Lecture Notes for Math 251: ODE and PDE. Lecture 25: 7.5 Homogeneous Linear Systems with Constant Coefficients

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1 Homogeneous Linear Systems with Constant Coefficients

Last Time: We studied linear independence, eigenvalues, and eigenvectors.

1.1 Solutions to Systems of Differential Equations

A two-dimensional equation has the form

$$x' = ax + by \tag{1}$$
$$y' = cx + dy \tag{2}$$

Suppose we have got our system written in matrix form

$$x' = Ax \tag{3}$$

How do we solve this equation? If A were a 1×1 matrix, i.e. a constant, and x were a vector with 1 component, the differential equation would be the separable equation

$$x' = ax \tag{4}$$

We know this is solved by

$$x(t) = ce^{at}. \tag{5}$$

One might guess, then, that in the $n \times n$ case, instead of a we have some other constant in the exponential, and instead of the constant of integration c we have some constant vector η. So our guess for the solution will be

$$x(t) = \eta e^{rt}. \tag{6}$$

Plugging the guess into the differential equation gives

$$\begin{aligned} r\eta e^{rt} &= A\eta e^{rt} && (7)\\ (A\eta - r\eta)e^{rt} &= 0 && (8)\\ (A - rI)\eta e^{rt} &= 0. && (9) \end{aligned}$$

Since $e^{rt} \neq 0$, we end up with the requirement that

$$(A - rI)\eta = 0 \tag{10}$$

This should seem familiar, it is the condition for η to be an eigenvector of A with eigenvalue r. Thus, we conclude that for (6) to be a solution of the original differential equation, we must have η an eigenvalue of A with eigenvalue r.

That tells us how to get some solutions to systems of differential equations, we find the eigenvalues and vectors of the coefficient matrix A, then form solutions using (6). But how will we form the general solution?

Thinking back to the second/higher order linear case, we need enough linearly independent solutions to form a fundamental set. As we noticed last lecture, if we have all simple eigenvalues, then all the eigenvectors are linearly independent, and so the solutions formed will be as well. We will handle the case of repeated eigenvalues later.

So we will find the fundamental solutions of the form (6), then take their linear combinations to get our general solution.

1.2 The Phase Plane

We are going to rely on qualitatively understanding what solutions to a linear system of differential equations look like, this will be important when considering nonlinear equations. We know the trivial solution $x = 0$ is always a solution to our homogeneous system $x' = Ax$. $x = 0$ is an example of an **equilibrium solution**, i.e. it satisfies

$$x' = Ax = 0 \tag{11}$$

and is a constant solution. We will assume our coefficient matrix A is nonsingular ($\det(A) \neq 0$), thus $x = 0$ is the only equilibrium solution.

The question we want to ask is whether other solutions move towards or away from this constant solution as $t \to \pm\infty$, so that we can understand the long term behavior of the system. This is no different than what we did when we classified equilibrium solutions for first order autonomous equations, we will generalize the ideas to systems of equations.

When we drew solution spaces then, we did so on the ty-plane. To do something analogous we would require three dimensions, since we would have to sketch both x_1 and x_2 vs. t. Instead, what we do is ignore t and think of our solutions as trajectories on the x_1x_2-plane. Then our equilibrium solution is the origin. The x_1x_2-plane is called the **phase plane**. We will see examples where we sketch solutions, called **phase portraits**.

1.3 Real, Distinct Eigenvalues

Lets get back to the equation $x' = Ax$. We know if λ_1 and λ_2 are real and distinct eigenvalues of the 2×2 coefficient matrix A associated with eigenvectors $\eta^{(1)}$ and $\eta^{(2)}$, respectively. We know from above $\eta^{(1)}$ and $\eta^{(2)}$ are linearly independent, as λ_1 and λ_2 are simple. Thus the solutions obtained from them using (6) will also be linearly independent, and in fact will form a fundamental set of solutions. The general solution is

$$x(t) = c_1 e^{\lambda_1 t}\eta^{(1)} + c_2 e^{\lambda_2 t}\eta^{(2)} \tag{12}$$

So if we have real, distinct eigenvalues, all that we have to do is find the eigenvectors, form the general solution as above, and use any initial conditions that may exist.

Example 1. Solve the following initial value problem

$$x' = \begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix} x \quad x(0) = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{13}$$

The first thing we need to do is to find the eigenvalues of the coefficient matrix.

$$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{vmatrix} -2-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} && (14)\\ &= \lambda^2 + \lambda - 6 && (15)\\ &= (\lambda - 2)(\lambda + 3) && (16) \end{aligned}$$

So the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -3$. Next we need the eigenvectors.

(1) $\lambda_1 = 2$

$$(A - 2I)\eta = 0 \tag{17}$$

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{18}$$

So we will want to find solutions to the system

$$\begin{aligned} -4\eta_1 + 2\eta_2 &= 0 && (19)\\ 2\eta_1 - \eta_2 &= 0. && (20) \end{aligned}$$

Using either equation we find $\eta_2 = 2\eta_1$, and so any eigenvector has the form

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ 2\eta_1 \end{pmatrix} \tag{21}$$

Choosing $\eta_1 = 1$ we obtain the first eigenvector

$$\eta^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{22}$$

(2) $\lambda_2 = -3$

$$(A+3I)\eta = 0 \tag{23}$$
$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{24}$$

So we will want to find solutions to the system

$$\eta_1 + 2\eta_2 = 0 \tag{25}$$
$$2\eta_1 + 4\eta_2 = 0. \tag{26}$$

Using either equation we find $\eta_1 = -2\eta_2$, and so any eigenvector has the form

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -2\eta_2 \\ \eta_2 \end{pmatrix}. \tag{27}$$

Choosing $\eta_2 = 1$ we obtain the second eigenvector

$$\eta^{(2)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}. \tag{28}$$

Thus our general solution is

$$x(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}. \tag{29}$$

Now let's use the initial condition to solve for c_1 and c_2. The condition says

$$\begin{pmatrix} 5 \\ 0 \end{pmatrix} = x(0) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} -2 \\ 1 \end{pmatrix}. \tag{30}$$

All that's left is to write out is the matrix equation as a system of equations and then solve.

$$c_1 - 2c_2 = 5 \tag{31}$$
$$2c_1 + c_2 = 0 \quad \Rightarrow c_1 = 1, c_2 = -2 \tag{32}$$

Thus the particular solution is

$$x(t) = e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2e^{-3t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}. \tag{33}$$

Example 2. Sketch the phase portrait of the system from Example 1.

In the last example we saw that the eigenvalue/eigenvector pairs for the coefficient matrix were

$$\lambda_1 = 2 \qquad \eta^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{34}$$
$$\lambda_2 = -3 \qquad \eta^{(2)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}. \tag{35}$$

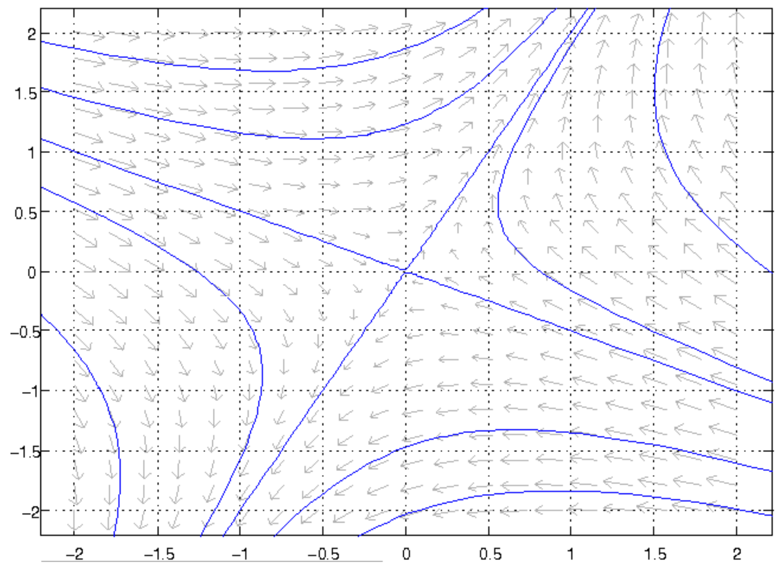

Figure 1: Phase Portrait of the saddle point in Example 1

The starting point for the phase portrait involves sketching solutions corresponding to the eigenvectors (i.e. with c_1 or $c_2 = 0$). We know that if $x(t)$ is one of these solutions

$$x'(t) = Ac_i e^{\lambda_i t}\eta^{(i)} = c_i \lambda_i e^{\lambda_i t}\eta^{(i)}. \tag{36}$$

This is just, for any t, a constant times the eigenvector, which indicates that lines in the direction of the eigenvector are these solutions to the system. There are called **eigensolutions** of the system.

Next, we need to consider the direction that these solutions move in. Let's start with the first eigensolution, which corresponds to the solution with $c_2 = 0$. The first eigenvalue is $\lambda_1 = 2 > 0$. This indicates that this eigensolution will grow exponentially, as the exponential in the solution has a positive exponent. The second eigensolution corresponds to $\lambda_2 = -3 < 0$, so the exponential in the appropriate solution is negative. Hence this solution will decay and move towards the origin.

What does the typical trajectory do (i.e. a trajectory where both $c_1, c_2 \neq 0$)? The general solution is

$$x(t) = c_1 e^{2t}\eta^{(1)} + c_2 e^{-3t}\eta^{(2)}. \tag{37}$$

Thus as $t \to \infty$, this solution will approach the positive eigensolution, as the component corresponding to the negative eigensolution will decay away. On the other hand, as $t \to -\infty$, the trajectory will asymptotically reach the negative eigensolution, as the positive eigensolution component will be tiny. The end result is the phase portrait as in Figure 1. When the phase portrait looks like this (which happens in all cases with eigenvalues of mixed signs), the equilibrium solution at the origin is classified as a **saddle point** and is **unstable**.

Example 3. Solve the following initial value problem.

$$\begin{aligned} x_1' &= 4x_1 + x_2 \quad x_1(0) = 6 && (38)\\ x_2' &= 3x_1 + 2x_2 \quad x_2(0) = 2 && (39) \end{aligned}$$

Before we can solve anything, we need to convert this system into matrix form. Doing so converts the initial value problem to

$$x' = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} x \quad x(0) = \begin{pmatrix} 6 \\ 2 \end{pmatrix}. \tag{40}$$

To solve, the first thing we need to do is to find the eigenvalues of the coefficient matrix.

$$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} && (41) \\ &= \lambda^2 - 6\lambda + 5 && (42) \\ &= (\lambda - 1)(\lambda - 5) && (43) \end{aligned}$$

So the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 5$. Next, we find the eigenvectors.

(1) $\lambda_1 = 1$

$$(A - I)\eta = 0 \tag{44}$$

$$\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{45}$$

So we will want to find solutions to the system

$$\begin{aligned} 3\eta_1 + \eta_2 &= 0 && (46) \\ 3\eta_1 + \eta_2 &= 0. && (47) \end{aligned}$$

Using either equation we find $\eta_2 = -3\eta_1$, and so any eigenvector has the form

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ -3\eta_1 \end{pmatrix} \tag{48}$$

Choosing $\eta_1 = 1$ we obtain the first eigenvector

$$\eta^{(1)} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}. \tag{49}$$

(2) $\lambda_2 = 5$

$$(A - 5I)\eta = 0 \tag{50}$$

$$\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{51}$$

So we will want to find solutions to the system

$$\begin{aligned} -\eta_1 + \eta_2 &= 0 && (52) \\ 3\eta_1 - 3\eta_2 &= 0. && (53) \end{aligned}$$

Using either equation we find $\eta_1 = \eta_2$, and so any eigenvector has the form

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_2 \\ \eta_2 \end{pmatrix}. \tag{54}$$

Choosing $\eta_2 = 1$ we obtain the second eigenvector

$$\eta^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{55}$$

Thus our general solution is

$$x(t) = c_1 e^t \begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{56}$$

Now using our initial conditions we solve for c_1 and c_2. The condition gives

$$\begin{pmatrix} 6 \\ 2 \end{pmatrix} = x(0) = c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{57}$$

All that is left is to write out this matrix equation as a system of equations and then solve

$$c_1 + c_2 = 6 \tag{58}$$

$$-3c_1 + c_2 = 2 \Rightarrow c_1 = 1, c_2 = 5 \tag{59}$$

Thus the particular solution is

$$x(t) = e^t \begin{pmatrix} 1 \\ -3 \end{pmatrix} + 5e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{60}$$

Example 4. Sketch the phase portrait of the system from Example 3.

In the last example, we saw that the eigenvalue/eigenvector pairs for the coefficient matrix were

$$\lambda_1 = 1 \qquad \eta^{(1)} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}. \tag{61}$$

$$\lambda_2 = 5 \qquad \eta^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{62}$$

We begin by sketching the eigensolutions (these are straight lines in the directions of the eigenvectors). Both of these trajectories move away from the origin, though, as the eigenvalues are both positive.

Since $|\lambda_2| > |\lambda_1|$, we call the second eigensolution the **fast eigensolution** and the first one the **slow eigensolution**. The term comes from the fact that the eigensolution corresponds to the eigenvalue with larger magnitude will either grow or decay more quickly than the other one.

As both grow in forward time, asymptotically, as $t \to \infty$, the fast eigensolution will dominate the typical trajectory, as it gets larger much more quickly than the slow eigensolution does. So

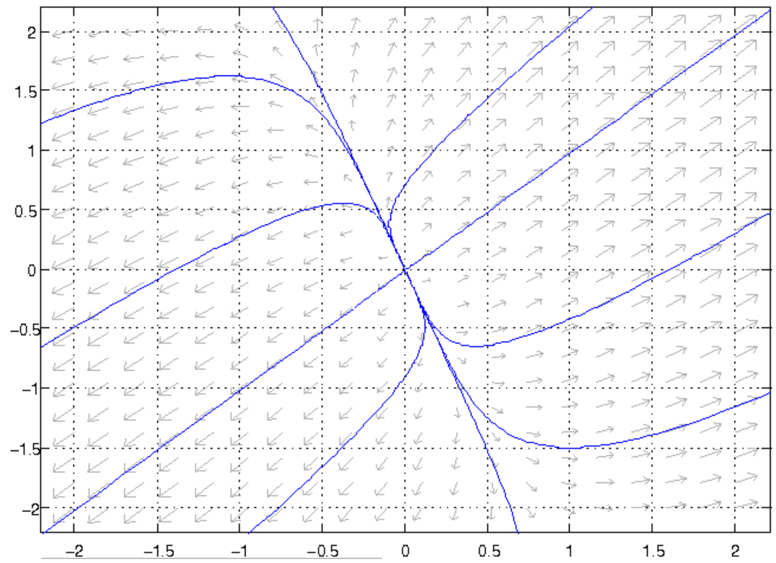

Figure 2: Phase Portrait of the unstable node in Example 2

in forward time, other trajectories will get closer and closer to the eigensolution corresponding to $\eta^{(2)}$. On the other hand, as $t \to -\infty$, the fast eigensolution will decay more quickly than the slow one, and so the eigensolution corresponding to $\eta^{(1)}$ will dominate in backwards time.

Thus the phase portrait will look like Figure 2. Whenever we have two positive eigenvalues, every solution moves away from the origin. We call the equilibrium solution at the origin, in this case, a **node** and classify it as being **unstable**.

Example 5. Solve the following initial value problem.

$$x_1' = -5x_1 + x_2 \quad x_1(0) = 2 \tag{63}$$
$$x_2' = 2x_1 - 4x_2 \quad x_2(0) = -1 \tag{64}$$

We convert this system into matrix form.

$$x' = \begin{pmatrix} -5 & 1 \\ 2 & -4 \end{pmatrix} x \quad x(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}. \tag{65}$$

To solve, the first thing we need to do is to find the eigenvalues of the coefficient matrix.

$$0 = \det(A - \lambda I) = \begin{vmatrix} -5-\lambda & 1 \\ 2 & -4-\lambda \end{vmatrix} \tag{66}$$
$$= \lambda^2 + 9\lambda + 18 \tag{67}$$
$$= (\lambda+3)(\lambda+6) \tag{68}$$

So the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -6$. Next, we find the eigenvectors.

(1) $\lambda_1 = -3$

$$(A+3I)\eta = 0 \tag{69}$$

$$\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{70}$$

So we will want to find solutions to the system

$$-2\eta_1 + \eta_2 = 0 \tag{71}$$

$$2\eta_1 - \eta_2 = 0. \tag{72}$$

Using either equation we find $\eta_2 = 2\eta_1$, and so any eigenvector has the form

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ 2\eta_1 \end{pmatrix} \tag{73}$$

Choosing $\eta_1 = 1$ we obtain the first eigenvector

$$\eta^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{74}$$

(2) $\lambda_2 = -6$

$$(A+6I)\eta = 0 \tag{75}$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{76}$$

So we will want to find solutions to the system

$$\eta_1 + \eta_2 = 0 \tag{77}$$

$$2\eta_1 + 2\eta_2 = 0. \tag{78}$$

Using either equation we find $\eta_1 = -\eta_2$, and so any eigenvector has the form

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\eta_2 \\ \eta_2 \end{pmatrix}. \tag{79}$$

Choosing $\eta_2 = 1$ we obtain the second eigenvector

$$\eta^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{80}$$

Thus our general solution is

$$x(t) = c_1 e^{-3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-6t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{81}$$

Now using our initial conditions we solve for c_1 and c_2. The condition gives

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} = x(0) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{82}$$

All that is left is to write out this matrix equation as a system of equations and then solve

$$\begin{aligned} c_1 - c_2 &= 2 & (83) \\ 2c_1 + c_2 &= -1 \Rightarrow c_1 = \frac{1}{3}, c_2 = -\frac{5}{3} & (84) \end{aligned}$$

Thus the particular solution is

$$x(t) = \frac{1}{3}e^{-3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{5}{3}e^{-6t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{85}$$

Example 6. Sketch the phase portrait of the system from Example 5.

In the last example, we saw that the eigenvalue/eigenvector pairs for the coefficient matrix were

$$\begin{aligned} \lambda_1 &= -3 \qquad \eta^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} & (86) \\ \lambda_2 &= -6 \qquad \eta^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. & (87) \end{aligned}$$

We begin by sketching the eigensolutions. Both of these trajectories decay towards the origin, since both eigenvalues are negative. Since $|\lambda_2| > |\lambda_1|$, the second eigensolution is the fast eigensolution and the first one the slow eigensolution. In the general solution, both exponentials are negative and so every solution will decay and move towards the origin. Asymptotically, as $t \to \infty$ the trajectory gets closer and closer to the origin, the slow eigensolution will dominate the typical trajectory, as it dies out less quickly than the fast eigensolution. So in forward time, other trajectories will get closer and closer to the eigensolution corresponding to $\eta^{(1)}$. On the other hand, as $t \to -\infty$, the fast solution will grow more quickly than the slow one, and so the eigensolution corresponding to $\eta^{(2)}$ will dominate in backwards time.

Thus the phase portrait will look like Figure 3. Whenever we have two negative eigenvalues, every solution moves toward the origin. We call the equilibrium solution at the origin, in this case, a **node** and classify it as being **asymptotically stable**.

HW 7.5 # 2,3,4,15,16

If you cannot plot them, describe the behavior as $t \to \infty$.

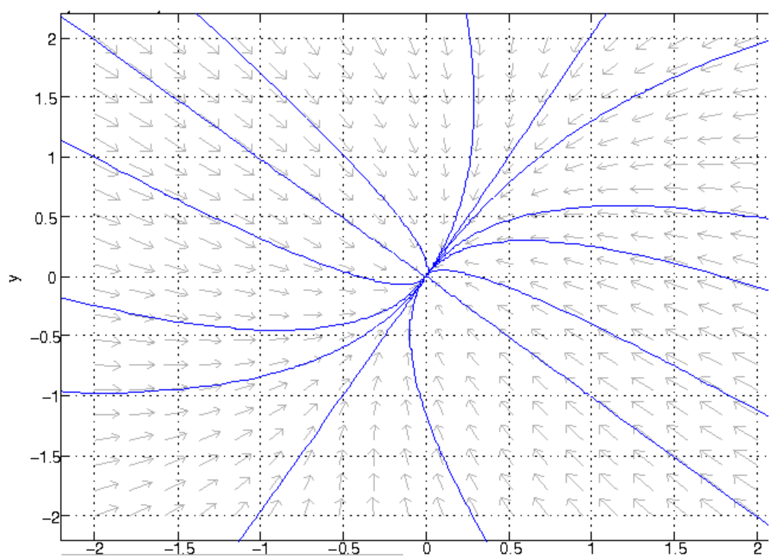

Figure 3: Phase Portrait of the Stable Node in Example 3