Hypergraph Coloring Complexes and Ehrhart f^*-Vectors

Felix Breuer

San Francisco State University
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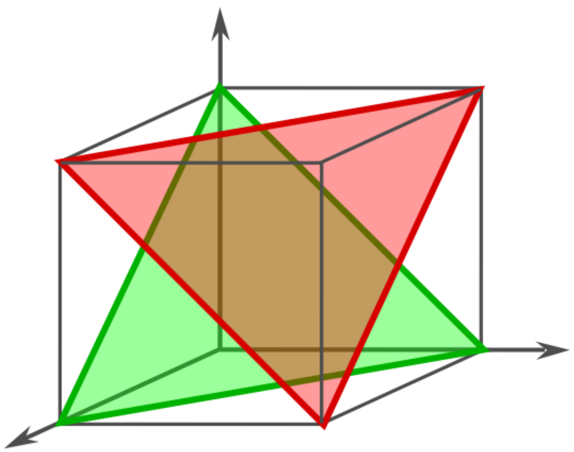
Ehrhart Theory...

$X \subset \mathbb{R}^d$ $\qquad k \in \mathbb{Z}_{>0}$ $\qquad L_X(k) = \#\mathbb{Z}^d \cap k \cdot X$

Ehrhart: If X is an integral polytopal complex, then $L_X(k)$ is a polynomial.

... has many Combinatorial Applications

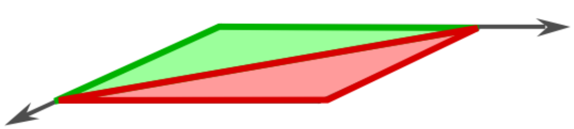
One example: Flow polynomial $\bar{\varphi}(k)$ of a graph G



Reciprocity

Theorem (B, Sanyal '12)
Combinatorial interpretation of $|\bar{\varphi}(-k)|$.

Theorem (Beck, B, Godkin, Martin '12)
...also works for flows on cell complexes.



Bounds

Theorem (B, Dall '11)
Constraints on coefficients of $\bar{\varphi}(k)$.

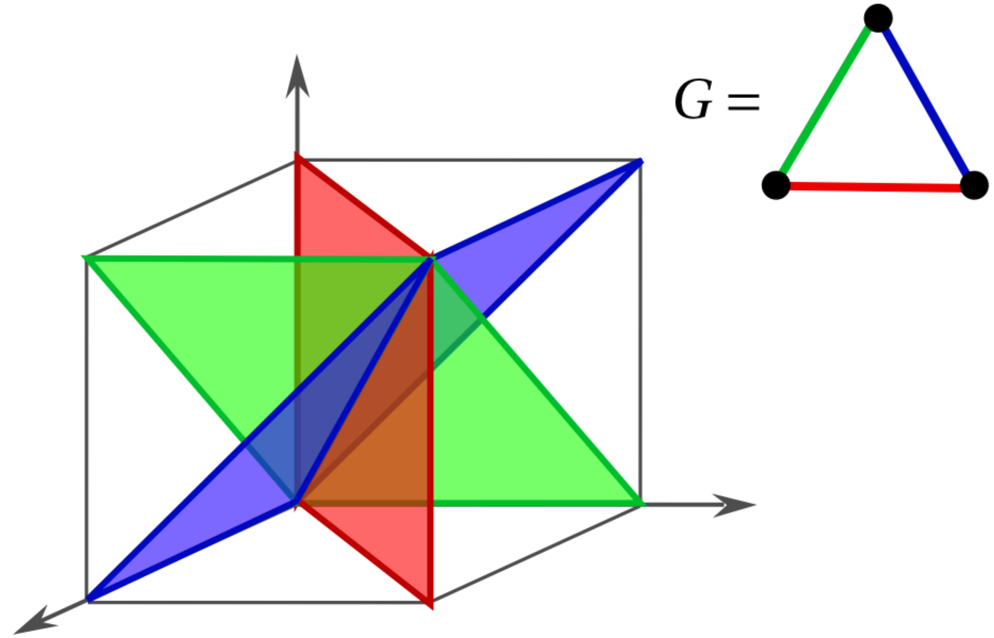
Hypergraph Colorings

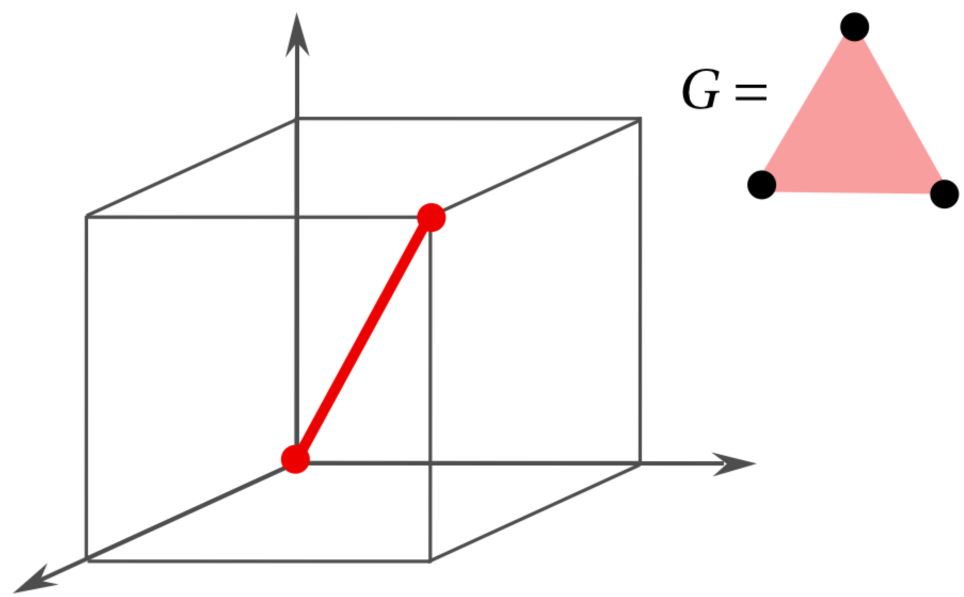
hypergraph $G = (V, E)$

$(k+1)$-coloring $c \in [0,k]^n \cap \mathbb{Z}^n$

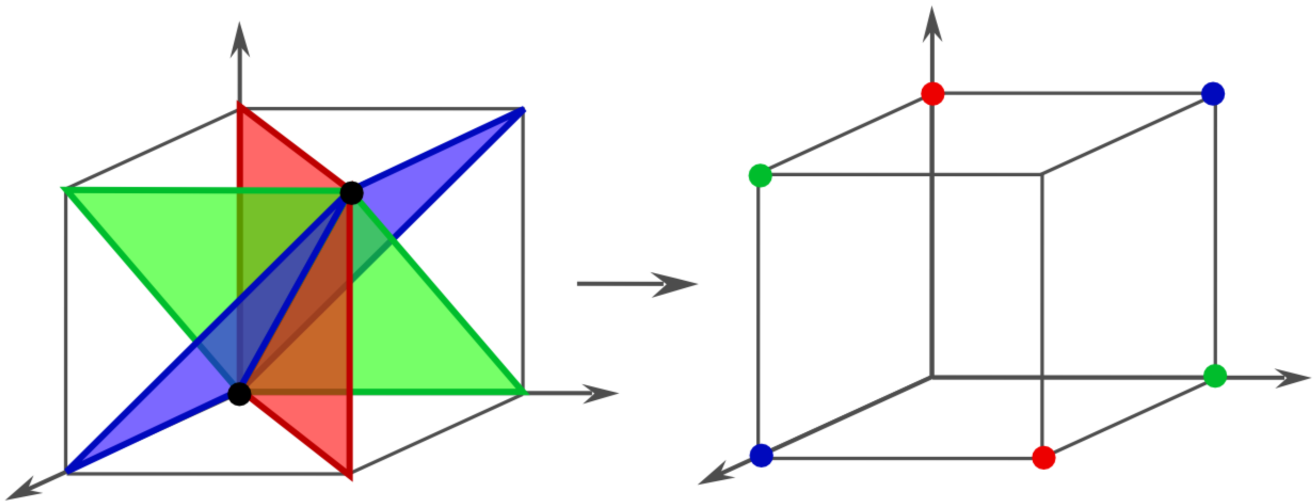
c is proper if
for all $e \in E$
there exist $v, w \in e$
such that $c_v \neq c_w$

$\chi_G(k) = \#$ proper k-colorings of G





Coloring Complex



remove cone points

↓

coloring complex Δ_G

$e \in E$

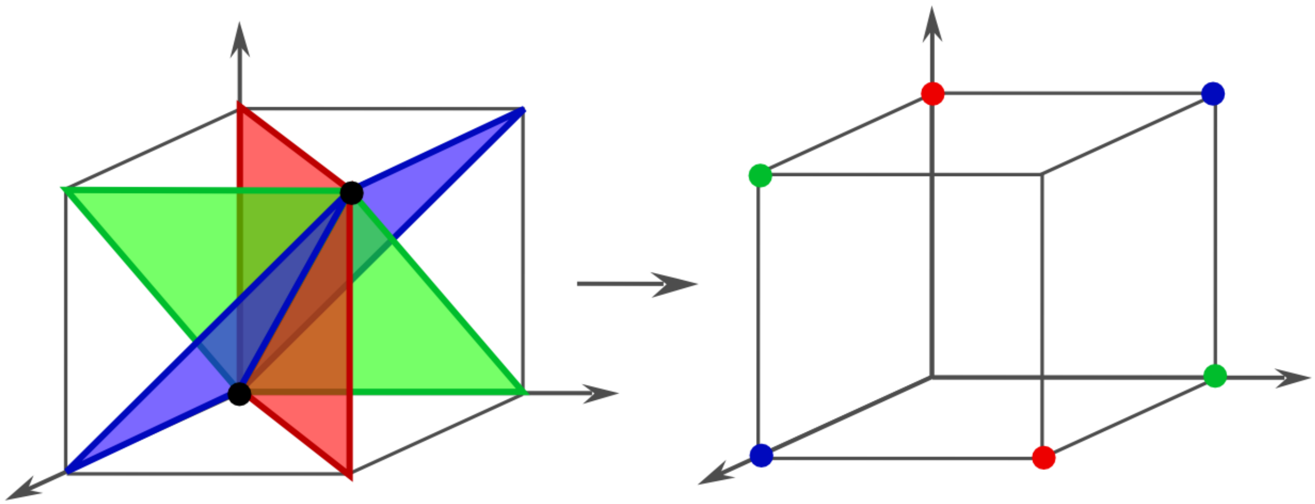
$H_e = \{x \mid x_a = x_b \; \forall a, b \in e\}$

triangulated cube of dim $|V| - |e| + 1$ → $P_e = [0,1]^V \cap H_e$

sphere of dim $|V| - |e| - 1$ → $Q_e = P_e \setminus \{0, 1\}$ ← remove 0,1 and incident faces

$$\Delta_G = \bigcup_{e \in E} Q_e$$

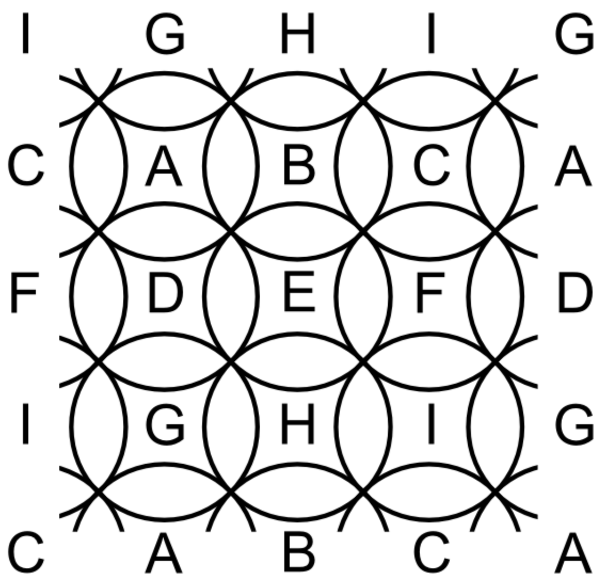
Coloring Complex - Properties

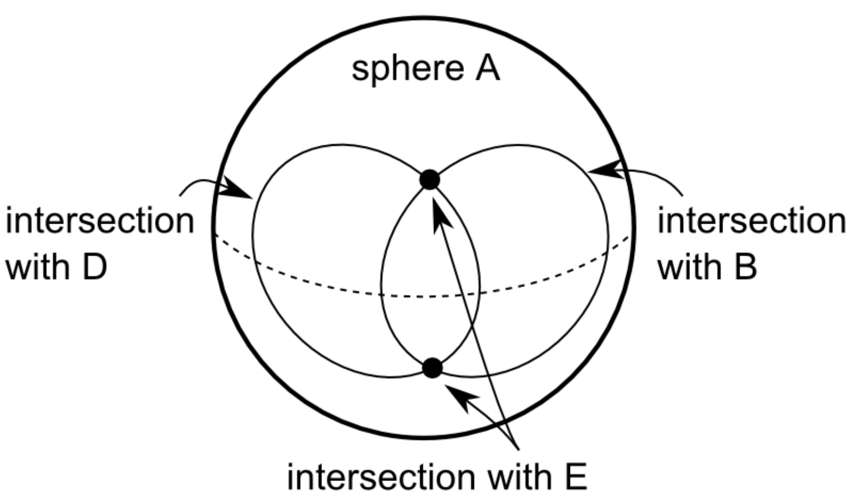


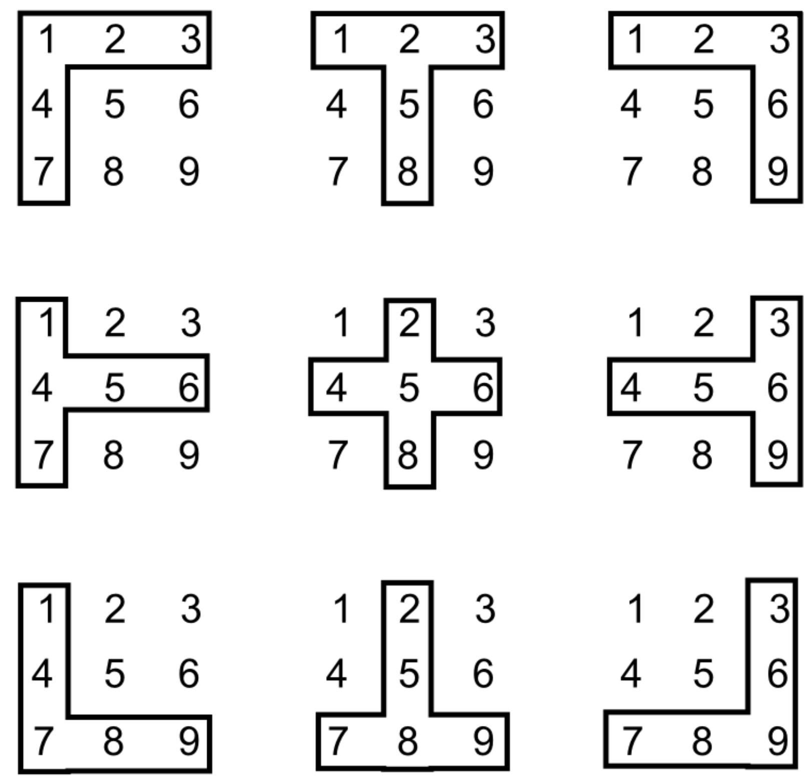
properties of Δ_G	graphs	hypergraphs
wedge of spheres	yes (Steingrimmson '01)	no
Cohen-Macaulay	yes (Jonsson '05)	no
shellable	yes (Hultman '07)	no
partitionable	yes	no
g-constraints	yes (Hersh, Swartz '07)	no
$h_i^* \geq 0$	yes	no
$f_i^* \geq 0$	yes	yes

(B, Dall, Kubitzke '11)

Coloring Complex - Not a Wedge of Spheres







h^*- and f^*-Vectors

Let p be a polynomial of degree d'. Let $d \geq d'$.

Then there exist h_i^* and f_i^* for $i = 0, \ldots, d$ such that

$$p(k) = \sum_{i=0}^{d} h_i^* \binom{k+d-i}{d} = \sum_{i=0}^{d} f_i^* \binom{k-1}{i}.$$

We define the h^*- and f^*-vectors by

$$h^*(p,d) = (h_0^*, \ldots, h_d^*),$$

$$f^*(p,d) = (f_0^*, \ldots, f_d^*).$$

The Geometry behind h^*- and f^*-Vectors

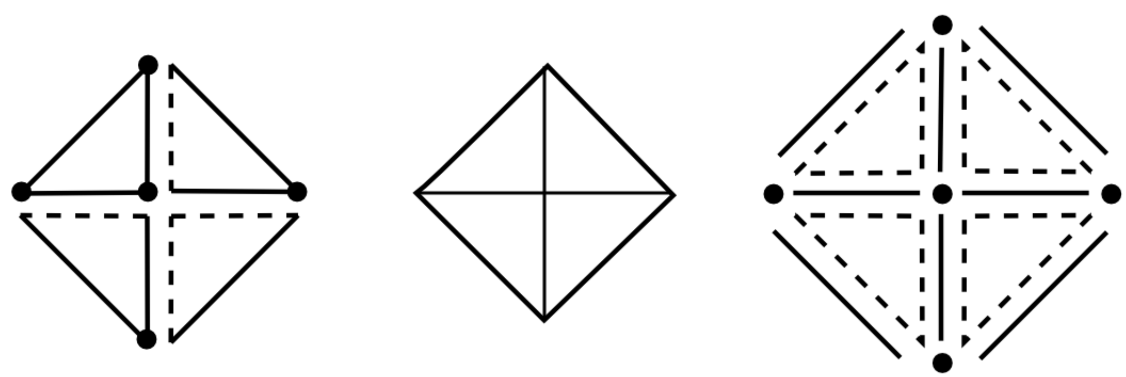
$$p(k) = \sum_{i=0}^{d} h_i^* \binom{k+d-i}{d} = \sum_{i=0}^{d} f_i^* \binom{k-1}{i}$$

Δ_i^d = d-dimensional unimodular simplex with i open facets

$$L_{\Delta_i^d}(k) = \binom{k+d-i}{d} \qquad\qquad L_{\Delta_{i+1}^i}(k) = \binom{k-1}{i}$$

Let K be an integral unimodular d-dimensional simplicial complex. Then

- $h(K) = h^*(L_K, d)$, if K is a shellable ball,
- $f(K) = f^*(L_K)$.



$h^*(L_{\diamond}, 2) = (1,2,1)$

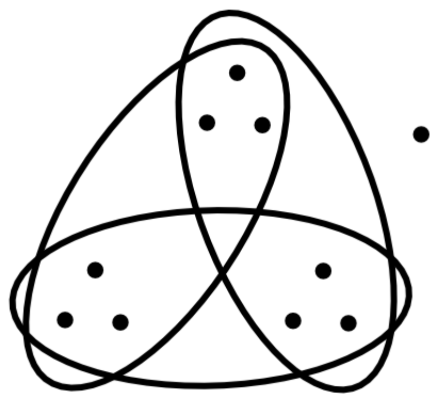
$f^*(L_{\diamond}) = (5,8,4)$

Non-negativity of h^*

Theorem (Stanley '80) If P is a d-dim integral polytope, then

$$0 \le h_i^*(L_P, d) \in \mathbb{Z} \quad \text{for all } i$$

Observation Not true for hypergraph coloring complexes!



$$\begin{aligned} h^*(\Delta_G, 3) &= 3 \cdot h^*(S^3, 3) - 2 \cdot h^*(S^0, 3) \\ &= 3 \cdot (0, 30, 60, 30) - 2 \cdot (2, -6, 6, -2) \\ &= (-4, 102, 168, 94) \end{aligned}$$

Observation $0 \le f_i^*(\Delta_G) \in \mathbb{Z}$ for all i for all hypergraphs G

Is this true for *all* integral simplicial complexes, even if they are not unimodular and not convex?

Non-negativity of f^*

Theorem (B '12)

There is a counting interpretation for the Ehrhart f^*-vector of any simplex.

partial polytopal complex = disjoint union of relatively open polytopes

Corollary (B '12)

$p(k) = L_X(k)$ for some integral partial polytopal complex X

if and only if

$0 \leq f_i^* \in \mathbb{Z}$ for all i.

In particular, integral polytopal complexes have a non-negative integral f^*-vector, even if they do not have unimodular triangulation and they are not convex.

Partitioning Cones

$v_1, \ldots, v_d \in \mathbb{Z}^d$ linearly independent

$x \in \operatorname{relint}(\operatorname{cone}_{\mathbb{R}}(v_1, \ldots, v_d)) \cap \mathbb{Z}^d$

with $x = \sum_i \lambda_i v_i$

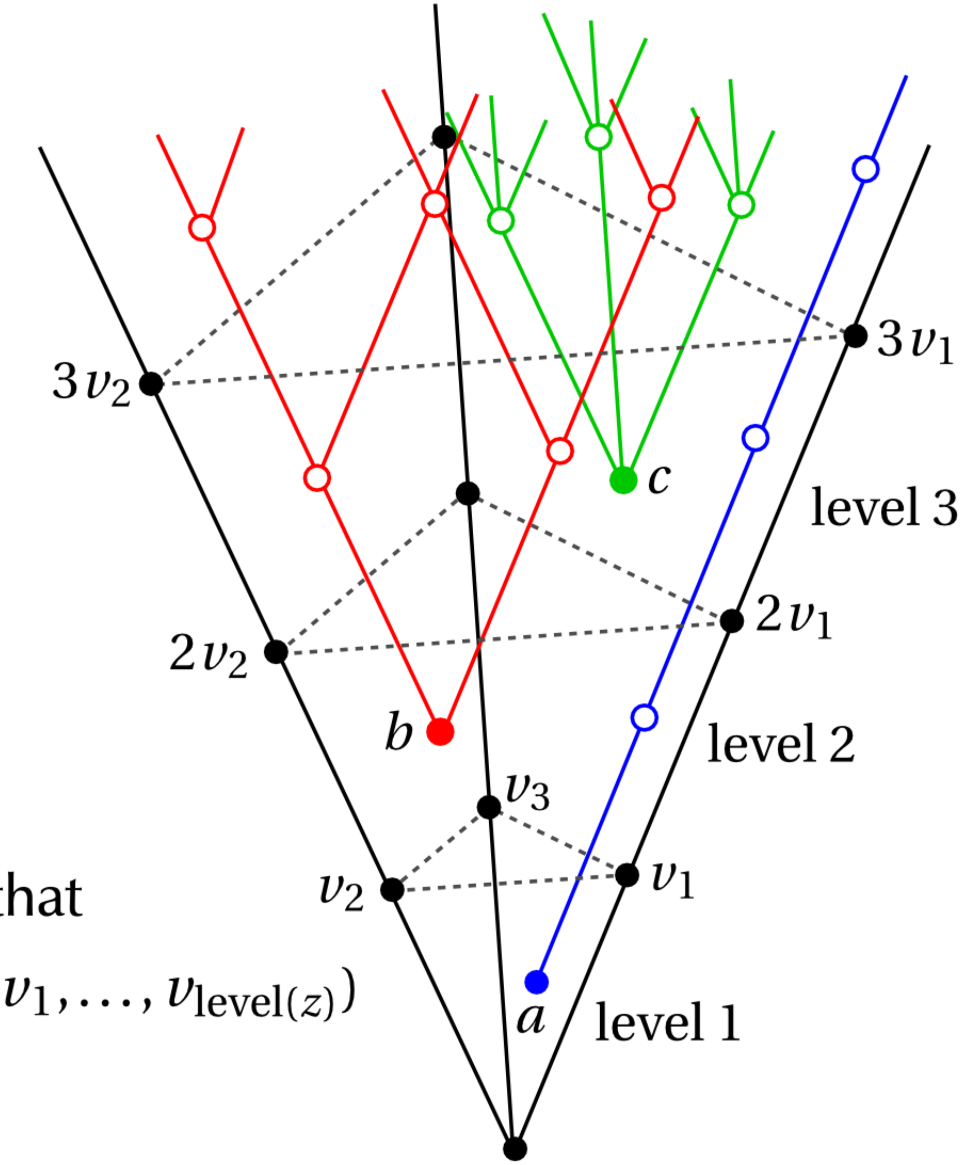
level$(x) :=$ the integer k such that

$$k - 1 < \sum_i \lambda_i \le k$$

x is **atomic** if there does *not* exist

$z \in \operatorname{relint}(\operatorname{cone}_{\mathbb{R}}(v_1, \ldots, v_d)) \cap \mathbb{Z}^d$ such that

level$(z) <$ level(x) and $x \in z + \operatorname{cone}_{\mathbb{Z}}(v_1, \ldots, v_{\operatorname{level}(z)})$



Theorem (B '12)

$$\operatorname{relint}(\operatorname{cone}_{\mathbb{R}}(v_1, \ldots, v_d)) \cap \mathbb{Z}^d = \dot{\bigcup_{z \text{ atomic}}} z + \operatorname{cone}_{\mathbb{Z}}(v_1, \ldots, v_{\operatorname{level}(z)})$$

$f_i^* = \#$ atomic integer points at level $i+1$ in the cone over $\Delta \times \{1\}$.