(s+1)^2+9} && (67) \end{aligned}$$

Now we can break our transform up into two pieces, one of which will correspond to the cosine and the other to the sine. At that point, fixing the numerators is the same as in the last few examples.

$$\begin{aligned} G(s) &= -3\frac{s+1}{(s+1)^2+9}+\frac{4}{3}\frac{3}{(s+1)^2+9} && (68) \\ g(s) &= -3e^{-t}\cos(3t)+\frac{4}{3}e^{-t}\sin(3t) && (69) \end{aligned}$$

(iii) $H(s) = \frac{s+2}{s^2-s-12}$

This seems identical to the last example, but there is a difference. We can immediately factor the denominator. This requires us to deal with the inverse transform differently. Factoring we see

$$H(s) = \frac{s+2}{(s+3)(s-4)}. \tag{70}$$

We know that if we have a linear denominator, that will correspond to an exponential. In this case we have two linear factors. This by itself is not the denominator of an particular Laplace Transform, but we know a method for turning certain rational functions with factored denominators into a sum of more simple radical functions with those factors in each denominator. This method is **Partial Fractions Decomposition**.

We start by writing

$$H(s) = \frac{s+2}{(s+3)(s-4)} = \frac{A}{s+3}+\frac{B}{s-4}. \tag{71}$$

Multiply through by $(s+3)(s-4)$:

$$s+2 = A(s-4) = B(s+3). \tag{72}$$

This must be true for any value of s. As a result, we have two methods for determining A and B.

Method 1: Match coefficients on functions of s just like in the method of undetermined coefficients

$$\begin{aligned} s: \quad 1 &= A+B && (73) \\ 1: \quad 2 &= -4A+3B && (74) \end{aligned}$$

Factor in Denominator	Partial Fractions Term
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\ldots+\frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$

TABLE 25.1. Translation from factored denominator to partial fractions.

Solving the system of two equations in two unknowns we get $A=\frac{1}{7}$ and $B=\frac{6}{7}$.

Method 2: Choose values of s (since it must hold for all s) and solve for A and B.

$$s=-3: \quad -1 = -7A \quad \Rightarrow \quad A=\frac{1}{7} \tag{75}$$

$$s=4: \quad 6 = 7B \quad \Rightarrow \quad B=\frac{6}{7} \tag{76}$$

Thus, our transform can be written as

$$H(s)=\frac{\frac{1}{7}}{s+3}+\frac{\frac{6}{7}}{s-4} \tag{77}$$

and taking the inverse transforms, we get

$$h(t)=\frac{1}{7}e^{-3t}+\frac{6}{7}e^{4t} \tag{78}$$

REMARK: We could have done the the last example by completing the square. However, this would have left us with expressions involving the hyperbolic sine, $\sinh$, and the hyperbolic cosine, $\cosh$. These are interesting functions which can be written in terms of exponentials, but it will be much easier for us to work directly with the exponentials. So we are better off just doing partial fractions even though it's slightly more work.

Partial Fractions and completing the square are a part of life when it comes to Laplace Transforms. Being good at this technique helps when solving IVPs, since most answers have sines, cosines, and exponentials.

Here is a quick review of partial fractions. The first step is to factor the denominator as much as possible. Then using the table above, we can find each of the terms for our partial fractions decomposition. This table is not exhaustive, but we will only have these factors in most if not all cases.

Example 14. Find the inverse transform of each of the following.

(i) $F(s)=\frac{2s+1}{(s-2)(s+3)(s-1)}$

The form of the decomposition will be

$$G(s)=\frac{A}{s-2}+\frac{B}{s+3}+\frac{C}{s-1} \tag{79}$$

since all the factors in our denominator are linear. Putting the right hand side over a common denominator and setting numerators equal, we have

$$2s + 1 = A(s+3)(s-1) + B(s-2)(s-1) + C(s-2)(s+3) \tag{80}$$

We can again use one of the two methods in Partial Fractions, where we choose key values of s that will isolate the coefficients.

$$\begin{aligned} s = 2: \quad 5 &= A(5)(1) \quad \Rightarrow \quad A = 1 &(81)\\ s = -3: \quad -5 &= B(-5)(-4) \quad \Rightarrow \quad B = -\frac{1}{4} &(82)\\ s = 1: \quad 3 &= C(-1)(4) \quad \Rightarrow \quad C = -\frac{3}{4} &(83)\end{aligned}$$

Thus, the partial fraction decomposition for this transform is

$$F(s) = \frac{1}{s-2} - \frac{\frac{1}{4}}{s+3} - \frac{\frac{3}{4}}{s-1}. \tag{84}$$

The inverse transform is

$$f(t) = e^{2t} - \frac{1}{4}e^{-3t} - \frac{3}{4}e^{t} \tag{85}$$

(ii) $G(s) = \frac{2-3s}{(s-2)(s^2+3)}$

Now we have a quadratic in the denominator. Looking at the table we see the form of the partial fractions decomposition will be

$$G(s) = \frac{A}{s-2} + \frac{Bs+C}{s^2+3}. \tag{86}$$

If we multiply through by $(s-2)(s^2+3)$, we get

$$2 - 3s = A(s^2+3) + (Bs+C)(s-2) \tag{87}$$

Notice that we cannot use the second method from the previous example, since there are 2 key values of s, but we have 3 constants. Thus we must compare coefficients

$$\begin{aligned} 2 - 3s &= A(s^2+3) + (Bs+C)(s-2) &(88)\\ &= As^2 + 3A + Bs^2 - 2Bs + Cs - 2C &(89)\\ &= (A+B)s^2 + (-2B+C)s + (3A-2C) &(90)\end{aligned}$$

We have the following system of equations to solve.

$$\begin{aligned} s^2: \quad A + B &= 0 &(91)\\ s: \quad -2B + C &= -3 \quad \Rightarrow \quad A = -\frac{8}{7} \ B = \frac{8}{7} \ C = -\frac{5}{7} &(92)\\ s^0 = 1: \quad 3A - 2C &= 2 &(93)\end{aligned}$$

Thus the partial fraction decomposition is

$$G(s) = -\frac{\frac{8}{7}}{s-2} + \frac{\frac{8}{7}s}{s^2+3} - \frac{\frac{5}{7}}{s^2+3} \tag{94}$$

$$= -\frac{8}{7}\frac{1}{s-2} + \frac{8}{7}\frac{2}{s^2+3} - \frac{5}{7\sqrt{3}}\frac{\sqrt{3}}{s^2+3} \tag{95}$$

and the inverse transform is

$$g(t) = -\frac{8}{7}e^{2t} + \frac{8}{7}\cos(\sqrt{3}t) - \frac{5}{7\sqrt{3}}\sin(\sqrt{3}t). \tag{96}$$

(iii) $H(s) = \frac{2}{s^3(s-1)}$
The partial fraction decomposition in this case is

$$H(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s-1}. \tag{97}$$

Multiplying through by the denominator of H gives

$$2 = As^2(s-1) + Bs(s-1) + C(s-1) + Ds^3 \tag{98}$$

$$= As^3 - As^2 + Bs^2 - Bs + Cs - C + Ds^3 \tag{99}$$

and we have to solve the system of equations

$$s^3: \quad A + D = 0 \tag{100}$$

$$s^2: \quad -A + B = 0 \tag{101}$$

$$s: \quad -B + C = 0 \quad \Rightarrow \quad A = -2 \;\; B = -2 \;\; C = -2 \;\; D = 2 \tag{102}$$

$$1: \quad -C = 2 \tag{103}$$

Thus our partial fractions decomposition becomes

$$H(s) = -\frac{2}{s} - \frac{2}{s^2} - \frac{2}{s^3} + \frac{2}{s-1} \tag{104}$$

$$= -\frac{2}{s} - \frac{2}{s^2} - \frac{2}{2!}\frac{2!}{s^3} + \frac{2}{s-1} \tag{105}$$

and the inverse transform is

$$h(t) = -2 - 2t - t^2 + 2e^t \tag{106}$$

HW 6.1 # 2, 4, 12, 15, (18 optional), 22, 24