Lecture Notes for Math 251: ODE and PDE. Lecture 22: 6.5 Dirac Delta and Laplace Transforms

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1 Dirac Delta and the Laplace Transform

Last Time: We studied differential equations with discontinuous forcing functions. Now we want to look at when that forcing function is a **Dirac Delta**. When we considered step functions we considered these to be like switches which get turned on after a given amount of time. In applications this represented a new external force applied at a certain time. What if instead we wanted to apply a large force over a short period of time? In applications this represents a hammer striking an object once or a force applied only at one instant. We want to introduce the Dirac Delta "function" to accomplish this goal.

1.1 The Dirac Delta

Paul Dirac (1902 - 1984) was a British Physicist who studied quantum Mechanics. Famous Quote on Poetry: "In Science one tries to tell people, in such a way as to be understood by everyone, something that no one ever knew before. But in poetry, it's exactly the opposite."

There are several ways to define the Dirac delta, but we will define it so it satisfies the following properties:

Definition 1. *(Dirac Delta)* The Dirac delta at $t = c$, denoted $\delta(t - c)$, satisfies the following:
(1) $\delta(t - c) = 0$, when $t \neq c$
(2) $\int_{c-\epsilon}^{c+\epsilon} \delta(t - c)dt = 1$, for any $\epsilon > 0$
(3) $\int_{c-\epsilon}^{c+\epsilon} f(t)\delta(t - c)dt = f(c)$, for any $\epsilon > 0$.

We can think of $\delta(t - c)$ as having an "infinite" value at $t = c$, so that its total energy is 1, all concentrated at a single point. So think of the Dirac delta as an instantaneous impulse at time $t = c$. The second and third properties will not work when the limits are the endpoints of any interval including $t = c$. This function is zero everywhere but one point, and yet the integral is 1. This is why it is a "function", the Dirac delta is not really a function, but in higher mathematics it is referred to as a **generalized function** or a **distribution**.

1.2 Laplace Transform of the Dirac Delta

Before we try solving IVPs with Dirac Deltas, we will need to know its Laplace Transform. By definition

$$\mathcal{L}\{\delta(t-c)\} = \int_0^\infty e^{-st}\delta(t-c)dt = e^{-cs} \tag{1}$$

by the third property of the Dirac delta. Notice that this requires $c > 0$ or the integral would just vanish. Now let's try solving some IVPs involving Dirac deltas

Example 2. Solve the following initial value problem

$$y'' + 3y' - 10y = 4\delta(t-2), \quad y(0) = 2, \quad y'(0) = -3 \tag{2}$$

We begin by taking the Laplace Transform of both sides of the equation.

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + 3(sY(s) - y(0)) - 10Y(s) &= 4e^{-2s} &(3)\\ (s^2 + 3s - 10)Y(s) - 2s - 3 &= 4e^{-2s} &(4)\end{aligned}$$

So

$$\begin{aligned} Y(s) &= \frac{4e^{-2s}}{(s+5)(s-2)} + \frac{2s+3}{(s+5)(s-2)} &(5)\\ &= Y_1(s)e^{-2s} + Y_2(s) &(6)\end{aligned}$$

By partial fractions decomposition we have

$$\begin{aligned} Y_1(s) &= \frac{4}{(s+5)(s-2)} = \frac{4}{7}\frac{1}{s-2} - \frac{4}{7}\frac{1}{s+5} &(7)\\ Y_2(s) &= \frac{2s+3}{(s+5)(s-2)} = \frac{1}{s-2} + \frac{1}{s+5} &(8)\end{aligned}$$

Take inverse Laplace Transforms to get

$$\begin{aligned} y_1(t) &= \frac{4}{7}e^{2t} - \frac{4}{7}e^{-5t} &(9)\\ y_2(t) &= e^{2t} + e^{-5t} &(10)\end{aligned}$$

and the solution is then

$$\begin{aligned} y(t) &= y_1(t-2)u_2(t) + y_2(t) &(11)\\ &= u_2(t)\left(\frac{4}{7}e^{2(t-2)} - \frac{4}{7}e^{-5(t-2)}\right) + e^{2t} + e^{-5t} &(12)\\ &= u_2(t)\left(\frac{4}{7}e^{2t-4} - \frac{4}{7}e^{-5t-10}\right) + e^{2t} + e^{-5t} &(13)\end{aligned}$$

Notice that even though the exponential in the transform $Y(s)$ came originally from the delta, once we inverse the transform the corresponding term becomes a step function. This is generally the case, because there is a relationship between the step function $u_c(t)$ and the delta $\delta(t-c)$.

We begin with the integral

$$\begin{aligned} \int_{-\infty}^{t} \delta(u-c)du &= \begin{cases} 0, & t < c \\ 1, & t > c \end{cases} && (14) \\ &= u_c(t) && (15) \end{aligned}$$

The Fundamental Theorem of Calculus says

$$u_c'(t) = \frac{d}{dt}\left(\int_{-\infty}^{t} \delta(u-c)du\right) = \delta(t-c) \tag{16}$$

Thus the Dirac delta at $t = c$ is the derivative of the step function at $t = c$, which we can think of geometrically by remembering that the graph of $u_c(t)$ is horizonatal at every $t \neq c$, hence at those points $u_c'(t) = 0$ and it has a jump of one at $t = c$.

Example 3. Solve the following initial value problem.

$$y'' + 4y' + 9y = 2\delta(t-1) + e^t, \quad y(0) = 0, \quad y'(0) = -1 \tag{17}$$

First we Laplace Transform both sides and solve for $Y(s)$.

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) + 9Y(s) &= 2e^{-s} + \frac{1}{s-1} && (18) \\ (s^2 + 4s + 9)Y(s) + 1 &= 2e^{-s} + \frac{1}{s-1} && (19) \end{aligned}$$

Thus

$$\begin{aligned} Y(s) &= \frac{2e^{-s}}{s^2+4s+9} + \frac{1}{(s-1)(s^2+4s+9)} - \frac{1}{s^2+4s+9} && (20) \\ &= Y_1(s)e^{-s} + Y_2(s) = Y_3(s) && (21) \end{aligned}$$

Next, we have to prepare $Y(s)$ for the inverse transform. This will require completing the square for $Y_1(s)$ and $Y_3(s)$, while we will need to first use partial fractions decomposition on $Y_2(s)$. I leave the details to you to verify, so we now have everything in the correct form for the inverse transform

$$\begin{aligned}
Y_1(s) &= \frac{2}{s^2+4s+9} = \frac{2}{(s+2)^2+5} \\
&= \frac{2}{\sqrt{5}}\frac{\sqrt{5}}{(s+2)^2+5} \\
Y_2 &= \frac{1}{(s-1)(s^2+4s+9)} = \frac{1}{14}\left(\frac{1}{s-1} - \frac{s+5}{(s+2)^2+5}\right) \\
&= \frac{1}{14}\left(\frac{1}{s-1} - \frac{(s+2-2)+5}{(s+2)^2+5}\right) \\
&= \frac{1}{14}\left(\frac{1}{s-1} - \frac{s+2}{(s+2)^2+5} - \frac{3}{(s+2)^2+5}\right) \\
&= \frac{1}{14}\left(\frac{1}{s-1} - \frac{s+2}{(s+2)^2+5} - \frac{3}{\sqrt{5}}\frac{\sqrt{5}}{(s+2)^2+5}\right) \\
Y_3(s) &= \frac{1}{(s^2+4s+9)} = \frac{1}{(s+2)^2+5} \\
&= \frac{1}{\sqrt{5}}\frac{\sqrt{5}}{(s+2)^2+5}
\end{aligned}$$

So their inverse transforms are

$$\begin{aligned}
y_1(t) &= \frac{2}{\sqrt{5}}3^{-2t}\sin(\sqrt{5}t) && (22) \\
y_2(t) &= \frac{1}{14}(e^t - e^{-2t}\cos(\sqrt{5}t) - \frac{3}{\sqrt{5}}e^{-2t}\sin(\sqrt{5}t)) && (23) \\
y_3(t) &= \frac{1}{\sqrt{5}}e^{-2t}\sin(\sqrt{5}t). && (24)
\end{aligned}$$

Thus, since our original transformed function was

$$Y(s) = Y_1(s)e^{-s} + Y_2(s) - Y_3(s) \quad (25)$$

we obtain

$$\begin{aligned}
y(t) &= u_1(t)y_1(t-1) + y_2(t) - y_3(t) && (26) \\
&= u_1(t)\left(\frac{2}{\sqrt{5}}e^{-2t+2}\sin(\sqrt{5}t - \sqrt{5})\right) && (27) \\
&+ \frac{1}{14}\left(e^t - e^{-2t}\cos(\sqrt{5}t) - \frac{3}{\sqrt{5}}e^{-2t}\sin(\sqrt{5}t)\right) && (28) \\
&- \frac{1}{\sqrt{5}}e^{-2t}\sin(\sqrt{5}t). && (29)
\end{aligned}$$

Example 4. Solve the following initial value problem.

$$y' + 16y = 2u_3(t) + 5\delta(t-1), \quad y(0) = 1, \quad y'(0) = 2 \tag{30}$$

Again, we begin by taking the Laplace Transform of the entire equation and applying our initial conditions, we get

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + 16Y(s) &= \frac{2e^{-3s}}{s} + 5e^{-s} &(31)\\ (s^2+16)Y(s) - s - 2 &= \frac{2e^{-3s}}{s} + 5e^{-s} &(32) \end{aligned}$$

Solving for $Y(s)$

$$\begin{aligned} Y(s) &= \frac{2e^{-3s}}{s(s^2+16)} + \frac{5e^{-s}}{s^2+16} + \frac{s+2}{s^2+16} &(33)\\ &= Y_1(s)e^{-3s} + Y_2(s)e^{-s} + Y_3(s) &(34) \end{aligned}$$

The only one of these we need to use partial fractions on is the first one. The rest can be dealt with directly, all they need is a little modification.

$$\begin{aligned} Y_1(s) &= \frac{2}{s(s^2+16)} = \frac{1}{8}\frac{1}{s} - \frac{1}{8}\frac{s}{s^2+16} &(35)\\ Y_2(s) &= \frac{5}{s^2+16} = \frac{5}{4}\frac{4}{s^2+16} &(36)\\ Y_3(s) &= \frac{s+2}{s^2+16} = \frac{s}{s^2+16} + \frac{1}{2}\frac{4}{s^2+16} &(37) \end{aligned}$$

and so the associated inverse transforms are

$$\begin{aligned} y_1(t) &= \frac{1}{8} - \frac{1}{8}\cos(4t) &(38)\\ y_2(t) &= \frac{5}{4}\sin(4t) &(39)\\ y_3(t) &= \cos(4t) + \frac{1}{2}\sin(4t). &(40) \end{aligned}$$

Our solution is the inverse transform of

$$Y(s) = Y_1(s)e^{-3s} + Y_2(s)e^{-s} + Y_3(s) \tag{41}$$

and this will be

$$\begin{aligned} y(t) &= u_3(t)y_1(t-3) + u_1(t)y_2(t-1) + y_3(t) &(42)\\ &= u_3(t)\left(\frac{1}{8} - \frac{1}{8}\cos(4(t-3))\right) + \frac{5}{4}u_1(t)\sin(4(t-1)) + \cos(4t) + \frac{1}{2}\sin(4t) &(43)\\ &= u_3(t)\left(\frac{1}{8} - \frac{1}{8}\cos(4t-12)\right) + \frac{5}{4}u_1(t)\sin(4t-4) + \cos(4t) + \frac{1}{2}\sin(4t) &(44) \end{aligned}$$

HW 6.5 # 2,4,6,8,11