Lecture Notes for Math 251: ODE and PDE. Lecture 21: 6.4 Differential Equations with Discontinuous Forcing Functions

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1 Differential Equations With Discontinuous Forcing Functions

Last Time: We considered the Laplace Transforms of Step Functions.

$$\mathcal{L}\{u(t-c)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} \tag{1}$$

where $f(t-c)$ is the coefficient function of $u(t-c)$.

1.1 Inverse Step Functions

Now let's look at some inverse transforms. The previous formula's associated inverse transform is

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u(t-c)f(t-c), \tag{2}$$

where $f(t) = \mathcal{L}^{-1}\{F(s)\}$. So we need to be careful about the shifting. This time we shift at the end of the calculation, after finding the inverse transform of the coefficient of the exponential.

Example 1. Find the inverse Laplace Transform of the following.

(i) $F(s) = \frac{se^{-2s}}{(s-4)(s+3)}$

Whenever we do this, it is a good idea to ignore the exponential and determine the inverse transform of what is left over first. In this case we cannot split anything up, since there is only one exponential and no terms without an exponential. So we pull out the exponential and ignore it for the time being

$$F(s) = e^{-2s}\frac{s}{(s-4)(s+3)} = e^{-2s}H(s). \tag{3}$$

We want to determine $h(t) = \mathcal{L}^{-1}\{H(s)\}$. Once we have that, the definition of the inverse transform tells us that the inverse will be

$$f(t) = h(t-2)u(t-2). \tag{4}$$

Now we need to use partial fractions on $H(s)$ so that we can take its inverse transform. The form of the decomposition is

$$H(s) = \frac{A}{s-4} + \frac{B}{s+3} \tag{5}$$

So

$$s = A(s+3) + B(s-4) \tag{6}$$

We can use the quick method of picking key values of s

$$s = 4: \quad 4 = 7A \quad \Rightarrow \quad A = \frac{4}{7} \tag{7}$$

$$s = -3: \quad -3 = -7B \quad \Rightarrow \quad B = \frac{3}{7} \tag{8}$$

So the partial fraction decomposition is

$$H(s) = \frac{4}{7}\frac{1}{s-4} + \frac{3}{7}\frac{1}{s+3} \tag{9}$$

$$= \frac{4}{7}\frac{1}{s-4} + \frac{3}{7}\frac{1}{s+3}. \tag{10}$$

So we have

$$h(t) = \frac{4}{7}e^{4t} + \frac{3}{7}e^{-3t}. \tag{11}$$

Thus, since $f(t) = h(t-2)u(t-2)$,

$$f(t) = u(t-2)(\frac{4}{7}e^{4(t-2)} + \frac{3}{7}e^{-3(t-2)}) \tag{12}$$

$$= u(t-2)(\frac{4}{7}e^{4t-8} + \frac{3}{7}e^{-3t+6}). \tag{13}$$

(ii) $G(s) = \frac{2e^{-3s}+3e^{-7s}}{(s-3)(s^2+9)}$

We write

$$G(s) = (2e^{-3s} + 3e^{-7s})\frac{1}{(s-3)(s^2+9)} = (2e^{-3s} + 3e^{-7s})H(s). \tag{14}$$

We want to find the inverse transform of

$$H(s) = \frac{1}{(s-3)(s^2+9)}. \tag{15}$$

The partial fraction decomposition is

$$H(s) = \frac{A}{s-3} + \frac{Bs+C}{s^2+9} = \frac{A(s^2+9) + (Bs+C)(s-3)}{(s-3)(s^2+9)} \tag{16}$$

So we have the following

$$\begin{aligned} 1 &= A(s^2+0)+(Bs+C)(s-3) && (17)\\ &= (A+B)s^2+(-3B+C)s+(9A-3C) && (18)\end{aligned}$$

Setting the coefficients equal and solving gives

$$\begin{aligned} s^2: \quad A+B &= 0 && (19)\\ s: \quad -3B+C &= 0 \quad \Rightarrow \quad A=\frac{1}{18}\ \ B=-\frac{1}{18}\ \ C=-\frac{3}{18} && (20)\\ s^0: \quad 9A-3C &= 1 && (21)\end{aligned}$$

Substituting back into the transform, we get

$$\begin{aligned} H(s) &= \frac{1}{18}\left(\frac{1}{s-3}+\frac{-s-3}{s^2+9}\right) && (22)\\ &= \frac{1}{18}\left(\frac{1}{s-3}-\frac{s}{s^2+9}-\frac{3}{s^2+9}\right). && (23)\end{aligned}$$

Now, if we take the inverse transform, we get

$$h(t)=\frac{1}{18}(e^{3t}-\cos(3t)-\sin(3t)). \quad (24)$$

Returning to the original problem, we had

$$\begin{aligned} G(s) &= (2e^{-3s}+3e^{-7s})H(s) && (25)\\ &= 2e^{-3s}H(s)+3e^{-7s}H(s). && (26)\end{aligned}$$

We had to distribute $H(s)$ through the parenthesis and use the definition, since we must end up with each term containing one step function and one coefficient function. So

$$\begin{aligned} g(t) &= 2h(t-3)u(t-3)+3h(t-7)u(t-7)) && (27)\\ &= \frac{u(t-3)}{9}(e^{3(t-3)}-\cos(3(t-3))-\sin(3(t-3))) && (28)\\ &+ \frac{3u(t-7)}{18}(e^{3(t-7)}-\cos(3(t-7))-\sin(3(t-7))) && (29)\end{aligned}$$

(iii) $F(s)=\frac{e^{-4s}}{s^2(s+1)}$
We write

$$\begin{aligned} F(s) &= e^{-4s}\frac{1}{s^2(s+1)}=e^{-4s}H(s) && (30)\\ H(s) &= \frac{A}{s^2}+\frac{B}{s}+\frac{C}{s+1} && (31)\\ 1 &= A(s+1)+Bs(s+1)+Cs^2 && (32)\\ &= (B+C)s^2+(A+B)s+A && (33)\end{aligned}$$

So we have

$$B + C = 0 \quad (34)$$
$$A + B = 0 \quad \Rightarrow \quad A = 1 \;\; B = -1 \;\; C = 1 \quad (35)$$
$$A = 1 \quad (36)$$

Thus $H(s)$ and its inverse transform are

$$H(s) = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1} \quad (37)$$
$$h(t) = t - 1 + e^{-t} \quad (38)$$

Our original transform function was

$$F(s) = e^{-4s}H(s). \quad (39)$$

By the definition of the inverse transform, it will be

$$\begin{aligned} f(t) &= h(t-4)u(t-4) && (40)\\ &= ((t-4) - 1 + e^{-(t-4)})u(t-4) && (41)\\ &= (t - 5 + e^{4-t})u(t-4) && (42) \end{aligned}$$

(iv) $G(s) = \frac{s-e^{-2s}}{s^2+2s+5}$

In this case, we won't have to do partial fractions, since the denominator does not factor. Instead, we will have to complete the square and we get

$$G(s) = \frac{s - e^{-2s}}{(s+1)^2+4} = \frac{s}{(s+1)^2+4} - e^{-2s}\frac{1}{(s+1)^2+4} = G_1(s) - e^{-2s}G_2(s) \quad (43)$$

We need to treat $G_1(s)$ and $G_2(s)$ separately. $G_1(t)$ is almost fine, but we need the numerator to contain $s+1$ instead of s. We do this by adding and subtracting 1 from the numerator

$$\begin{aligned} G_1(s) &= \frac{s+1-1}{(s+1)^2+4} && (44)\\ &= \frac{s+1}{(s+1)^2+4} - \frac{1}{(s+1)^2+4} && (45)\\ &= \frac{s+1}{(s+1)^2+4} - \frac{1}{2}\frac{2}{(s+1)^2+4} && (46)\\ g_1(t) &= e^{-t}\cos(2t) - \frac{1}{2}e^{-t}\sin(2t). && (47) \end{aligned}$$

For $G_2(t)$, all we have to of is to make the numerator 2 instead of 1.

$$\begin{aligned} G_2(s) &= \frac{1}{(s+1)^2+4} && (48)\\ &= \frac{1}{2}\frac{2}{(s+1)^2+4} && (49)\\ g_2(t) &= \frac{1}{2}e^{-t}\sin(2t) && (50) \end{aligned}$$

Our original transform was

$$G(s) = G_1(s) - e^{-2s}G_2(s) \tag{51}$$

By the definition of the inverse transform

$$\begin{aligned} g(t) &= g_1(t) - g_2(t-2)u(t-2) && (52)\\ &= e^{-t}\cos(2t) - \frac{1}{2}e^{-t}\sin(2t) - \frac{1}{2}e^{-(t-2)}\sin(2(t-2))u(t-2) && (53)\\ &= e^{-t}\cos(2t) - \frac{1}{2}e^{-t}\sin(2t) - \frac{1}{2}e^{2-t)}\sin(2t-4))u(t-2) && (54) \end{aligned}$$

HW 6.3 # 19,21

1.2 Solving IVPs with Discontinuous Forcing Functions

We want to now actually solve IVPs by using the Laplace Transform and our experience with step functions.

Example 2. Find the solution to the initial value problem

$$y'' + y = g(t), \quad y(0) = 0, \quad y'(0) = 1 \tag{55}$$

where

$$g(t) = \begin{cases} 2t & \text{if } 0 \leq t < 1 \\ 2 & \text{if } 1 \leq t < \infty \end{cases} \tag{56}$$

The Laplace Transform of the left side and using ICs we get

$$\mathcal{L}\{y'' + y\} = s^2Y(s) - sy(0) - y'(0) + Y(s) = (s^2+1)Y(s) - 1 \tag{57}$$

From techniques in the previous section we know

$$\mathcal{L}\{g(s)\} = G(s) = \frac{(2-2e^{-s})}{s^2} \tag{58}$$

So combining these two

$$(s^2+1)Y(s) - 1 = \frac{2-2e^{-s}}{s^2} \tag{59}$$

Solving for $Y(s)$ we get

$$Y(s) = \frac{2-2e^{-s}}{s^2(s^2+1)} + \frac{1}{s^2+1} \tag{60}$$

Note that if we use partial fractions decomposition

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1} \tag{61}$$

So the equation becomes

$$Y(s) = \left(\frac{2-2e^{-s}}{s^2} - \frac{2-2e^{-s}}{s^2+1}\right) + \frac{1}{s^2+1} \tag{62}$$

$$= \frac{2}{s^2} - \frac{2e^{-s}}{s^2} + \frac{2e^{-s}}{s^2+1} - \frac{1}{s^2+1} \tag{63}$$

From our table we know the inverse of $1/s^2$ is the function t, and the inverse transform of $1/(s^2+1)$ is $\sin(t)$. Thus

$$y(t) = 2t - 2(t-1)u(t-1) + 2u(t-1)\sin(t-1) - \sin(t) \tag{64}$$

The function $y(t)$ can also be written as

$$y(t) = \begin{cases} 2t - \sin(t) & \text{if } 0 \le t < 1 \\ 2 + 2\sin(t-1) - \sin(t) & \text{if } 1 \le t < \infty \end{cases} \tag{65}$$

Example 3. Actually $\#6$ in Homework Section of book. Solve the following IVP

$$y'' + 3y' + 2y = u_2(t), \quad y(0) = 0, \quad y'(0) = 1 \tag{66}$$

Take the Laplace Transform of both sides of the ODE

$$s^2Y(s) - sy(0) - y'(0) + 3[sY(s) - y(0)] + 2Y(s) = \frac{e^{-2s}}{s} \tag{67}$$

Applying the initial conditions,

$$s^2Y(s) + 3sY(s) + 2Y(s) - 1 = \frac{e^{-2s}}{s} \tag{68}$$

Solving for the transform

$$Y(s) = \frac{1}{s^2+3s+2} + \frac{e^{-2s}}{s(s^2+3s+2)}. \tag{69}$$

Using Partial Fractions Decomposition

$$\frac{1}{s^2+3s+2} = \frac{1}{s+1} - \frac{1}{s+2} \tag{70}$$

and

$$\frac{1}{s(s^2+3s+2)} = \frac{1}{2}\left[\frac{1}{s} + \frac{1}{s+2} - \frac{2}{s+1}\right] \tag{71}$$

Taking the inverse transform term by term the solution is

$$y(t) = e^{-t} - e^{-2t} + \left[\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\right]u_2(t). \tag{72}$$

Example 4. $\#5$ in the Homework Section of the book. Solve the following IVP

$$y'' + 3y' + 2y = f(t), \quad y(0) = 0, \quad y'(0) = 0 \tag{73}$$

where

$$f(t) = \begin{cases} 1 & \text{if } 0 \leq t < 10 \\ 0 & \text{if } 10 \leq t < \infty \end{cases} \tag{74}$$

Finding the Laplace Transform of both sides of the ODE we have

$$s^2Y(s) - sy(0) - y'(0) + 3[sY(s) - y(0)] + 2Y(s) = \mathcal{L}\{f(t)\} \tag{75}$$

Applying the Initial Conditions

$$s^2Y(s) + 3sY(s) + 2Y(s) = \mathcal{L}\{f(t)\} \tag{76}$$

The transform of the forcing function is

$$\mathcal{L}\{f(t)\} = \frac{1}{s} - \frac{e^{-10s}}{s}. \tag{77}$$

Solving for the transform gives

$$Y(s) = \frac{1}{s(s^2 + 3s + 2)} - \frac{e^{-10s}}{s(s^2 + 3s + 2)}. \tag{78}$$

Using Partial Fractions Decomposition

$$\frac{1}{s(s^2 + 3s + 2)} = \frac{1}{2}\left[\frac{1}{s} + \frac{1}{s+2} - \frac{2}{s+1}\right]. \tag{79}$$

Thus

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2 + 3s + 2)}\right] = \frac{1}{2} + \frac{e^{-2t}}{2} - e^{-t}. \tag{80}$$

So using the inverse theorem for Step Functions

$$\mathcal{L}^{-1}\left[\frac{e^{-10s}}{s(s^2 + 3s + 2)}\right] = \left[\frac{1}{2} + \frac{e^{-2t}}{2} - e^{-t}\right]u_{10}(t). \tag{81}$$

Hence the solution to the IVP is

$$y(t) = \frac{1}{2}[1 - u_{10}(t)] + \frac{e^{-2t}}{2} - e^{-t} - \frac{1}{2}[e^{-(2t-20)} - 2e^{-(t-10)}]u_{10}(t). \tag{82}$$

HW 6.4 # 2,4,7,9,11