Lecture Notes for Math 251: ODE and PDE. Lecture 27: 7.8 Repeated Eigenvalues

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1 Repeated Eigenvalues

Last Time: We studied phase portraits and systems of differential equations with complex eigenvalues.

In the previous cases we had distinct eigenvalues which led to linearly independent solutions. Thus, all we had to do was calculate those eigenvectors and write down solutions of the form

$$x_i(t) = \eta^{(i)} e^{\lambda_i t}. \tag{1}$$

When we have an eigenvalue of multiplicity 2, however, Theorem 3 from a previous lecture tells us that we could have either one or two eigenvectors up to linear independence. If we have two, we are ok, if not then we have more work to do.

1.1 A Complete Eigenvalue

We call a repeated eigenvalue **complete** if it has two distinct (linearly independent) eigenvectors. Suppose our repeated eigenvalue λ has two linearly independent eigenvectors $\eta^{(1)}$ and $\eta^{(2)}$. Then we can proceed as before and our general solution is

$$
\begin{aligned}
x(t) &= c_1 e^{\lambda t} \eta^{(1)} + c_2 e^{\lambda t} \eta^{(2)} & (2) \\
&= e^{\lambda t} (c_1 \eta^{(1)} + c_2 \eta^{(2)}). & (3)
\end{aligned}
$$

It is a basic fact from linear algebra that given two linearly independent vectors such as $\eta^{(1)}$ and $\eta^{(2)}$, we can form any other two-dimensional vector out of a linear combination of these two. So any vector function of the form $x(t) = e^{\lambda t} \eta$ is a solution. As discussed earlier, this can only happen if η is an eigenvector of the coefficient matrix with eigenvalue λ. The conclusion, is that if λ has two linearly independent eigenvectors, every vector is an eigenvector.

This only happens if the coefficient matrix A is a scalar multiple of the identity matrix, since we need
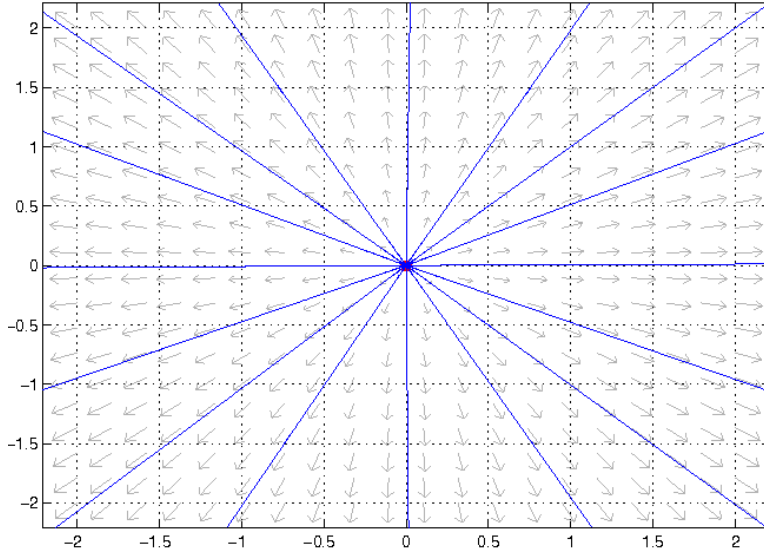
$$A\eta = \lambda \eta = \lambda I \eta \tag{4}$$

Figure 1: Phase Portrait of a star node

for every vector η. Thus, this case only arises when

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \tag{5}$$

or when the original system is

$$x_1 = \lambda x_1 \tag{6}$$
$$x_2 = \lambda x_2 \tag{7}$$

What does the phase portrait look like in this case? Since every vector is an eigenvector, every trajectory that is not a constant solution at the origin is an eigensolution and hence a straight line. We call such the equilibrium solution, in this case a **star node** and its stability is determined by the sign of λ.

(1) If $\lambda > 0$ all the eigensolutions grow away from the origin and the origin is unstable.

(2) If $\lambda < 0$ every eigensolution decays to the origin and the origin is asymptotically stable. We get Figure **??**.

This is a fairly degenerate situation that will not come up in any further discussion, but is important to keep in mind when it happens.

1.2 A Defective Eigenvalue

The other possibility is that λ only has a single eigenvector η up to linear independence. In this case, to form the general solution we need two linearly independent solutions, but we only have one.

$$x_1(t) = e^{\lambda t}\eta \tag{8}$$

In this case, we say that λ is **defective** or **incomplete**. What should we do in this case?

We had a similar problem in the second order linear case. When we ran into this situation there, we were able to work around it by multiplying the solution be t. What if we try that here?

$$x(t) = te^{\lambda t}\eta \tag{9}$$

is a solution

$$x' = Ax \tag{10}$$
$$\eta e^{\lambda t} + \lambda\eta te^{\lambda t} = A\eta te^{\lambda t} \tag{11}$$

Matching coefficients, in order for this guess to be a solution we require

$$\eta = 0 \quad \Rightarrow \quad \eta = 0 \tag{12}$$
$$\lambda\eta e^{\lambda t} = A\eta e^{\lambda t} \quad \Rightarrow \quad (A - \lambda I)\eta = 0 \tag{13}$$

Thus we need η to be an eigenvector of A, which we knew it was, but we also need $\eta = 0$, which cannot be since an eigenvector by definition is nonzero. We need another approach.

The problem with the last attempt is we ended up with a term that did not have a t, but rather just an exponential in it, and this term caused us to require $\eta = 0$. A possible fix might be to add in another term in our guess that only involves an exponential and some other vector ρ. Let's guess that the form of the solution is

$$x(t) = te^{\lambda t}\eta + e^{\lambda t}\rho \tag{14}$$

and see what conditions on ρ we can derive.

$$x' = Ax \tag{15}$$
$$\lambda\eta te^{\lambda t} + \eta e^{\lambda t} + \lambda\rho e^{\lambda t} = A(\eta te^{\lambda t} + \rho e^{\lambda t}) \tag{16}$$
$$(\eta + \lambda\rho + \lambda\eta t)e^{\lambda t} = A\eta te^{\lambda t} + A\rho e^{\lambda t} \tag{17}$$

Thus, setting the coefficients equal and we have

$$A\eta = \lambda\eta \quad \Rightarrow \quad (A - \lambda I)\eta = 0 \tag{18}$$
$$\eta + \lambda\rho = A\rho \quad \Rightarrow \quad (A - \lambda I)\rho = \eta \tag{19}$$

The first condition only tells us that η is an eigenvector of A, which we knew. But the second condition is more useful. It tells us that $(A - \lambda I)\rho = \eta$, then

$$x(t) = \eta te^{\lambda t} + \rho e^{\lambda t} \tag{20}$$

will be a solution to the differential equation.

A vector ρ satisfying

$$(A - \lambda I)\rho = \eta \tag{21}$$

is called a **generalized eigenvector**, while $(A - \lambda I)\rho \neq 0$, it is not hard to verify

$$(A - \lambda I)^2\rho = 0 \tag{22}$$

So as long as we can reproduce a generalized eigenvector ρ, this formula will give a second solution and we can form a general solution.

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Example 1. Find the general solution to the following problem.

$$x' = \begin{pmatrix} -6 & -5 \\ 5 & 4 \end{pmatrix} x \tag{23}$$

The first thing we need to do is to find the eigenvalues of the coefficient matrix.

$$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{vmatrix} -6 - \lambda & -5 \\ 5 & 4 - \lambda \end{vmatrix} \tag{24} \\ &= \lambda^2 + 2\lambda + 1 \tag{25} \\ &= (\lambda + 1)^2 \tag{26} \end{aligned}$$

So we have a repeated eigenvalue of $\lambda = -1$. Due to the form of the matrix, we can also figure out from our previous discussion that λ will only have one eigenvector up to linear independence. Let's calculate it by solving $(A + I)\eta = 0$.

$$\begin{pmatrix} -5 & -5 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{27}$$

So the system of equations we want to solve is

$$\begin{aligned} -5\eta_1 - 5\eta_2 &= 0 \tag{28} \\ 5\eta_1 + 5\eta_2 &= 0. \tag{29} \end{aligned}$$

This is solved by anything of the form $\eta_1 = -\eta_2$. So if we choose $\eta_2 = 1$, we get the eigenvector of

$$\eta = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{30}$$

This isn't enough. We also need to find a generalized eigenvector ρ. So we need to solve $(A+I)\rho = \eta$, or

$$\begin{pmatrix} -5 & -5 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \rho_1 = \frac{1}{5} - \rho_2 \tag{31}$$

So our generalized eigenvector has the form

$$\rho = \begin{pmatrix} \frac{1}{5} - \rho_2 \\ \rho_2 \end{pmatrix}. \tag{32}$$

Choose $\rho_2 = 0$ gives

$$\rho = \begin{pmatrix} \frac{1}{5} \\ 0 \end{pmatrix}. \tag{33}$$

Thus our general solution is

$$\begin{aligned} x(t) &= c_1 e^{\lambda t} \eta + c_2 (t e^{\lambda t} \eta + e^{\lambda t} \rho) \tag{34} \\ &= c_1 e^{-t} \eta \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \left[t e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^{-t} \begin{pmatrix} \frac{1}{5} \\ 0 \end{pmatrix} \right]. \tag{35} \end{aligned}$$

Example 2. Sketch the Phase Portrait for the system in Example 1.

We begin by drawing our eigensolution. Note that in this case, we only have one, unlike the case where we had a (nondegenerate) node. The eigensolution is the straight line in the direction $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, as indicated in Figure **??**. As the eigenvalue is negative, the solution will decay towards the origin.

But what happens to the other trajectories? First, let's just consider the general solution

$$x(t) = c_1 e^{-t} \eta \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \left[t e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^{-t} \begin{pmatrix} \frac{1}{5} \\ 0 \end{pmatrix} \right] \tag{36}$$

with $c_2 \neq 0$. All three terms have the same exponential, but as $t \to \pm\infty$ the te^{-t} term will have a larger magnitude than the other two. Thus in both the forward and backward time, we would get that the trajectories will become parallel to the single eigensolution.

Now, since $\lambda < 0$ and exponentials decay faster than polynomials grow, we can see that as $t \to \infty$, every solution will decay to the origin. So the origin will be asymptotically stable. We also call the origin a **degenerate node** in this case, since it behaves like the a node, but has a single eigensolution.

Consider a node with two close eigenvalues. Then try to imagine what happens to the general solution as we bring the eigenvalues together. The eigensolutions will collapse together, but the non-eigensolution trajectories would keep their asymptotic behavior with regard to this collapsed eigensolution.

Notice that, as illustrated in Figure **??** we end up with a large degree of rotation of the solution. The solution has to turn around to be able to be asymptotic to the solution in both forward and backward time. This is because degenerate nodes are the borderline case between nodes and spirals. Suppose our characteristic equation is

$$0 = \lambda^2 + b\lambda + c. \tag{37}$$

The eigenvalues are then, by the quadratic formula

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4c}}{2a} \tag{38}$$

The discriminant of this equation is positive in the node case and negative in the spiral/center cases. We get degenerate nodes when the solutions transition between these two cases and the discriminant becomes zero. So for degenerate nodes the solutions are trying to wind around in a spiral, but they do not quite make it due to the lack of complexity of the eigenvalue.

But how do we know the direction of rotation? We do the same thing we did in the spiral case, compute the tangent vector at a point. Combined with our knowledge of the stability of the origin, will tell us how the non-eigensolutions must turn.

Let's start by considering the point $(1, 0)$. At this point,

$$\begin{pmatrix} -6 & -5 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -6 \\ 5 \end{pmatrix}. \tag{39}$$
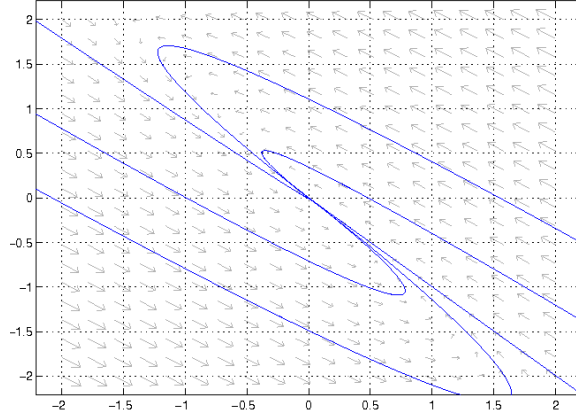
Figure 2: Phase Portrait for the asymptotically stable degenerate node in Example 1

Since we know our solution is asymptotically stable, the tangent vector can only point up and to the left if the solution rotates counterclockwise as they start to approach the origin.

Example 3. Find the general solution to the following system.

$$x' = \begin{pmatrix} 12 & 4 \\ -16 & -4 \end{pmatrix} x \tag{40}$$

The first thing we need to do is to find the eigenvalues of the coefficient matrix.

$$0 = \det(A - \lambda I) = \begin{vmatrix} 12 - \lambda & 4 \\ -16 & -4 - \lambda \end{vmatrix} \tag{41}$$

$$= \lambda^2 - 8\lambda + 16 \tag{42}$$

$$= (\lambda - 4)^2 \tag{43}$$

So we have a repeated eigenvalue of $\lambda = 4$. Let's calculate it by solving $(A - 4I)\eta = 0$.

$$\begin{pmatrix} 8 & 4 \\ -16 & -8 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{44}$$

So the system of equations we want to solve is

$$8\eta_1 + 4\eta_2 = 0 \tag{45}$$

$$-16\eta_1 - 8\eta_2 = 0. \tag{46}$$

This is solved by anything of the form $\eta_2 = -2\eta_1$. So if we choose $\eta_1 = 1$, we get the eigenvector of

$$\eta = \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \tag{47}$$

We also need to find a generalized eigenvector ρ. So we need to solve $(A - 4I)\rho = \eta$, or

$$\begin{pmatrix} 8 & 4 \\ -16 & -8 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \Rightarrow \quad \rho_2 = \frac{1}{4} - 2\rho_1 \tag{48}$$
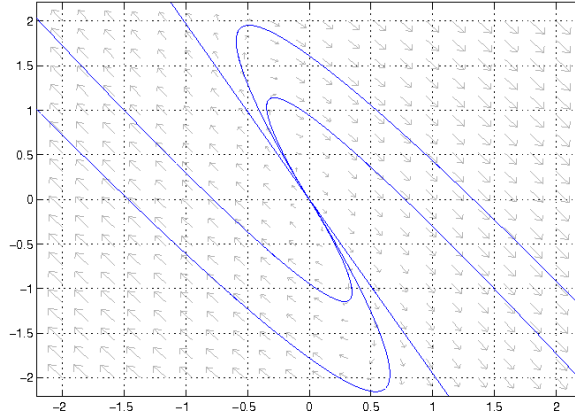
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Figure 3: Phase Portrait of the unstable degenerate node in Example 3.

So our generalize eigenvector has the form

$$\rho = \begin{pmatrix} \rho_1 \\ \frac{1}{4} - 2\rho_1 \end{pmatrix}. \tag{49}$$

Choose $\rho_1 = 0$ gives

$$\rho = \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix}. \tag{50}$$

Thus our general solution is

$$\begin{aligned} x(t) &= c_1 e^{\lambda t}\eta + c_2(te^{\lambda t}\eta + e^{\lambda t}\rho) & (51) \\ &= c_1 e^{-t}\rho = \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2(te^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + e^{-t}\begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix}. & (52) \end{aligned}$$

Example 4. Sketch the Phase Portrait of the system in Example 3.

Everything is completely analogous to the previous example's phase portrait. We sketch the eigensolution, and note that it will grow away from the origin as $t \to \infty$, so the origin will in this case be an unstable degenerate node.

Typical trajectories will once again come out of the origin parallel to the eigensolution and rotate around to be parallel to them again, and all we would need to do is to calculate the direction of rotation by computing the tangent vector at a point or two. At $(1, 0)$, we would get

$$x' = \begin{pmatrix} 16 \\ 12 \end{pmatrix}. \tag{53}$$

which can only happen given that the solutions are growing if the direction of rotation is clockwise. Thus we get Figure 3.

Note in the last 3 sections 7.5,7.6, 7.8 we have covered the information in Section 9.1, which is sketching phase portraits, and identifying the three distinct cases for 1. Real Distinct Eigenvalues,

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2. Complex Eigenvalues, and 3. Repeated Eigenvalues.

HW 7.8 # 2,3c,4,8a