Lecture Notes for Math 251: ODE and PDE. Lecture 20: 6.2 Solutions of Initial Value Problems

Shawn D. Ryan

Spring 2012

1 Laplace Transform for Derivatives

Last Time: We thoroughly studied Laplace and Inverse Laplace Transforms

Recall the following formula from a previous lecture

$$\mathcal{L}\{f^{(n)}\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - ... - s f^{(n-2)}(0) - f^{(n-1)}(0). \tag{1}$$

We will be dealing exclusively with second order differential equations, so we will need to remember

$$\begin{aligned} \mathcal{L}\{y'\} &= sY(s) - y(0) & (2) \\ \mathcal{L}\{y''\} &= s^2 Y(s) - sy(0) - y'(0) & (3) \end{aligned}$$

You should be familiar with the general formula from lecture 6.1.

REMARK: Notice that we must have our initial conditions at $t = 0$ to use Laplace Transforms.

Example 1. Solve the following IVP using Laplace Transforms

$$y'' - 5y' - 6y = 5t \quad y(0) = -1 \quad y'(0) = 2 \tag{4}$$

We begin by transforming both sides of the equation:

$$\begin{aligned} \mathcal{L}\{y''\} - 5\mathcal{L}\{y'\} - 6\mathcal{L}\{y\} &= 5\mathcal{L}\{t\} & (5) \\ s^2 Y(s) - sy(0) - y'(0) - 5sY(s) + 5y(0) - 6Y(s) &= \frac{5}{s^2} & (6) \\ (s^2 - 5s - 6)Y(s) + s - 2 - 5 &= \frac{5}{s^2}. & (7) \end{aligned}$$

As we have already begun doing, now we solve for $Y(s)$.

$$\begin{aligned} Y(s) &= \frac{5}{s^2(s^2 - 5s - 6)} + \frac{7 - s}{s^2 - 5s - 6} & (8) \\ &= \frac{5}{s^2(s-6)(s+1)} + \frac{7 - s}{(s-6)(s+1)} & (9) \\ &= \frac{5 + 7s^2 - s^3}{s^2(s-6)(s+1)} & (10) \end{aligned}$$

We now have an expression for $Y(s)$, which is the Laplace Transform of the solution $y(t)$ to the initial value problem. We have simplified as much as possible now we need partial fractions decomposition to take the inverse transform

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-6} + \frac{D}{s+1}. \tag{11}$$

Multiplying through by $s^2(s-6)(s+1)$ gives

$$6 + 7s^2 + s^3 = As(s-6)(s+1) + B(s-6)(s+1) + Cs^2(s+1) + Ds^2(s-6) \tag{12}$$

We can find the constant by choosing key values of s

$$\begin{aligned}
s = 0: \quad 6 &= -6B \quad \Rightarrow \quad B = -1 && (13)\\
s - 6: \quad 42 &= 252C \quad \Rightarrow \quad C = \frac{1}{6} && (14)\\
s = -1: \quad 14 &= -7D \quad \Rightarrow \quad A = \frac{1}{12} && (15)
\end{aligned}$$

So

$$\begin{aligned}
Y(s) &= \frac{1}{12}\frac{1}{s} - \frac{1}{s^2} + \frac{1}{6}\frac{1}{s-6} - \frac{1}{2}\frac{1}{s+1} && (16)\\
y(t) &= \frac{1}{12} - t + \frac{1}{6}e^{6t} - \frac{1}{2}e^{-t}. && (17)
\end{aligned}$$

EXERCISE: Solve the IVP in the previous example using the Method of Undetermined Coefficients. Do you get the same thing? Which method took less work?

Example 2. Solve the following initial value problem

$$y'' + 2y' + 5y = \cos(t) - 10t, \quad y(0) = 0, \quad y'(0) = 1 \tag{18}$$

We begin by transforming the entire equation and solving for $Y(s)$.

$$\begin{aligned}
\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} &= \mathcal{L}\{\cos(t)\} - 10\mathcal{L}\{t\} && (19)\\
s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + 5Y(s) &= \frac{s}{s^2+1} - \frac{10}{s^2} && (20)\\
(s^2 + 2s + 5)Y(s) - 1 &= \frac{s}{s^2+1} - \frac{10}{s^2} && (21)
\end{aligned}$$

So we have

$$\begin{aligned}
Y(s) &= \frac{s}{(s^2+1)(s^2+2s+5)} - \frac{10}{s^2(s^2+2s+5)} + \frac{1}{s^2+2s+5} && (22)\\
&= Y_1(s) + Y_2(s) + Y_3(s). && (23)
\end{aligned}$$

Now we will have to take inverse transforms. This will require doing partial fractions on the first two pieces.

Let's start with the first one

$$Y_1(s) = \frac{s}{(s^2+1)(s^2+2s+5)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+5} \tag{24}$$

Multiply through by $(s^2+1)(s^2+2s+5)$

$$\begin{aligned} 2 &= As(s^2+2s+5) + B(s^2+2s+5) + Cs(s^2+1) + D(s^2+1) && (25)\\ &= (A+C)s^3 + (2A+B+D)s^2 + (5A+2B+C)s + (5B+D) && (26) \end{aligned}$$

This gives us the following system of equations, which we solve

$$\begin{aligned} A+C &= 0 && (27)\\ 2A+B+D &= 0 && (28)\\ 5A+2B+C &= 1 \quad \Rightarrow \quad A=\frac{1}{5} \;\; B=\frac{1}{10} \;\; C=-\frac{1}{5} \;\; D=-\frac{1}{2} && (29)\\ 5B+D &= 0 && (30) \end{aligned}$$

Thus our first term becomes

$$Y_1(s) = \frac{1}{5}\frac{s}{s^2+1} + \frac{1}{10}\frac{1}{s^2+1} - \frac{1}{5}\frac{s}{s^2+2s+5} - \frac{1}{2}\frac{1}{s^2+2s+5}. \tag{31}$$

We will hold off on taking the inverse transform until we solve for Y_2.

$$Y_2(s) = -\frac{10}{s^2(s^2+2s+5)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+2s+5} \tag{32}$$

We multiply through by $s^2(s^2+2s+5)$

$$\begin{aligned} -10 &= As(s^2+2s+5) + B(s^2+2s+5) + Cs^3 + Ds^2 && (33)\\ &= (A+C)s^3 + (2A+B+D)s^2 + (5A+2B)s + 5B && (34) \end{aligned}$$

This gives the following system of equations

$$\begin{aligned} A+C &= 0 && (35)\\ 2A+B+D &= 0 && (36)\\ 5A+2B &= 0 \quad \Rightarrow \quad A=\frac{4}{5} \;\; B=-2 \;\; C=-\frac{4}{5} \;\; D=\frac{2}{5} && (37)\\ 5B &= -10 && (38) \end{aligned}$$

Thus we have

$$Y_2(s) = \frac{4}{5}\frac{1}{s} - \frac{2}{s^2} - \frac{4}{5}\frac{s}{s^2+2s+5} + \frac{2}{5}\frac{1}{s^2+2s+5}. \tag{39}$$

Let's return to our original function

$$\begin{aligned} Y(s) &= Y_1(s) + Y_2(s) + Y_3(s) && (40)\\ &= \frac{1}{5}\frac{s}{s^2+1} + \frac{1}{10}\frac{1}{s^2+1} + \frac{4}{5}\frac{1}{s} - \frac{2}{s^2} + \left(-\frac{1}{5} - \frac{4}{5}\right)\frac{s}{s^2+2s+5} + \left(-\frac{1}{2} + \frac{2}{5} + 1\right)\frac{1}{s^2+2s+5} && (41)\\ &= \frac{1}{5}\frac{s}{s^2+1} + \frac{1}{10}\frac{1}{s^2+1} + \frac{4}{5}\frac{1}{s} - \frac{2}{s^2} - \frac{s}{(s+1)^2+4} + \frac{9}{10}\frac{1}{(s+1)^2+4} && (42) \end{aligned}$$

Now we have to adjust the last two terms to make them suitable for the inverse transform. Namely, we need to have $s+1$ in the numerator of the second to last term, and 2 in the numerator of the last term.

$$= \frac{1}{5}\frac{s}{s^2+1} + \frac{1}{10}\frac{1}{s^2+1} + \frac{4}{5}\frac{1}{s} - \frac{2}{s^2} - \frac{s+1-1}{(s+1)^2+4} + \frac{9}{10}\frac{1}{(s+1)^2+4} \tag{43}$$

$$= \frac{1}{5}\frac{s}{s^2+1} + \frac{1}{10}\frac{1}{s^2+1} + \frac{4}{5}\frac{1}{s} - \frac{2}{s^2} - \frac{s+1}{(s+1)^2+4} + \frac{19}{10}\frac{1}{(s+1)^2+4} \tag{44}$$

$$= \frac{1}{5}\frac{s}{s^2+1} + \frac{1}{10}\frac{1}{s^2+1} + \frac{4}{5}\frac{1}{s} - \frac{2}{s^2} - \frac{s+1}{(s+1)^2+4} + \frac{19}{20}\frac{2}{(s+1)^2+4} \tag{45}$$

(46)

So our solution is

$$y(t) = \frac{1}{5}\cos(t) + \frac{1}{10}\sin(t) + \frac{4}{5} - 2t - e^{-t}\cos(2t) + \frac{19}{20}e^{-t}\sin(2t). \tag{47}$$

We could have done both the preceding examples using the method of Undetermined Coefficients. In fact, it would have been a lot less work.

The following examples are to be done after learning Section 6.3. Let's do some involving step functions, which is where Laplace Transforms work great.

Example 3. Solve the following initial value problem.

$$y'' - y' + 6y = 2 - u_2(t)e^{2t-4} \quad y(0) = 0 \quad y'(0) = 0 \tag{48}$$

As before, we begin by transforming everything. Before we do that, however, we need to write the coefficient function $u_2(t)$ as a function evaluated at $t-2$.

$$y'' - 5y' + 6y = 2 - u_2(t)e^{2(t-2)} \tag{49}$$

Now we can transform

$$\mathcal{L}\{y''\} - 5\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} = 2\mathcal{L}\{1\} - \mathcal{L}\{u_2(t)e^{2(t-2)}\} \tag{50}$$

$$s^2Y(s) - sy(0) - y'(0) - 5sY(s) + 5y(0) - 6Y(s) = \frac{2}{s} - e^{-2s}\mathcal{L}\{e^{2t}\} \tag{51}$$

$$(s^2 - 5s + 6)Y(s) = \frac{2}{s} - e^{-2s}\frac{1}{s-2} \tag{52}$$

So we end up with

$$Y(s) = \frac{2}{s(s-3)(s-2)} - e^{-2s}\frac{1}{(s-3)(s-2)^2} \tag{53}$$

$$= Y_1(s) + e^{-2s}Y_2(s). \tag{54}$$

Since one of the terms has an exponential, we will need to deal with each term separately. I'll leave it to you to check the partial fractions.

$$\begin{aligned} Y_1(s) &= \frac{1}{3}\frac{1}{s} + \frac{2}{3}\frac{1}{s-3} - \frac{1}{s-2} && (55)\\ Y_2(s) &= -\frac{1}{s-3} + \frac{1}{s-2} + \frac{1}{(s-2)^2} && (56) \end{aligned}$$

Thus we have

$$Y(s) = \frac{1}{2}\frac{1}{s} + \frac{2}{3}\frac{1}{s-3} - \frac{1}{s-2} + e^{-2s}\left(-\frac{1}{s-3} + \frac{1}{s-2} + \frac{1}{(s-2)^2}\right) \qquad (57)$$

and taking the inverse transforms

$$\begin{aligned} y(t) &= \frac{1}{2} + \frac{2}{3}e^{3t} - e^{2t} + u_2(t)\left(-e^{3(t-2)} + e^{2(t-2)} + (t-2)e^{2(t-2)}\right) && (58)\\ &= \frac{1}{2} + \frac{2}{3}e^{3t} - e^{2t} + u_2(t)\left(-e^{3t-6} - e^{2t-4} + te^{2t-4}\right) && (59) \end{aligned}$$

once we observe that $\mathcal{L}\{\frac{1}{(s-a)^2}\} = te^{at}$.

Example 4. Solve the following initial value problem

$$y'' + 4y = 8 + tu_4(t), \quad y(0) = 0, \quad y'(0) = 0 \qquad (60)$$

We need to first write the coefficient function of $u_4(t)$ in the form $h(t-4)$ for some function $h(t)$. So we write $h(t-4) = t = t-4+4$ and conclude $h(t) = t+4$. So our equation is

$$y'' + 4y = 8 + \big((t-4)+4\big)u_4(t). \qquad (61)$$

Now, we want to Laplace Transform everything.

$$\begin{aligned} \mathcal{L}\{y''\} + 4\mathcal{L}\{y\} &= 8\mathcal{L}\{1\} + \mathcal{L}\{(t-4)+4)u_4(t)\} && (62)\\ s^2Y(s) - sy(0) - y'(0) + 4Y(s) &= \frac{8}{s} + e^{-4s}\mathcal{L}\{t+4\} && (63)\\ (s^2+4)Y(s) &= \frac{8}{s} + e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right) && (64) \end{aligned}$$

So we have

$$\begin{aligned} Y(s) &= \frac{8}{s(s^2+4)} + e^{-4s}\left(\frac{1}{s^2(s^2+4)} + \frac{4}{s(s^2+4)}\right) && (65)\\ &= \frac{8}{s(s^2+4)} + e^{-4s}\frac{1+4s}{s^2(s^2+4)} && (66)\\ &= Y_1(s) + e^{-4s}Y_2(s). && (67) \end{aligned}$$

where we have consolidated the two fractions being multiplied by the exponential to reduce the number of partial fraction decompositions we need to compute. After doing partial fractions (leaving the details for you), we have

$$Y_1(s) = \frac{2}{s} - \frac{2s}{s^2+4} \tag{68}$$

and

$$Y_2(s) = \frac{1}{s} + \frac{1}{4}\frac{1}{s^2} - \frac{s}{s^2+4} - \frac{1}{4}\frac{1}{s^2+4}. \tag{69}$$

so

$$\begin{aligned} Y(s) &= \frac{2}{s} - \frac{2s}{s^2+4} + e^{-4s}\left(\frac{1}{s} + \frac{1}{4}\frac{1}{s^2} - \frac{s}{s^2+4} - \frac{1}{4}\frac{1}{s^2+4}\right) && (70) \\ &= \frac{2}{s} - 2\frac{s}{s^2+4} + e^{-4s}\left(\frac{1}{s} + \frac{1}{4}\frac{1}{s^2} - \frac{s}{s^2+4} - \frac{1}{8}\frac{2}{s^2+4}\right) && (71) \\ & && (72) \end{aligned}$$

and the solution is

$$\begin{aligned} y(t) &= 2 - 2\cos(2t) + u_4(t)\left(1 - \frac{1}{4}(t-4) - \cos(2(t-4)) - \frac{1}{8}\sin(2(t-4))\right) && (73) \\ &= 2 - 2\cos(2t) + u_4(t)\left(\frac{1}{4}t - \cos(2t-8) - \frac{1}{8}\sin(2t-8)\right). && (74) \end{aligned}$$

HW 6.2 # 1, 3, 6, 8, 10, 12, 13, 16, 21, 23