Lecture Notes for Math 251: ODE and PDE. Lecture 27: 9.1 The Phase Plane: Linear Systems

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1 Phase Portrait Review

Last Time: We studied phase portraits and systems of differential equations with repeated eigenvalues.

Note in the last 3 sections 7.5, 7.6, 7.8 we have covered the information in Section 9.1, which is sketching phase portraits, and identifying the three distinct cases for 1. Real Distinct Eigenvalues, 2. Complex Eigenvalues, and 3. Repeated Eigenvalues. Think of this section as a good review.

In Chapter 1 and Section 2.5 we considered the autonomous equations

$$\frac{dy}{dt} = f(y) \tag{1}$$

Consider the simplest system, a second order linear homogeneous system with constant coefficients. Such a system has the form

$$\frac{dx}{dt} = Ax \tag{2}$$

where A is a 2×2 matrix. We spent three sections solving these types of systems. Recall we seek solutions of the form $x = \eta e^{rt}$, then if we substitute this into the equation we found

$$(A - rI)\eta = 0 \tag{3}$$

Thus r is an eigenvalue and η the corresponding eigenvector.

Definition 1. Points where $Ax = 0$ correspond to equilibrium or constant solutions, and are called **critical points**. Note if A is nonsingular, then the only critical point is $x = 0$.

We must consider the five possible situations we could be in.

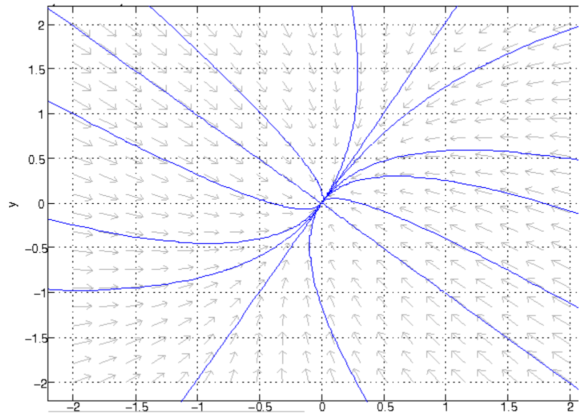

Figure 1: Phase Portrait of the Nodal Sink

1.1 Case I: Real Unequal Eigenvalues of the Same Sign

The general solution of $x' = Ax$ is

$$x = c_1\eta^{(1)}e^{\lambda_1 t} + c_2\eta^{(2)}e^{\lambda_2 t} \tag{4}$$

where λ_1 and λ_2 are either both positive or both negative. Suppose first that $\lambda_1 < \lambda_2 < 0$. Both exponentials decay, so as $t \to \infty$, then $x(t) \to 0$ regardless of the values of c_1 and c_2. Note the eigenvalue with the bigger magnitude, $|\lambda_i|$, will determine where the trajectories are directed to. So the trajectories will tend towards $\eta^{(1)}$.

Definition 2. The type of critical point where all solutions decay to the origin is a **node** or **nodal sink**.

If λ_1 and λ_2 are both positive and $0 < \lambda_2 < \lambda_1$, then the trajectories have the same pattern as the previous case but as $t \to \infty$ the solutions blow up so all arrows change direction and move away from the origin. The critical point is still called a **node** or **nodal source**. Notice it is a source because trajectories come from it and leave, whereas the nodal sink before sucked all trajectories towards itself.

1.2 Case II: Real Eigenvalues of Opposite Signs

The general solution of $x' = Ax$ is

$$x = c_1\eta^{(1)}e^{\lambda_1 t} + c_2\eta^{(2)}e^{\lambda_2 t} \tag{5}$$

where $\lambda_1 > 0$ and $\lambda_2 < 0$. Notice as $t \to \infty$ the second term decays to zero and the first eigenvector becomes dominant. So as time goes to infinity all trajectories asymptotically approach $\eta^{(1)}$. The only solutions that approach 0 are the ones which start on $\eta^{(2)}$. This is because $c_1 = 0$ and all terms would decay as time increases.

Definition 3. The origin where some solutions tend towards it and some tend away is called a **saddle point**.

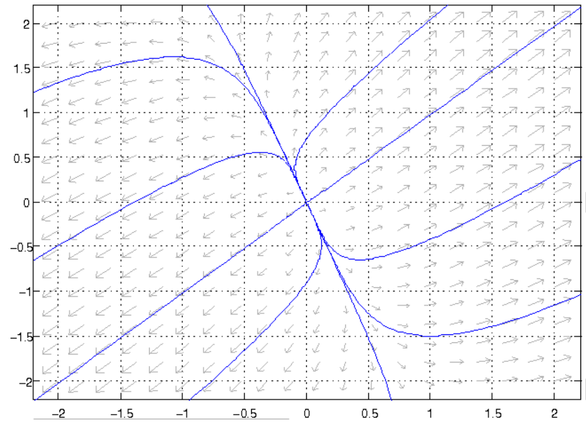

Figure 2: Phase Portrait of the Nodal Source

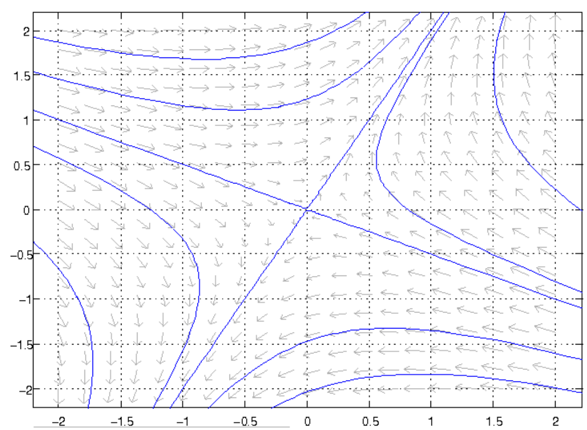

Figure 3: Phase Portrait of the saddle point

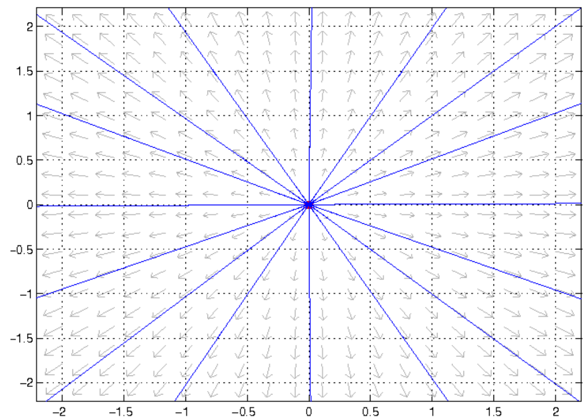

Figure 4: Phase Portrait of a star node

1.3 Case III: Repeated Eigenvalues

Here $\lambda_1 = \lambda_2 = \lambda$. We have two subcases.

1.3.1 Case IIIa: Two Independent Eigenvectors

The general solution of $x' = Ax$ is

$$x = c_1\eta^{(1)}e^{\lambda t} + c_2\eta^{(2)}e^{\lambda t} \tag{6}$$

where the eigenvectors are linearly independent. Every trajectory lies on a straight line through the origin. If $\lambda < 0$ all solutions decay to the origin, if $\lambda > 0$ then all solutions move away from the origin.

Definition 4. In either case, the critical point is called a **proper node** or a **star point**.

1.3.2 Case IIIb: One Independent Eigenvector

The general solution in this case is

$$x = c_1\eta e^{\lambda t} + c_2(\eta t e^{\lambda t} + \rho e^{\lambda t}). \tag{7}$$

where η is the eigenvector and ρ is the generalized eigenvector. For t large, $c_2\eta t e^{\lambda t}$ dominates. Thus as $t \to \infty$ every trajectory approaches the origin tangent to the line through the eigenvector. If the $\lambda > 0$ the trajectories move away from the origin, and if $\lambda < 0$ the trajectories moved towards the origin.

Definition 5. When a repeated eigenvalue has only a single independent eigenvector, the critical point is called an **improper** or **degenerate node**.

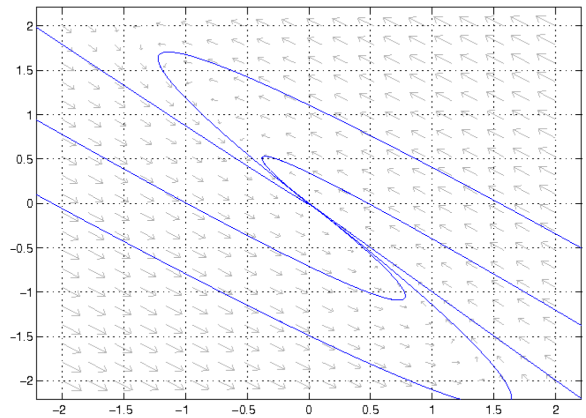

Figure 5: Phase Portrait for the asymptotically stable degenerate node

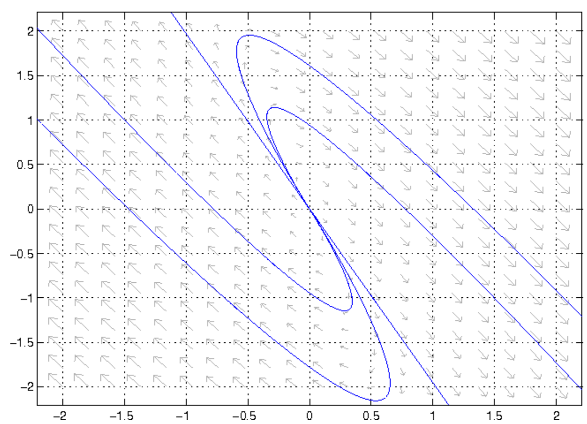

Figure 6: Phase Portrait of the unstable degenerate node

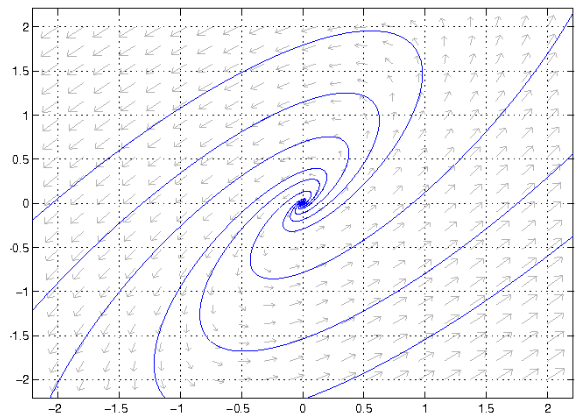

Figure 7: Phase Portrait of the unstable spiral

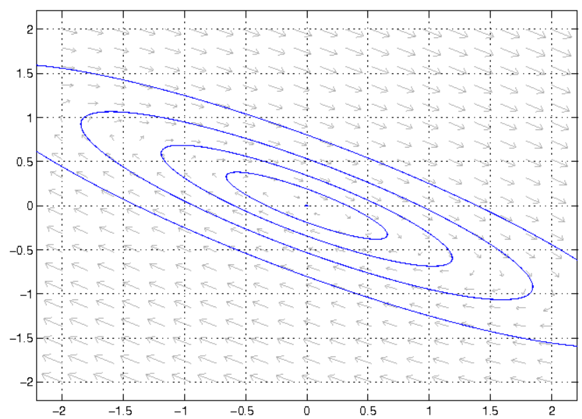

Figure 8: Phase Portrait of the center point

1.4 Case IV: Complex Eigenvalues

Suppose the eigenvalues are $\alpha \pm i\beta$, where α and β are real. In this case critical points are called **spiral point**. Depending on if the trajectories move toward or away from the origin it could be characterized as a **spiral sink** or **source**.

In the phase portrait we either spiral towards or away from the origin. If the real part $\alpha > 0$, then trajectories spiral away from the origin. If the real part $\alpha < 0$, then the trajectories spiral towards the origin.

1.5 Case V: Pure Imaginary Eigenvalues

Here $\alpha = 0$ and $\lambda = \pm\beta i$. In this case the critical point is called a **center**, because the trajectories are concentric circles around the origin. We can determine the direction of the circle by finding the tangent vector at a point like $(1, 0)$.

Eigenvalues	Type of Critical Point	Stability
$\lambda_1 > \lambda_2 > 0$	Node Source	Unstable
$\lambda_1 < \lambda_2 < 0$	Node Sink	Asymptotically Stable
$\lambda_2 < 0 < \lambda_1$	Saddle Point	Unstable
$\lambda_1 = \lambda_2 > 0$	Proper or Improper Node	Unstable
$\lambda_1 = \lambda_2 < 0$	Proper or Improper Node	Asymptotically Stable
$\lambda_1, \lambda_2 = \alpha + i\beta$		
$\alpha > 0$	Spiral Source	Unstable
$\alpha < 0$	Spiral Sink	Asymptotically Stable
$\lambda = \pm i\beta$	Center	Stable

Table 1: Table 9.1: Stability Properties of Linear Systems $x' = Ax$ with $\det(A - rI) = 0$ and $\det(A) \neq 0$.

2 Summary and Observations

1. After a long time, each trajectory exhibits one of only three types of behavior. As $t \to \infty$, each trajectory approaches the critical point $x = 0$, repeatedly traverses a closed curve around the critical point, or becomes unbounded.

2. For each point there is only one trajectory. The trajectories do not cross each other. The only solutions passing through the critical point are $x = 0$, all other solutions only approach the origin as $t \to \infty$ or $-\infty$.

3. In each of the five cases we have one of three situations:

(1) All trajectories approach the critical point $x = 0$ as $t \to \infty$. This is the case if the eigenvalues are real and negative or complex with negative real part. The origin is nodal or a spiral sink.

(2) All trajectories remain bounded but do not approach the origin as $t \to \infty$. This is the case if the eigenvalues are pure imaginary. The origin is a center.

(3) Some trajectories, and possibly all trajectories except $x = 0$, become unbounded as $t \to \infty$. This is the case if at least one of the eigenvalues is positive or if the eigenvalues have positive real part. The origin is a nodal source, a spiral source, or a saddle point.