Lecture Notes for Math 251: ODE and PDE. Lecture 21: 6.3 Step Functions

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Spring 2012

1 Step Functions

Last Time: We thoroughly studied the solution process for IVPs using Laplace Transform Methods.

When we originally defined the Laplace Transform we said we could use it for piecewise continuous functions, but so far we have only used it for continuous functions. We would like to solve differential equations with forcing functions that were not continuous, but had isolated points of discontinuity where the forcing function jumped from one value to another abruptly. An example is a mechanical vibration where we add some extra force later on in the process. Without Laplace Transforms we would have to split these IVPs into several different problems, with initial conditions derived from previous solutions.

1.1 Step Functions

Consider the following function

$$u(t-c) = u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases} \tag{1}$$

This function is called the **step** or **Heavyside function** at c. It represents a jump at $t = c$ from zero to one at $t = c$. One can think of a step function as a switch that turns on its coefficient at a specific time. The step function takes only values of 0 or 1, but is easy enough to give it any value desired by changing its coefficient. $4u_c(t)$ has the value 4 at $t = c$ and beyond. We can produce a switch that turns off at $t = c$,

$$1 - u_c(t) = \begin{cases} 1 - 0 = 1 & \text{if } t < c \\ 1 - 1 = 0 & \text{if } t \geq c \end{cases} \tag{2}$$

it will exhibit the desired behavior. This function will get any contribution we want prior to $t = c$. This is the key to writing a piecewise continuous function as a single expression instead of a system of cases.

Example 1. Write the following function in terms of step functions.

$$f(t) = \begin{cases} 9 & \text{if } t < 2 \\ -6 & \text{if } 2 \leq t < 6 \\ 25 & \text{if } 6 \leq t < 9 \\ 7 & \text{if } 9 \leq t \end{cases} \tag{3}$$

There are three jumps in this function at $t = 2$, $t = 6$, and $t = 9$. So we will need a total of three step functions, each of which will correspond to one of these jumps. In terms of step functions,

$$f(t) = 9 - 15u(t-2) + 31u(t-6) - 18u(t-9) \tag{4}$$

How did we come up with this?

When $t < 2$, all of the step functions have the value 0. So the only contributing term in our expression for $f(t)$ is 9, and on this region $f(t) = 9$.

On the next interval, $2 \leq t < 6$, we want $f(t) = -6$. The first step function $u(t-2)$ is on, while the others are off. Notice that 9 is still present and as a result, the coefficient of $u(t-2)$ will need to cause the sum of it and 9 to equal -6. So the coefficient must be -15.

On the third interval, $6 \leq t < 9$, we now have two functions turned on while the last one is off. The first two terms contribute $9 - 15u(t-2)$. Thus the coefficient of $u(t-6)$ will need to combine with these to give our desired value of $f(t) = 25$. So it must be 31.

The last term, we have the interval $9 \leq t$. Now all the switches are turned on and the coefficient of $u(t-9)$, the step function corresponding to the final jump, should move us from our previous value of 25 to our new value, $f(t) = 7$. As a result, it must be -18.

We are not just interested in situations where our forcing function takes constant values on intervals. In the case of mechanical vibrations of the sort we considered earlier, we might want to add in a new external force which is sinusoidal.

So consider the following piecewise continuous function: $g(t) = u(t-c)f(t-c)$, where $f(t)$ is some function. We shift it by c, the starting point of the step function, to indicate that we want it to start working at $t = c$ instead of $t = 0$, which it would normally. Think of this graphically, to get the graph of $g(t)$, what we want to do is take the graph of $f(t)$, starting at $t = 0$, and push it to start at $t = c$ with the value of 0 prior to this time. This requires shifting the argument of f by c.

1.2 Laplace Transform

What is the Laplace Transform $\mathcal{L}\{g(t)\}$?

$$\begin{aligned} \mathcal{L}\{g(t)\} &= \mathcal{L}\{u(t-c)f(t-c)\} && (5) \\ &= \int_0^\infty u(t-c)e^{-st}f(t-c)dt && (6) \\ &= \int_c^\infty e^{-st}f(t-c) \quad \text{using the definition of the step function} && (7) \end{aligned}$$

Now this looks almost like a Laplace Transform, except that the integral starts at $t = c$ instead of $t = 0$. So we introduce a new variable $y = t - c$ to shift the integral back to starting at 0.

$$\begin{aligned} G(s) &= \int_0^\infty e^{-s(u+c)} f(u)du & (8) \\ &= e^{-sc}\int_0^\infty e^{-su} f(u)du & (9) \\ &= e^{-sc}F(s) \quad \text{using the notation} F(s) = \mathcal{L}\{f(u)\}. & (10) \end{aligned}$$

Notice that the Laplace Transform we end up with is the Laplace Transform of the **original** function $f(t)$ multiplied by an exponential related to the step function's "on" time, even though we had shifted the function by c to begin with. Summarizing, we have the formula

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-sc}F(s) = e^{-sc}\mathcal{L}\{f(t)\}. \tag{11}$$

It is **critical** that we write the function to be transformed in the correct form, as a different function shifted by c, before we transform it using the above equation. We compute the transform of $f(t)$ NOT the shifted function $f(t-c)$. This is the most common mistake initially.

We can get a formula for a step function by itself. To do so, we consider a step function multiplied by the constant function $f(t) = 1$. In this case $f(t-c) = 1$. So

$$\mathcal{L}\{u_c(t)\} = \mathcal{L}\{u_c(t)\cdot 1\} = e^{-cs}\mathcal{L}\{1\} = \frac{1}{s}e^{-cs}. \tag{12}$$

Example 2. Find the Laplace Transforms of each of the following

(i) $f(t) = 10u_6(t) + 3(t-4)^2u_4(t) - (1 + e^{10-2t})u_5(t)$

Recall that we must write each piece in the form $u_c(t)h(t-c)$ before we take the transform. If it is not in that form, we have to put it in that form first. There are three terms in $f(t)$. We will use the linearity of the Laplace Transform to treat them separately, then add them together in the end. Write

$$f(t) = f_1(t) + f_2(t) + f_3(t) \tag{13}$$

$f_1(t) = 10u_6(t)$, so it is just a constant multiplied by a step function. We can thus us Equation (**??**) to determine its Laplace Transform

$$\mathcal{L}\{f_1(t)\} = 10\mathcal{L}\{u_6(t)\} = \frac{10e^{-6s}}{s} \tag{14}$$

$f_2(t) = 3(t-4)^2u_4(t)$, so we have to do two things: 1. Write it as a function shifted by 4 (if not in that form already) and 2. Isolate the function that was shifted and transform it. In this case, we are good: we can write $f_2(t) = h(t-4)u_4(t)$, with $h(t) = 3t^2$. Thus

$$\mathcal{L}\{f_2(t)\} = e^{-4s}\mathcal{L}\{3t^2\} = 3e^{-4s}\frac{2}{s^3} = \frac{6e^{-4s}}{s^3}. \tag{15}$$

Finally, we have $f_3(t) = -(1 + e^{10-2t})u_5(t)$. Again, we have to express it as a function shifted by 5 and then identify the unshifted function so that we may transform it. This can be accomplished by rewriting

$$f_3(t) = -(1 + e^{-2(t-5)})u_5(t), \tag{16}$$

so, writing $g_3(t) = h(t-5)u_5(t)$, we have $h(t) = -(1+e^{-2t})$ as the unshifted coefficient function. Thus

$$\mathcal{L}\{f_3(t)\} = e^{-5s}\mathcal{L}\{-(1+e^{-2t})\} = -e^{-5s}\left(\frac{1}{s} + \frac{1}{s+2}\right). \tag{17}$$

Putting it all together,

$$F(s) = \frac{10e^{-6s}}{s} + \frac{6e^{-4s}}{s^3} - e^{-5s}\left(\frac{1}{s} + \frac{1}{s+2}\right). \tag{18}$$

(ii) $g(t) = t^2u_2(t) - \cos(t)u_7(t)$

In the last example, it turned out that all of the coefficient functions were pre-shifted (the most we had to do was pull out a constant to see that). In this example, that is definitely not the case. So what we want to do is to write each of our coefficient functions as the shift (by whichever constant is appropriate for that step function) of a different function. The idea is that we add and subtract the desired quantity, then simplify, keeping the correct shifted term.

So, let's write $g(t) = g_1(t) + g_2(t)$ where

$$g_1(t) = t^2u_2(t) \tag{19}$$
$$g_1(t) = (t-2+2)^2u_2(t) \tag{20}$$

This is not quite there yet, use the Associative Property of Addition

$$g_1(t) = ((t-2)+2)^2u_2(t) \tag{21}$$
$$= ((t-2)^2 + 4(t-2) + 4)u_2(t). \tag{22}$$

Now we can see that $g_1(t) = h(t-2)u_2(t)$, where $h(t) = t^2 + 4t + 4$.

$$\mathcal{L}\{g_1(t)\} = e^{-2s}\mathcal{L}\{t^2 + 4t + 4\} = e^{-2s}(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}) \tag{23}$$

The second term is similar. We start with

$$g_2(t) = -\cos(t)u_7(t) = -\cos((t-7)+7)u_7(t) \tag{24}$$

Here we need to use the trig identity

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b). \tag{25}$$

This yields

$$g_2(t) = -(\cos(t-7)\cos(7) - \sin(t-7)\sin(7))u_7(t) \tag{26}$$

Since $\cos(7)$ and $\sin(7)$ are just constants we get (after using the linearity of the Laplace Transform)

$$\mathcal{L}\{g_2(t)\} = -e^{-7s}(\cos(7)\mathcal{L}\{\cos(t)\} - \sin(7)\mathcal{L}\{\sin(t)\}) \tag{27}$$
$$= -e^{-7s}(\frac{s\cos(7)}{s^2+1} - \frac{\sin(7)}{s^2+1}). \tag{28}$$

Piecing back together we get

$$G(s) = e^{-2s}(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}) - e^{-7s}(\frac{s\cos(7) - \sin(7)}{s^2+1}). \tag{29}$$

(iii) $f(t) = \begin{cases} t^3 & \text{if } t < 4 \\ t^3 + 2\sin(\frac{t}{12} - \frac{1}{3}) & \text{if } 4 \leq t \end{cases}$

The first step is to write $f(t)$ as a single equation using step functions.

$$f(t) = t^3 + 2\sin(\frac{t}{12} - \frac{1}{3})u_4(t) \tag{30}$$

Next, we want to write the coefficients of $u_4(t)$ as another function shifted by 4.

$$f(t) = t^3 + 2\sin(\frac{1}{12}(t-4))u_4(t). \tag{31}$$

Since everything is shifted, we have

$$\begin{aligned} F(s) &= \mathcal{L}\{t^3\} + 2e^{-4s}\mathcal{L}\{\sin(\frac{1}{12}t)\} && (32) \\ &= \frac{3!}{s^4} + 2e^{-4s}\frac{\frac{1}{12}}{s^2 + (\frac{1}{12})^2} && (33) \\ &= \frac{6}{s^4} + \frac{e^{-4s}}{12(s^2 + \frac{1}{144})} && (34) \\ &= \frac{6}{s^4} + \frac{e^{-4s}}{12s^2 + \frac{1}{12}}. && (35) \end{aligned}$$

(iv) $g(t) = \begin{cases} t & \text{if } t < 2 \\ 3 + (t-2)^2 & \text{if } 2 \leq t \end{cases}$

First, we need to write $g(t)$ using step functions.

$$g(t) = t + (-t + 3 + (t-2)^2)u_2(t). \tag{36}$$

Notice that we had to subtract t from the coefficient of $u_2(t)$ in order to make $g(t)$ have the correct value when $t \geq 2$. However, this means that the coefficient function of $u_2(t)$ is no longer properly shifted. As a result, we need to add and subtract 2 from that t to make it have the proper form.

$$\begin{aligned} g(t) &= t + (-(t - 2 + 2) + 3 + (t-2)^2)u_2(t) && (37) \\ &= t + (-(t-2) - 2 + 3 + (t-2)^2)u_2(t) && (38) \\ &= t + (-(t-2) + 1 + (t-2)^2)u_2(t) && (39) \end{aligned}$$

So, we have

$$\begin{aligned} G(s) &= \mathcal{L}\{t\} + e^{-2s}(\mathcal{L}\{-t\} + \mathcal{L}\{1\} + \mathcal{L}\{t^2\}) && (40) \\ &= \frac{1}{s^2} + e^{-2s}(-\frac{1}{s^2} + \frac{1}{s} + \frac{2}{s^3}). && (41) \end{aligned}$$

As you can see, taking the Laplace Transforms of the functions involving step functions can be a bit more complicated than taking Laplace Transforms of original functions. It still is not too bad, we have to make sure our coefficient functions are appropriately shifted.

HW 6.3 # 1, 2, 3, 4, 5, 7, 8, 9, 13, 15, 16, 19, 21