Lecture Notes for Math 251: ODE and PDE. Lecture 23: 7.1 Introduction to Systems of Differential Equations

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1 Systems of Differential Equations

Last Time: We finished the chapter on Laplace Transforms by studying impulse functions and the Dirac delta. Now we start Chapter 7: Systems of Two Linear Differential Equations.

1.1 7.1 Systems of Differential Equations

To this point we have focused on solving a single equation, but may real world systems are given as a system of differential equations. An example is Population Dynamics, Normally the death rate of a species is not a constant but depends on the population of predators. An example of a system of first order linear equations is

$$x_1' = 3x_1 + x_2 \tag{1}$$
$$x_2' = 2x_1 - 4x_2 \tag{2}$$

We call a system like this **coupled** because we need to know what x_1 is to know what x_2 is and vice versa. It is important to note that there will be a lot of similarities between our discussion and the previous sections on second and higher order linear equations. This is because any higher order linear equation can be written as a system of first order linear differential equations.

Example 1. Write the following second order differential equation as a system of first order linear differential equations

$$y'' + 4y' - y = 0, \quad y(0) = 2, \quad y'(0) = -2 \tag{3}$$

All that is required to rewrite this equation as a first order system is a very simple change of variables. In fact, this is **ALWAYS** the change of variables to use for a problem like this. We set

$$x_1(t) = y(t) \tag{4}$$
$$x_2(t) = y'(t) \tag{5}$$

Then we have

$$x_1' = y' = x_2 \tag{6}$$
$$x_2' = y'' = y - 4y' = x_1 - 4x_2 \tag{7}$$

Notice how we used the original differential equation to obtain the second equation. The first equation, $x_1' = x_2$, is always something you should expect to see when doing this. All we have left to do is to convert the initial conditions.

$$x_1(0) = y(0) = 2 \tag{8}$$
$$x_2(0) = y'(0) = -2 \tag{9}$$

Thus our original initial value problem has been transformed into the system

$$x_1' = x_2, \quad x_1(0) = 2 \tag{10}$$
$$x_2' = x_1 - 4x_2, \quad x_2(0) = -2 \tag{11}$$

Let's do an example for higher order linear equations.

Example 2.

$$y^{(4)} + ty''' - 2y'' - 3y' - y = 0 \tag{12}$$

as a system of first order differential equations.

We want to use a similar change of variables as the previous example. The only difference is that since our equation in this example is fourth order we will need four new variables instead of two.

$$x_1 = y \tag{13}$$
$$x_2 = y' \tag{14}$$
$$x_3 = y'' \tag{15}$$
$$x_4 = y''' \tag{16}$$

Then we have

$$x_1' = y' = x_2 \tag{17}$$
$$x_2' = y'' = x_3 \tag{18}$$
$$x_3' = y''' = x_4 \tag{19}$$
$$x_4' = y^{(4)} = y + 3y' + 2y'' - ty''' = x_1 + 3x_2 + 2x_3 - tx_4 \tag{20}$$

as our system of equations. To be able to solve these, we need to review some facts about systems of equations and linear algebra.

2 7.2 Review of Matrices

2.1 Systems of Equations

In this section we will restrict our attention only to the linear algebra that might come up when studying systems of differential equations. This is far from a complete treatment, so if you're curious, taking a linear algebra course is either mandatory for your major or a good idea to be more well-rounded.

Suppose we start with a system of n equations with n unknowns $x_1, x_2, ..., x_n$.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 && (21)\\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 && (22)\\ . \quad . &\quad . && (23)\\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n &= b_n && (24)\\ & && (25) \end{aligned}$$

Here's the basic fact about linear systems of equations with the same number of unknowns as equations.

Theorem 3. *Given a system of n equations with n unknowns, there are three possibilities for the number of solutions:*
(1) No Solutions
(2) Exactly One Solution
(3) Infinitely Many Solutions

Definition 4. A system of equations is called **nonhomogeneous** if at least one $b_i \neq 0$. If every $b_i = 0$, the system is called **homogeneous**. A homogeneous system has the following form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= 0 && (26)\\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= 0 && (27)\\ . \quad . &\quad . && (28)\\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n &= 0 && (29)\\ & && (30) \end{aligned}$$

Notice that there is always at least one solution given by

$$x_1 = x_2 = ... = x_n = 0 \tag{31}$$

This solution is called the **trivial solution**. This means that it is impossible for a homogeneous system to have zero solutions, and Theorem 1 can be modified as follows

Theorem 5. *Given a homogeneous system of n equations with n unknowns, there are two possibilities for the number of solutions:*
(1) Exactly one solution, the trivial solution
(2) Infinitely many non-zero Solutions in addition to the trivial solution.

2.2 Linear Algebra

While we could solve the homogeneous and nonhomogeneous systems directly, we have very powerful tools available to us. The main objects of study in linear algebra are **Matrices** and **Vectors**.

An $n \times n$ matrix (referred to as an n-dimensional matrix) is an array of numbers with n rows and n columns. It is possible to consider matrices with different numbers of rows and columns, but this is more general than we need for this course. An $n \times n$ matrix has the form

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \tag{32}$$

There is one special matrix we will need to be familiar with. This is the n-dimensional **Identity Matrix**.

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \tag{33}$$

We will focus on 2×2 matrices in this class. The principles we discuss extend to higher dimensions, but computationally 2×2 matrices are much easier.

Matrix addition and subtraction are fairly straightforward. Do everything componentwise. If we multiply by a constant, **Scalar Multiplication**, we multiply each component by a constant.

Example 6. Given the matrices

$$A = \begin{pmatrix} 3 & 1 \\ -2 & 5 \end{pmatrix} \quad B = \begin{pmatrix} -2 & 0 \\ 1 & 4 \end{pmatrix} \tag{34}$$

compute $A - 2B$.

The first thing to do is compute $2B$

$$2B = 2\begin{pmatrix} -2 & 0 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 2 & 8 \end{pmatrix} \tag{35}$$

Then we have

$$A - 2B = \begin{pmatrix} 3 & 1 \\ -2 & 5 \end{pmatrix} - \begin{pmatrix} -4 & 0 \\ 2 & 8 \end{pmatrix} \tag{36}$$

$$= \begin{pmatrix} 7 & 1 \\ -4 & -3 \end{pmatrix}. \tag{37}$$

Notice that these operations require the dimensions of the matrices to be equal. A **vector** is a one-dimensional array of numbers. For example

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{38}$$

is a vector of n unknowns. We can think of a vector as a $1 \times n$ or an $n \times 1$ dimensional matrix with regards to matrix operations.

We can multiply two matrices A and B together by "multiplying" each row in A by each column of B. That is, to find the element in the ith row and the jth column, we multiply the corresponding elements in the ith row of the first matrix and jth column of the second matrix and add these products together.

Example 7. Compute AB, where

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 2 & -3 \end{pmatrix}. \tag{39}$$

So

$$AB = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & -3 \end{pmatrix} \tag{40}$$

$$= \begin{pmatrix} 1(0) + 2(2) & 1(1) + 2(-3) \\ -1(0) + 3(2) & -1(1) + 3(-3) \end{pmatrix} \tag{41}$$

$$= \begin{pmatrix} 4 & -5 \\ 6 & -10 \end{pmatrix} \tag{42}$$

Notice that $AB \neq BA$ in general. Matrix Multiplication is **NOT** commutative. We must pay special attention to the dimensions of the matrices being multiplied. If the number of columns of A do not match the number of rows of B, we cannot compute AB. Also, the identity matrix I_n is the identity for matrix multiplication, i.e. $I_n A = A I_n = A$ for any matrix A.

In particular, we can multiply an n-dimensional matrix over a vector with n-components together as in the following example

Example 8. Compute

$$\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 4 \end{pmatrix} \tag{43}$$

We proceed by "multiplying" each row in the matrix by the vector.

$$\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2(-1) + -1(4) \\ 3(-1) + 2(4) \end{pmatrix} \tag{44}$$

$$= \begin{pmatrix} -6 \\ 5 \end{pmatrix} \tag{45}$$

REMARK: Multiplication of a matrix with a vector yields another vector. We then have an interpretation of a matrix A as a linear function on vectors.

Definition 9. *(Determinants)* Every square $(n \times n)$ matrix has a number associated to it, called the **determinant**. We will not learn how to compute determinants for $n > 2$, as the process gets more and more complicated as n increases. The standard notation for the determinant of a matrix is

$$\det(A) = |A| \tag{46}$$

For a 2×2 matrix, the determinant is computed using the following formula

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \tag{47}$$

that is, the determinant is the product of the main diagonal minus the product of the off diagonal.

Example 10. Compute the determinants of

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \tag{48}$$

There is not much to do here but us the definition.

$$\begin{aligned} \det(A) &= 2(2) - 3(1) = 4 - 3 = 1 && (49) \\ \det(B) &= 1(4) - 2(2) = 4 - 4 = 0 && (50) \end{aligned}$$

We call a matrix A **singular** if $\det(A) = 0$ and **nonsingular** otherwise. In the previous example, the first matrix was nonsingular while the second was singular.

Determinants give us important information about the existence of an inverse for a given matrix. The inverse of a matrix A, denoted A^{-1}, satisfies

$$AA^{-1} = A^{-1}A = I_n \tag{51}$$

Inverses do not necessarily exist for a given matrix.

Theorem 11. *Given a matrix A,*
(1) If A is nonsingular an inverse, A^{-1}, will exist.
(2) If A is singular, no inverse, A^{-1}, will exist.

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{52}$$

Definition 12. The **Transpose** of a matrix is switching the rows and columns so that $a_{ij} = a^T_{ji}$.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \tag{53}$$

HW 7.2 # 2ac,3ac,4,8,10,11,22,25
Hint: For conjugation $\bar{A}$, just conjugate each term. Also $A^* = \bar{A}^T$. Finally, $(x, y) = x^T \bar{y}$.