Lecture Notes for Math 251: ODE and PDE. Lecture 24: 7.3 Systems of Linear Algebraic Equations; Linear Independence, Eigenvalues and Eigenvectors

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1 Introduction to 2×2 Matrices

Last Time: We studied basic concepts in Linear Algebra necessary for solving systems of differential equations.

1.1 7.3 Systems of Linear Algebraic Equations; Linear Independence, Eigenvalues and Eigenvectors

We return our attention now to the system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 && (1)\\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 && (2)\\ &\vdots && (3)\\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n &= b_n && (4)\\ & && (5) \end{aligned}$$

To express this system of equations in matrix form, we start by writing both sides as vectors.

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n \end{pmatrix} \tag{6}$$

Notice that the left side of the equation can be rewritten as a matrix-vector product.

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \tag{7}$$

We can simplify this notation by writing

$$A\mathbf{x} = b \tag{8}$$

where $\mathbf{x}$ is the vector whose entries are variables in the system, A is the matrix of coefficients of the system (called the **coefficient matrix**), and $\mathbf{b}$ is the vector whose entries are the right-hand side of the equations. We call Equation (8) the **matrix form** of the system of equations.

We know that the system of equations has zero one or infinitely many solutions. Suppose $\det(A) \neq 0$, i.e. A is nonsingular. Then Equation (8) has only one solution

$$\mathbf{x} = A^{-1}\mathbf{b}. \tag{9}$$

So we can rewrite our earlier Theorem from last lecture as

Theorem 1. *Given the system of equations* (8)*,*
(1) If $\det(A) \neq 0$*, there is exactly one solution,*
(2) If $\det(A) = 0$*, there are either zero or infinitely many solutions.*

Recall that if the system were homogeneous, each $b_i = 0$, we always have the trivial solution $x_i = 0$. Denoting the vector with entries all 0 by $\mathbf{0}$, the matrix form of a homogeneous system is

$$A\mathbf{x} = \mathbf{0} \tag{10}$$

Thus we can express the earlier Theorem 2 from last lecture as

Theorem 2. *Given the homogeneous system of equations,*
(1) If $\det(A) \neq 0$*, there is exactly one solution* $\mathbf{x} = \mathbf{0}$*,*
(2) If $\det(A) = 0$*, there will be infinitely many nonzero solutions.*

1.2 Eigenvalues and Eigenvectors

The following is probably the most important aspect of linear algebra. We have already observed if we multiply a vector by a matrix, we get another vector, i.e.,

$$A\eta = \mathbf{y} \tag{11}$$

A natural question to ask is when $\mathbf{y}$ is just a scalar multiple of η. In other words, for what vectors η is multiplication by A equivalent to scaling η, or

$$A\eta = \lambda\eta \tag{12}$$

If (12) is satisfied for some constant λ and some vector η, we call η an **eigenvector** of A with **eigenvalue** λ. We first notice if $\eta = 0$, (12) will be satisfied for any λ. We are not interested in that case, so in general we will assume $\eta \neq 0$.

So how can we find solutions to (12)? Start by rewriting it, recalling that I is the 2×2 identity matrix.

$$\begin{aligned} A\eta &= \lambda\eta && (13) \\ A\eta - \lambda I\eta &= \mathbf{0} && (14) \\ (A - \lambda I)\eta &= \mathbf{0} && (15) \end{aligned}$$

We had to multiply λ by the identity I before we could factor it out. This is because we cannot subtract a constant from a matrix. The last equation is the matrix form for a homogeneous system of equations. By Theorem 3 form last lecture, if $A - \lambda I$ is nonsingular ($\det(A) \neq 0$), the only solution is the trivial solution $\eta = \mathbf{0}$, which we have already said we are not interested in. On the other hand, if $A - \lambda I$ is singular, we will have infinitely many nonzero solutions. Thus the condition that we will need to find any eigenvalues and eigenvectors that may exist for A is for

$$\det(A - \lambda I) = 0 \tag{16}$$

It is a basic fact that this equation is an nth degree polynomial if A is an $n \times n$ matrix. This is called the **characteristic equation** of the matrix A.

As a result, the Fundamental Theorem of Algebra tells us that an $n \times n$ matrix A has n eigenvalues, counting multiplicities. To find them, all we have to do is to find the roots of an nth degree polynomial, which is no problem for small n. Suppose we have found these eigenvalues. What can we conclude about their associated eigenvectors?

Definition 3. We call k vectors $\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_k}$ **linearly independent** if the only constants $c_1, c_2, ..., c_k$ satisfying

$$c_1\mathbf{x_1} + c_2\mathbf{x_2} + ... + c_k\mathbf{x_k} = 0 \tag{17}$$

are $c_1 = c_2 = ... = c_k = 0$. This definition should look familiar. This is an identical definition to our earlier definition of linear independence for functions.

Theorem 4. *If $\lambda_1, \lambda_2, ..., \lambda_n$ is the complete list of eigenvalues of A, including multiplicities, then*
(1) If λ occurs only once in the list it is called ***simple***
(2) If λ occurs $k > 1$ times it has multiplicity k
(3) If $\lambda_1, \lambda_2, ..., \lambda_k (k \leq n)$ are the simple eigenvalues of A with corresponding eigenvectors $\eta^{(\mathbf{1})}, \eta^{(\mathbf{2})}, ..., \eta^{(\mathbf{k})}$, then these eigenvectors $\eta^{(\mathbf{i})}$ are linearly independent.
(4) If λ is an eigenvalue with multiplicity k, then λ will have anywhere from 1 to k linearly independent eigenvectors.

This fact should look familiar from our discussion of second and higher order equations. This theorem tells us when we have linearly independent eigenvectors, which is useful when trying to solve systems of differential equations. Now once we have the eigenvalues, how do we find the associated eigenvectors?

Example 5. Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}. \tag{18}$$

The first thing we need to do is to find the roots of the characteristic equation of the matrix

$$A - \lambda I = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3-\lambda & 4 \\ 2 & 1-\lambda \end{pmatrix} \tag{19}$$

This is

$$0 = \det(A - \lambda I) = (3-\lambda)(1-\lambda) - 8 = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1). \tag{20}$$

This the two eigenvalues of A are $\lambda_1 = 5$ and $\lambda_2 = -1$. Now to find the eigenvectors we need to plug each eigenvalue into $(A - \lambda I)\eta = 0$ and solve for η.

(1) $\lambda_1 = 5$

In this case, we have the following system

$$\begin{pmatrix} -2 & 4 \\ 2 & -4 \end{pmatrix} \eta = \mathbf{0} \tag{21}$$

Next, we will write out components of the two vectors and multiply through

$$\begin{pmatrix} -2 & 4 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{22}$$

$$\begin{pmatrix} -2\eta_1 + 4\eta_2 \\ 2\eta_1 - 4\eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{23}$$

For this vector equation to hold, the components must match up. So we have got to find a solution to the system

$$-2\eta_1 + 4\eta_2 = 0 \tag{24}$$

$$2\eta_1 - 4\eta_2 = 0. \tag{25}$$

Notice that these are the same equation, but differ by a constant, in this case -1. This will always be the case if we have found our eigenvalues correctly, since we know that $A - \lambda I$ is singular and so our system should have infinitely many solutions.

Since the equations are basically the same we need to choose one and obtain a relation between eigenvector components η_1 and η_2. Let's choose the first. This gives

$$2\eta_1 = 4\eta_2. \tag{26}$$

and so we have $\eta_1 = 2\eta_2$. As a result, any eigenvector corresponding to $\lambda_1 = 5$ has the form

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 2\eta_2 \\ \eta_2 \end{pmatrix}. \tag{27}$$

There are infinitely many vectors of this form, we need only one. We can select one by choosing a value for η_2. The only restriction is we do not want to pick $\eta_2 = 0$, since then $\eta = \mathbf{0}$, which we want to avoid. We may choose, for example, $\eta_2 = 1$, and then we have

$$\eta^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \tag{28}$$

(2) $\lambda_2 = -1$

In the previous case we went into more detail than we will in future examples. The process is the same. Plugging λ_2 into $(A - \lambda I)\eta = 0$ gives the system

$$\begin{aligned} \begin{pmatrix} 4 & 4 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} && (29) \\ \begin{pmatrix} 4\eta_1 + 4\eta_2 \\ 2\eta_1 + 2\eta_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} && (30) \end{aligned}$$

The two equations corresponding to this vector equation are

$$\begin{aligned} 4\eta_1 + 4\eta_2 &= 0 && (31) \\ 2\eta_1 + 2\eta_2 &= 0. && (32) \end{aligned}$$

Once again, these differ by a constant factor. Solving the first equation we find

$$\eta_1 = -\eta_2 \tag{33}$$

and so any eigenvector has the form

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\eta_2 \\ \eta_2 \end{pmatrix}. \tag{34}$$

We can choose $\eta_2 = 1$, giving us a second eigenvector of

$$\eta^{(\mathbf{2})} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{35}$$

Summarizing the eigenvalue/eigenvector pairs of A are

$$\begin{aligned} \lambda_1 &= 5 \qquad \eta^{(\mathbf{1})} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} && (36) \\ \lambda_2 &= -1 \qquad \eta^{(\mathbf{2})} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. && (37) \end{aligned}$$

REMARK: We could have ended up with any number of different values for our eigenvectors $\eta^{(\mathbf{1})}$ and $\eta^{(\mathbf{2})}$, depending on the choices we made at the end. However, they would have only differed by a multiplicative constant.

Example 6. Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 2 & -1 \\ 5 & 4 \end{pmatrix}. \tag{38}$$

The characteristic equation for this matrix is

$$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} && (39) \\ &= (2-\lambda)(4-\lambda) + 5 && (40) \\ &= \lambda^2 - 6\lambda + 13 && (41) \end{aligned}$$

By completing the square (or quadratic formula), we see that the roots are $r_{1,2} = 3 \pm 2i$. If we get complex eigenvalues, to find the eigenvectors we proceed as we did in the previous example.

(1) $\lambda_1 = 3 + 2i$

Here the matrix equation

$$(A - \lambda I)\eta = 0 \tag{42}$$

becomes

$$\begin{aligned} \begin{pmatrix} -1-2i & -1 \\ 5 & 1-2i \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} && (43) \\ \begin{pmatrix} (-1-2i)\eta_1 - \eta_2 \\ 5\eta_1 + (1-2i)\eta_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} && (44) \end{aligned}$$

So the pair of equations we get are

$$\begin{aligned} (-1-2i)\eta_1 - \eta_2 &= 0 && (45) \\ 5\eta_1 + (1-2i)\eta_2 &= 0. && (46) \end{aligned}$$

It is not as obvious as the last example, but these two equations are scalar multiples. If we multiply the first equation by $-1+2i$, we recover the second. Now we choose one of these equations to work with. Let's use the first. This gives us that $\eta_2 = (-1-2i)\eta_1$, so any vector has the form

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ (-1-2i)\eta_1 \end{pmatrix}. \tag{47}$$

Choosing $\eta_1 = 1$ gives a first eigenvector of

$$\eta^{(\mathbf{1})} = \begin{pmatrix} 1 \\ -1-2i \end{pmatrix} \tag{48}$$

(2) $\lambda_1 = 3 - 2i$

Here the matrix equation

$$(A - \lambda I)\eta = \mathbf{0} \tag{49}$$

becomes

$$\begin{aligned} \begin{pmatrix} -1+2i & -1 \\ 5 & 1+2i \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} && (50) \\ \begin{pmatrix} (-1+2i)\eta_1 - \eta_2 \\ 5\eta_1 + (1+2i)\eta_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} && (51) \end{aligned}$$

So the pair of equations we get are

$$\begin{aligned} (-1+2i)\eta_1 - \eta_2 &= 0 && (52) \\ 5\eta_1 + (1+2i)\eta_2 &= 0 && (53) \end{aligned}$$

Let's use the first equation again. This gives us that $\eta_2 = (-1+2i)\eta_1$, so any eigenvector has the form

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ (-1+2i)\eta_1 \end{pmatrix}. \tag{54}$$

Choosing $\eta_1 = 1$ gives a second eigenvector of

$$\eta^{(2)} = \begin{pmatrix} 1 \\ -1+2i \end{pmatrix}. \tag{55}$$

To summarize, A has the following eigenvalue/eigenvector pairs

$$\lambda_1 = 3 - 2i \qquad \begin{pmatrix} 1 \\ -1-2i \end{pmatrix} \tag{56}$$

$$\lambda_2 = 3 + 2i \qquad \begin{pmatrix} 1 \\ -1+2i \end{pmatrix} \tag{57}$$

REMARK: Notice that the eigenvalues came in **complex conjugate** pairs, i.e. in the form $a \pm bi$. This is always the case for complex roots, as we can easily see from the quadratic formula. Moreover, the complex entries in the eigenvectors were also complex conjugates, and the real entries were the same up to multiplication by a constant. This is always the case as long as A does not have any complex entries.

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