Lecture Notes for Math 251: ODE and PDE. Lecture 29: 9.5 Predator-Prey Equations

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Last Time: We studied Autonomous systems, stability, and linearization of almost linear systems.

1 9.5 Predator-Prey Equations

Recall from Chapter 1 we discussed equations representing populations of animals which grow without predators. Now we want to consider a system where one species (predator) preys on the other species (prey), while the prey lives on another food source.

An example is a closed forest where foxes prey on rabbits and rabbits eat vegetation. At a lake, ladybugs are predators and aphids are prey.

We denote by x and y the populations of prey and predators respectively, at time t. We need the following assumptions to construct our model.

(1) In the absence of a predator, the prey grows at a rate proportional to the current population. So $\frac{dx}{dt} = ax$, when $y = 0$.

(2) In the absence of the prey, the predator dies out. Thus $\frac{dy}{dt} = -cy$ where $c > 0$ when $x = 0$.

(3) Then number of encounters between predator and prey is proportional to the product of their populations. Each such encounter tends to promote the growth of the predator and to inhibit the growth of the prey. Thus the growth rate of the predator is increased by a term of the form γxy, while the growth rate of the prey is decreased by a term $-\alpha xy$, where γ and α are positive constants.

From these assumptions we can form the following equations

$$\frac{dx}{dt} = ax - \alpha xy = x(a - \alpha y), \quad (1)$$

$$\frac{dy}{dt} = -cy + \gamma xy = y(-c + \gamma y) \quad (2)$$

The constants $a, c, \alpha,$ and γ are all positive constants. a and c are the growth rate of the prey and the death rate of the predators respectively. α and γ are measures of the effect of interaction between the two species. Equations (**??**)-(**??**) are known as the **Lotka-Volterra** equations.

The goal is to determine the qualitative behavior of the solutions (trajectories) of the system (**??**)-(**??**) for arbitrary positive initial values of x and y.

Example 1. Describe the solutions to the system

$$\frac{dx}{dt} = x - 0.5xy = F(x,y) \quad (3)$$
$$\frac{dy}{dt} = -0.75y + 0.25xy = G(x,y) \quad (4)$$

for x and y positive.

First we begin with techniques learned in 9.2 and find the critical points. The critical points of the system are the solutions of the equations

$$x - 0.5xy = 0, \quad -0.75y + 0.25xy = 0 \quad (5)$$

So the critical points are $(0,0)$ and $(3,2)$.

Next using 9.3, we want to consider the local behavior of the solutions near each critical point. At $(0,0)$ consider the Jacobian

$$J = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 1 - 0.5y & -0.5x \\ 0.25y & -0.75 + 0.25x \end{pmatrix} \quad (6)$$

For $(0,0)$ the Jacobian is

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -0.75 \end{pmatrix} \quad (7)$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -0.75$, since they are real of opposite signs Theorem 5 from Section 9.3 says we have a saddle point and thus $(0,0)$ is unstable.

At $(3,2)$ the Jacobian is

$$J = \begin{pmatrix} 0 & -1.5 \\ 0.5 & 0 \end{pmatrix} \quad (8)$$

The eigenvalues $\lambda_1 = \frac{\sqrt{3}i}{2}$ and $\lambda_2 = -\frac{\sqrt{3}i}{2}$. Since the eigenvalues are pure imaginary, $(3,2)$ is a center and thus a stable critical point. Besides solutions starting on coordinate axes, all solutions will circle around $(3,2)$.

Using this example consider the general system (**??**)-(**??**). First find the critical points of the system which are solutions to

$$x(a - \alpha y) = 0, \quad y(-c + \gamma x) = 0 \quad (9)$$

so the critical points are $(0,0)$ and $(\frac{c}{\gamma}, \frac{a}{\alpha})$. We first examine the solutions near each critical point. The Jacobian is

$$J = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} a - \alpha y & -\alpha x \\ \gamma y & -c + \gamma x \end{pmatrix} \quad (10)$$

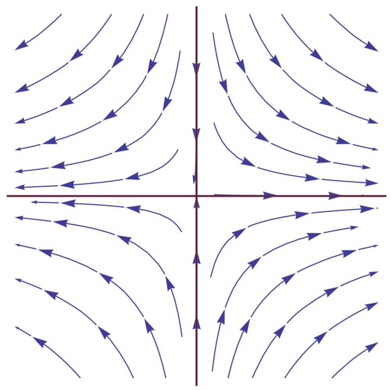

Figure 1: Direction Field for the critical point $(0, 0)$

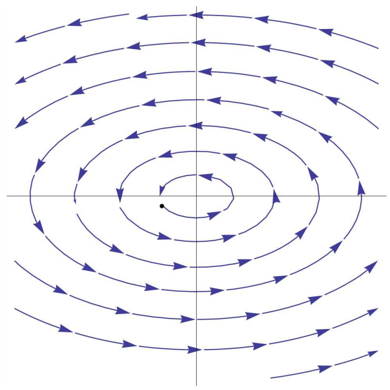

Figure 2: Direction Field for the critical point $(\frac{c}{\gamma}, \frac{a}{\alpha})$

Consider $(0, 0)$, which has Jacobian

$$\begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix} \tag{11}$$

The eigenvalues are a and $-c$. Recall a and c are both positive real constants. So the eigenvalues are real values of opposite signs, thus $(0, 0)$ is a saddle point and unstable. This makes sense since if we move slightly away from zero and introduce predators and prey, then the system would take off and the prey population would grow away from 0, while if we introduced a predator and no prey it would decay to zero.

Next consider $(\frac{c}{\gamma}, \frac{a}{\alpha})$. The Jacobian is

$$\begin{pmatrix} 0 & -\frac{\alpha c}{\gamma} \\ \frac{\gamma a}{\alpha} & 0 \end{pmatrix} \tag{12}$$

which has eigenvalues $\lambda_{1,2} = \pm i\sqrt{ac}$. So the eigenvalues are pure imaginary and the critical point is a stable center. The direction field for all predator prey equations will have those components and the direction fields will look like Figure 1 and Figure 2.

If we use the fact they are separable equations and solve for the trajectories we find that the solutions are ellipses, so $(\frac{c}{\gamma}, \frac{a}{\alpha})$ is a center.

This is what we expect, because a predator-prey system should be cyclic. When the predator population gets to high there is not enough prey and predators die out. When the prey population is high the system can support more predators. These systems are always cyclic. The solutions to this system are

$$x = \frac{c}{\gamma} + \frac{c}{\gamma} K \cos(\sqrt{ac}t + \Phi) \quad (13)$$

$$y = \frac{a}{\alpha} + \frac{a}{\alpha} K \sin(\sqrt{ac}t + \Phi) \quad (14)$$

where K and Φ are determined by the initial conditions.

From these solutions we get four main conclusions about predator prey models:

(1) The sizes of the predator and prey populations vary sinusoidally with period $\frac{2\pi}{\sqrt{ac}}$. This period of oscillation is independent of initial conditions.

(2) The predator and prey populations are out of phase by one quarter of a cycle. The prey leads and the predator lags as can be seen by plotting these due to $\sin$ and $\cos$.

(3) The amplitudes of the oscillations are $\frac{Kc}{\gamma}$ for the prey and $\frac{a\sqrt{c}K}{\alpha\sqrt{a}}$ for the predator and hence depend on the initial conditions as well as the parameters of the problem.

(4) The average populations of predator and prey over one complete cycle are $\frac{c}{\gamma}$ and $\frac{a}{\alpha}$, respectively. These are the same as the equilibrium populations.

One criticism of this model is in the absence of a predator the prey grows without bound, which is not realistic due to a finite amount of food available for the prey. This is a basic model which is decently accurate and a building block for many advanced models.

HW 9.5 # 1,2,4,5

Extra to think about: In 4 why is the critical point $(\frac{9}{8}, 0)$ unstable from a real-world point of view. Think about what it would mean for the predator-prey system.

Hint: It is much easier to do a) in each question after you do b) and c).