Lecture Notes for Math 251: ODE and PDE. Lecture 28: 9.2 Autonomous Systems and Stability and 9.3 Almost Linear Systems

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Spring 2012

Last Time: We did an in depth review of phase portraits and constructing them give a system of equations.

1 9.2 Autonomous Systems and Stability

What is an Autonomous System?

Definition 1. A system of two simultaneous differential equations of the form

$$\frac{dx}{dt} = F(x,y), \quad \frac{dy}{dt} = G(x,y) \tag{1}$$

where F and G are continuous and have continuous partial derivatives in some domain D. From Theorem 7.1 we know there exists a unique solution $x = \phi(t), y = \psi(t)$ of the system satisfying the initial conditions

$$x(t_0) = x_0, \quad y(t_0) = y_0 \tag{2}$$

The property that makes the system **autonomous** is that F and G only depend on x and y and not t.

1.1 Stability and Instability

Consider the autonomous system of the form

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}) \tag{3}$$

Definition 2. The points where $\mathbf{f}(\mathbf{x}) = 0$ are the critical points, which correspond to constant or equilibrium solutions of the autonomous system.

Definition 3. A critical point $\mathbf{x}^0$ is said to be **stable** if, given any $\epsilon > 0$, there is a $\delta > 0$ such that every solution $x = \phi(t)$, which at $t = 0$ satisfies

$$||\phi(0) - x^0|| < \delta \tag{4}$$

exists for all positive t and satisfies

$$||\phi(t) - x^0|| < \epsilon \tag{5}$$

for all $t \geq 0$. It's **asymptotically stable** if

$$||\phi(0) - x^0|| < \delta \tag{6}$$

then

$$\lim_{t\to\infty} \phi(t) = x^0 \tag{7}$$

Finally, it is **unstable** if a solution does not approach a critical point as $t \to \infty$.

Example 4. Find the critical points of

$$\frac{dx}{dt} = -(x-y)(1-x-y), \quad \frac{dy}{dt} = x(2+y). \tag{8}$$

We find the critical points by solving the algebraic equations

$$(x-y)(1-x-y) = 0 \tag{9}$$
$$x(2+y) = 0 \tag{10}$$

One way to satisfy the second equation is to choose $x = 0$. Then the first equation becomes $y(1-y) = 0$, so $y = 0$ or $y = 1$. Now let's choose $y = -2$, then the first equation becomes $(x+2)(3-x) = 0$ so $x = -2$ or $x = 3$. So the four critical points are $(0,0)$, $(0,1)$, $(-2,-2)$, and $(3,-2)$.

2 9.3 Locally Linear Systems

We start with a few key theorem in this section. Consider the linear system

$$x' = Ax \tag{11}$$

Theorem 5. *The critical point $x = 0$ of the linear system above is asymptotically stable if the eigenvalues r_1, r_2 are real and negative or have negative real part; stable, but not asymptotically stable if r_1 and r_2 are pure imaginary; unstable if r_1 and r_2 are real and either positive or if they have positive real part.*

2.1 Introduction to Nonlinear Systems

The general form of the two dimensional system of differential equations is

$$x_1' = f_1(x_1, x_2) \tag{12}$$
$$x_2' = f_2(x_1, x_2) \tag{13}$$

For systems like this it is hard to find trajectories analytically, as we did for linear systems. Thus we need to discuss the behavior of these solutions.

There are some features of nonlinear phase portraits that we should be aware of:

(1) The **fixed** or **critical points** which are the equilibrium or steady-state solutions. These correspond to points x satisfying $\mathbf{f}(\mathbf{x}) = 0$. So x_1 and x_2 are zeroes for both f_1 and f_2.
(2) The **closed orbits**, which correspond to solutions that are periodic for both x_1 and x_2.
(3) How trajectories are arranged, new fixed points and closed orbits.
(4) The stability or instability of fixed points and closed orbits, which of these attract nearby trajectories and which repel them?

Theorem 6. (Existence and Uniqueness) *Consider the initial value problem*

$$\mathbf{x}' = \mathbf{f}(x) \quad x(0) = x_0 \tag{14}$$

If f is continuous and its partial derivatives on some region containing x_0, then the initial value problem has a unique solution $x(t)$ on some interval near $t = 0$.

Note: The theorem asserts that no two trajectories can intersect.

2.2 Linearization around Critical Points

To begin we always start by finding the critical points, which correspond to the equilibrium solutions of the system. If the system is linear the only critical point is the origin, $(0, 0)$. Nonlinear systems can have many fixed points and we want to determine the behavior of the trajectories near these points. Consider,

$$x' = f(x, y) \tag{15}$$
$$y' = g(x, y) \tag{16}$$

The goal of linearization is to use our knowledge of linear systems to conclude what we can about the phase portrait near (x_0, y_0). We will try to approximate our nonlinear system by a linear system, which we can then classify. Since (x_0, y_0) is a fixed point, and the only fixed point of a linear system is the origin, we will want to change variables so that (x_0, y_0) becomes the origin of the new coordinate system. Thus, let

$$u = x - x_0 \tag{17}$$
$$v = y - y_0. \tag{18}$$

We need to rewrite our differential equation in terms of u and v.

$$\begin{aligned} u' &= x' && (19)\\ &= f(x,y) && (20)\\ &= f(x_0+u, y_0+v) && (21)\end{aligned}$$

The natural thing to do is a Taylor Expansion of f near (x_0, y_0).

$$\begin{aligned} &= f(x_0,y_0) + u\frac{\partial f}{\partial x}(x_0,y_0) + v\frac{\partial f}{\partial y}(x_0,y_0) + \text{higher order terms} && (22)\\ &= u\frac{\partial f}{\partial x}(x_0,y_0) + v\frac{\partial f}{\partial y}(x_0,y_0) + H.O.T. && (23)\end{aligned}$$

recall that $f(x_0,y_0) = 0$ (since it is a fixed point). To simplify notation we will sometimes write $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ which are evaluated at (x_0, y_0), but it is important to keep this in mind. The partial derivatives are numbers not functions. Also, recall we are considering what happens very close to our fixed point, u and v are both small, and hence the higher order terms are smaller still and will be disregarded in computations. By a similar computation we have

$$v' = u\frac{\partial g}{\partial x} + v\frac{\partial g}{\partial y} + H.O.T. \tag{24}$$

Ignoring the small higher order terms, we can write this system of rewritten differential equations in matrix form. The **linearized system** near (x_0, y_0) is

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \\ \frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial y}(x_0,y_0) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{25}$$

We will use, from this point on, the notation $f_x = \frac{\partial f}{\partial x}$. The matrix

$$A = \begin{pmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{pmatrix} \tag{26}$$

is called the **Jacobian Matrix** at (x_0, y_0) of the vector-valued function $\mathbf{f}(x) = \begin{pmatrix} f(x_1,x_2) \\ g(x_1,x_2) \end{pmatrix}$. In multivariable calculus, the Jacobian matrix is appropriate analogue of the single variable calculus derivative. We then study this linear system with standard techniques.

HW 9.2 # 5a-10a

HW 9.3 # 5abc,7abc,8abc,9abc