Lecture Notes for Math 251: ODE and PDE. Lecture 26: 7.6 Complex Eigenvalues

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Spring 2012

1 Complex Eigenvalues

Last Time: We studied phase portraits and systems of differential equations with real eigenvalues.

We are looking for solutions to the equation $x' = Ax$. What happens when the eigenvalues are complex?

We still have solutions of the form

$$x = \eta e^{\lambda t} \tag{1}$$

where η is an eigenvector of A with eigenvalue λ. However, we want real-valued solutions, which we will not have if they remain in this form.

Our strategy will be similar in this case: we'll use Euler's formula to rewrite

$$e^{(a+ib)t} = e^{at}\cos(bt) + e^{at} i \sin(bt) \tag{2}$$

then we will write out one of our solutions fully into real and imaginary parts. It will turn out that each of these parts gives us a solution, and in fact they will also form a fundamental set of solutions.

Example 1. Solve the following initial value problem.

$$x' = \begin{pmatrix} 3 & 6 \\ -2 & -3 \end{pmatrix} x \quad x(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \tag{3}$$

The first thing we need to do is to find the eigenvalues of the coefficient matrix.

$$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{vmatrix} 3-\lambda & 6 \\ -2 & -3-\lambda \end{vmatrix} && (4) \\ &= \lambda^2 + 3 && (5) \\ & && (6) \end{aligned}$$

So the eigenvalues are $\lambda_1 = \sqrt{3}i$ and $\lambda_2 = -\sqrt{3}i$. Next we need the eigenvectors. It turns out we will only need one. Consider $\lambda_1 = \sqrt{3}i$.

$$(A - \sqrt{3}iI)\eta^* = 0 \tag{7}$$

$$\begin{pmatrix} 3-\sqrt{3}i & 6 \\ -2 & -3-\sqrt{3}i \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{8}$$

The system of equations to solve is

$$(3-\sqrt{3}i)\eta_1 + 6\eta_2 = 0 \tag{9}$$

$$-2\eta_1 + (-3-\sqrt{3}i)\eta_2 = 0. \tag{10}$$

We can use either equation to find solutions, but lets solve the second one. This gives $\eta_1 = \frac{1}{2}(-3-\sqrt{3}i)\eta_2$. Thus any eigenvector has the form

$$\eta = \begin{pmatrix} \frac{1}{2}(-3-\sqrt{3}i)\eta_2 \\ \eta_2 \end{pmatrix} \tag{11}$$

and choosing $\eta_2 = 2$ yields the first eigenvector

$$\eta^{(1)} = \begin{pmatrix} -3-\sqrt{3}i \\ 2 \end{pmatrix}. \tag{12}$$

Thus we have a solution

$$x_1(t) = e^{\sqrt{3}it} \begin{pmatrix} -3-\sqrt{3}i \\ 2 \end{pmatrix} \tag{13}$$

Unfortunately, this is complex-valued, and we would like a real-valued solution. We he had a similar problem in the chapter on second order equations. What did we do then? We use Euler's formula to expand this imaginary exponential into sine and cosine terms, then split the solution into real and imaginary parts. This gave two fundamental solutions we needed.

We will do the same thing here. Using Euler's Formula to expand

$$e^{\sqrt{3}it} = \cos(\sqrt{3}t) + i\sin(\sqrt{3}t). \tag{14}$$

then multiply it through the eigenvector. After separating into real and complex parts using the basic matrix arithmetic operations, it will turn out that each of these parts is a solution. They are linearly independent and give us a fundamental set of solutions.

$$\begin{aligned} x_1(t) &= (\cos(\sqrt{3}t) + i\sin(\sqrt{3}t)) \begin{pmatrix} -3-\sqrt{3}i \\ 2 \end{pmatrix} && (15) \\ &= \begin{pmatrix} (-3\cos(\sqrt{3}t) - 3i\sin(\sqrt{3}t) - \sqrt{3}i\cos(\sqrt{3}t) + \sqrt{3}\sin(\sqrt{3}t) \\ 2\cos(\sqrt{3}t) + 2i\sin(\sqrt{3}t) \end{pmatrix} && (16) \\ &= \left(\begin{pmatrix} -3\cos(\sqrt{3}t) + \sqrt{3}\sin(\sqrt{3}t) \\ 2\cos(\sqrt{3}t) \end{pmatrix} + i \begin{pmatrix} -3\sin(\sqrt{3}t) - \sqrt{3}\cos(\sqrt{3}t) \\ 2\sin(\sqrt{3}t) \end{pmatrix} \right) && (17) \\ &= u(t) + iv(t) && (18) \end{aligned}$$

Both $u(t)$ and $v(t)$ are real-valued solutions to the differential equation. Moreover, they are linearly independent. Our general solution is then

$$\begin{aligned} x(t) &= c_1 u(t) + c_2 v(t) && (19) \\ &= c_1 \begin{pmatrix} -3\cos(\sqrt{3}t) + \sqrt{3}\sin(\sqrt{3}t) \\ 2\cos(\sqrt{3}t) \end{pmatrix} + c_2 \begin{pmatrix} -3\sin(\sqrt{3}t) - \sqrt{3}\cos(\sqrt{3}t) \\ 2\sin(\sqrt{3}t) \end{pmatrix} && (20) \end{aligned}$$

Finally, we need to use the initial condition to get c_1 and c_2. It says

$$\begin{pmatrix} -2 \\ 4 \end{pmatrix} = x(0) = c_1 \begin{pmatrix} -3 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -\sqrt{3} \\ 0 \end{pmatrix}. \tag{21}$$

This translates into the system

$$\begin{aligned} -3c_1 - \sqrt{3}c_2 &= -2 && (22) \\ 2c_1 &= 4 \quad \Rightarrow c_1 = 2 \quad c_2 = -\frac{4}{\sqrt{3}}. && (23) \end{aligned}$$

Hence our particular solution is

$$x(t) = 2\begin{pmatrix} -3\cos(\sqrt{3}t) + \sqrt{3}\sin(\sqrt{3}t) \\ 2\cos(\sqrt{3}t) \end{pmatrix} - \frac{4}{\sqrt{3}} \begin{pmatrix} -3\sin(\sqrt{3}t) - \sqrt{3}\cos(\sqrt{3}t) \\ 2\sin(\sqrt{3}t) \end{pmatrix} \tag{24}$$

Example 2. Sketch the phase portrait of the system in Example 1.

The general solution to the system in Example 1 is

$$x(t) = c_1\left(\begin{pmatrix} -3\cos(\sqrt{3}t) + \sqrt{3}\sin(\sqrt{3}t) \\ 2\cos(\sqrt{3}t) \end{pmatrix} + c_2 \begin{pmatrix} -3\sin(\sqrt{3}t) - \sqrt{3}\cos(\sqrt{3}t) \\ 2\sin(\sqrt{3}t) \end{pmatrix}\right) \tag{25}$$

Every term in this solution is periodic, we have $\cos(\sqrt{3}t)$ and $\sin(\sqrt{3}t)$. Thus both x_1 and x_2 are periodic functions for any initial conditions. On the phase plane, this translates to trajectories which are closed, that is they form circles or ellipses. As a result, the phase portrait looks like Figure 1.

This is always the case when we have purely imaginary eigenvalues, as the exponentials turn into a combination of sines and cosines. In this case, the equilibrium solution is called a **center** and is **neutrally stable** or just **stable**, note that it is not asymptotically stable.

The only work left to do in these cases is to figure out the eccentricity and direction that the trajectory traveled. The eccentricity is difficult, and we usually do not care that much about it. The direction traveled is easier to find. We can determine whether the trajectories orbit the origin in a clockwise or counterclockwise direction by calculating the tangent vector x' at a single point. For example, at the point $(1,0)$ in the previous example, we have

$$x' = \begin{pmatrix} 3 & 6 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{26}$$

Thus at $(1,0)$, the tangent vector points down and to the right. This can only happen is the trajectories circle to origin in a clockwise direction.

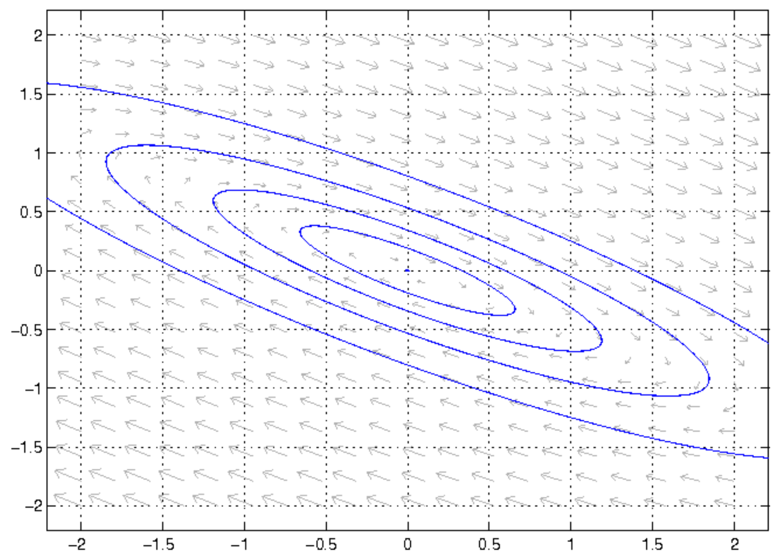

Figure 1: Phase Portrait of the center point in Example 1

Example 3. Solve the following initial value problem.

$$x' = \begin{pmatrix} 6 & -4 \\ 8 & -2 \end{pmatrix} x \quad x(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{27}$$

The first thing we need to do is to find the eigenvalues of the coefficient matrix.

$$0 = \det(A - \lambda I) = \begin{vmatrix} 6-\lambda & -4 \\ 8 & -2-\lambda \end{vmatrix} \tag{28}$$

$$= \lambda^2 - 4\lambda + 20 \tag{29}$$

$$\tag{30}$$

So the eigenvalues, using the Quadratic Formula are $\lambda_{1,2} = 2 \pm 4i$. Next we need the eigenvectors. It turns out we will only need one. Consider $\lambda_1 = 2 + 4i$.

$$(A - (2+4i)I)\eta^* = 0 \tag{31}$$

$$\begin{pmatrix} 4-4i & -4 \\ 8 & -4-4i \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{32}$$

The system of equations to solve is

$$(4-4i)\eta_1 - 4\eta_2 = 0 \tag{33}$$

$$8\eta_1 + (-4-4i)\eta_2 = 0. \tag{34}$$

We can use either equation to find solutions, but lets solve the first one. This gives $\eta_2 = (1-i)\eta_1$. Thus any eigenvector has the form

$$\eta = \begin{pmatrix} \eta_1 \\ (1-i)\eta_1 \end{pmatrix} \tag{35}$$

and choosing $\eta_1 = 1$ yields the first eigenvector

$$\eta^{(1)} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}. \tag{36}$$

Thus we have a solution

$$x_1(t) = e^{(2+4i)t} \begin{pmatrix} 1 \\ 1-i \end{pmatrix} \tag{37}$$

Using Euler's Formula to expand

$$\begin{aligned}
&= e^{2i}e^{4it} \begin{pmatrix} 1 \\ 1-i \end{pmatrix} && (38)\\
&= e^{2t}(\cos(4t) + i\sin(4t)) \begin{pmatrix} 1 \\ 1-i \end{pmatrix} && (39)\\
&= e^{2t} \begin{pmatrix} \cos(4t) + i\sin(4t) \\ \cos(4t) + i\sin(4t) - i\cos(4t) + \sin(4t) \end{pmatrix} && (40)\\
&= \left(\begin{pmatrix} \cos(4t) \\ \cos(4t) + \sin(4t) \end{pmatrix} + i \begin{pmatrix} \sin(4t) \\ \sin(4t) - \cos(4t) \end{pmatrix} \right) && (41)\\
&= u(t) + iv(t) && (42)
\end{aligned}$$

Our general solution is then

$$\begin{aligned}
x(t) &= c_1 u(t) + c_2 v(t) && (43)\\
&= c_1 e^{2t} \begin{pmatrix} \cos(4t) \\ \cos(4t) + \sin(4t) \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} \sin(4t) \\ \sin(4t) - \cos(4t) \end{pmatrix} && (44)
\end{aligned}$$

Finally, we need to use the initial condition to get c_1 and c_2. It says

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} = x(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \tag{45}$$

This translates into the system

$$\begin{aligned}
c_1 &= 1 && (46)\\
c_1 - c_2 &= 3 \quad \Rightarrow c_1 = 1 \quad c_2 = -2. && (47)
\end{aligned}$$

Hence our particular solution is

$$x(t) = e^{2t} \begin{pmatrix} \cos(4t) \\ \cos(4t) + \sin(4t) \end{pmatrix} - 2e^{2t} \begin{pmatrix} \sin(4t) \\ \sin(4t) - \cos(4t) \end{pmatrix} \tag{48}$$

Example 4. Sketch the phase portrait of the system in Example 3.

The only difference between the general solution to this example

$$x(t) = c_1 e^{2t} \begin{pmatrix} \cos(4t) \\ \cos(4t) + \sin(4t) \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} \sin(4t) \\ \sin(4t) - \cos(4t) \end{pmatrix} \tag{49}$$

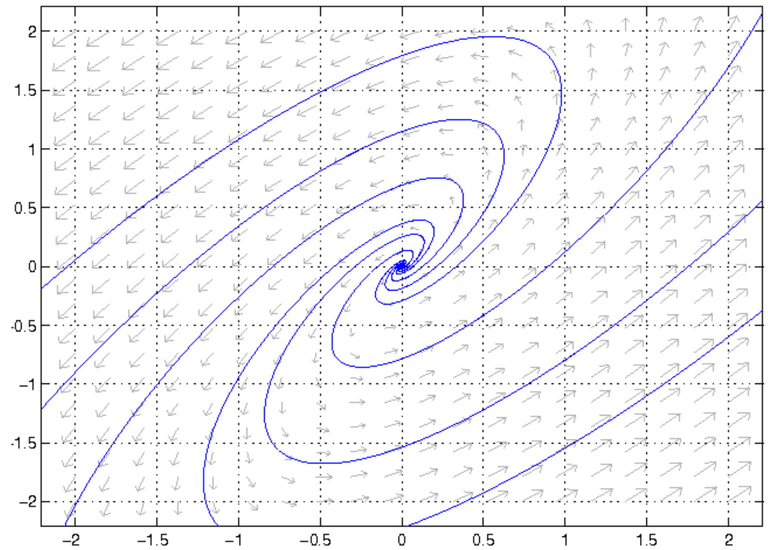

Figure 2: Phase Portrait of the unstable spiral in Example 3

and the one in Example 1 is the exponential sitting out front of the periodic terms. This will make the solution quasi-periodic, rather than actually periodic. The exponential, having a positive exponent, will cause the solution to grow as $t \to \infty$ away from the origin. The solution will still rotate, however, as the trig terms will cause the oscillation. Thus, rather than forming closed circles or ellipses, the trajectories will spiral out of the origin.

As a result, when we have complex eigenvalues $\lambda_{1,2} = a \pm bi$, we call the solution **spiral**. In this case, as the real part a (which affects the exponent) is positive, and the solution grows, the equilibrium at the center is unstable. If a is negative, then spiral would decay into the origin, and the equilibrium would have been asymptotically stable.

So what is there to calculate if we recognize we have a stable/unstable spiral? We still need to know the direction of rotation. This requires, as with the center, that we calculate the tangent vector at a point or two. In this case, the tangent vector at the point $(1, 0)$ is

$$x' = \begin{pmatrix} 6 & -4 \\ 7 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix}. \tag{50}$$

Thus the tangent vector at $(1, 0)$ points up and to the right. Combined with the knowledge that the solution is leaving the origin, this can only happen if the direction of rotation of the spiral is counterclockwise. We obtain a picture as in Figure 2.

HW 7.6 # 2,3,5,6,10,28