Multiplier Ideals of Certain Binomial Ideals

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The definition of multiplier ideals

Definition

Let X be a smooth variety X over $\mathbb{C}$ and let $\mathcal{I}$ be a nontrivial coherent ideal sheaf on X. Choose a log resolution $\pi : Y \to X$ of $\mathcal{I}$. Let $\mathcal{F}$ be the normal crossings divisor such that $\mathcal{I} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-\mathcal{F})$. Then for all positive real numbers λ, the **multiplier ideal sheaf of $\mathcal{I}^\lambda$ on X** is

$$\begin{aligned}\mathcal{J}(\mathcal{I}^\lambda) &= \pi_*\mathcal{O}_Y\left(K_{Y/X} - \lfloor \lambda\mathcal{F} \rfloor\right) \\ &= \bigcap_E \left\{ f \in \mathcal{O}_X \mid \operatorname{ord}_E(f) \geq \lfloor \operatorname{ord}_E(I)\lambda - \operatorname{ord}_E(J_{\mathcal{O}_E/\mathcal{O}_X}) \rfloor \right\}\end{aligned}$$

where the intersection ranges over the exceptional divisors E and $J_{\mathcal{O}_E/\mathcal{O}_X}$ is the Jacobian ideal of $\mathcal{O}_E$ over $\mathcal{O}_X$.

Howald's theorem

Theorem (Howald 2001)

Let $X = \mathbb{A}^n$. If $I \subset \mathbb{C}[x_1, x_2, \ldots, x_n] = \mathbb{C}[\mathbf{x}]$ is a monomial ideal with Newton polyhedron $\mathrm{Newt}(I)$*, then, for all $\lambda > 0$,*

$$\mathcal{J}(I^\lambda) = (\mathbf{x}^\mathbf{v} \in \mathbb{C}[\mathbf{x}] \mid \mathbf{v} + \mathbf{1} \in \text{ the interior of } \lambda \, \mathrm{Newt}(I)).$$

Note that the fan of the normalized blowup of I is the normal fan to its Newton polyhedron $\mathrm{Newt}(I)$. So, Howald's formula tells us that, for monomial ideals,

$$\mathcal{J}(I^\lambda) = \bigcap_\nu \{ f \in \mathbb{C}[\mathbf{x}] \mid \nu(f) \geq \lfloor \nu(I)\lambda - \nu(J_{R_\nu/\mathbb{C}[\mathbf{x}]}) \rfloor \}$$

where the intersection ranges over the Rees valuations of I.

Notation & setup

We assume the standard notation of toric geometry as found in Fulton's book.

Let S_τ be a normal affine semigroup and let $\varphi : S_\tau \twoheadrightarrow \Gamma$ be a surjective homomorphism of affine semigroups. Let $Z^\Gamma \subset X_\tau$ be the corresponding inclusion of (not necessarily normal) affine toric varieties. Let $\sigma = \tau \cap (\ker(\varphi \otimes \mathbb{R}))^\perp$, let X_{σ,N_σ} be the toric variety associated to the cone σ and the lattice $N_\sigma = N \cap \mathbb{R}\sigma$.

[We assume Γ has a trivial unit group. We also assume Z^Γ is not contained in a smooth toric hypersurface of X_τ. We can use adjunction/inversion of adjuntion to reduce to this case when computing multiplier ideals.]

González Pérez & Teissier's theorem on partial embedded resolutions

Theorem (González Pérez & Teissier 2002)

If Z^Γ is not contained in a smooth toric hypersurface of X_τ and Σ is any subdivision of τ containing σ, then:

1. *The strict transform Z^Γ_Σ of Z^Γ via the toric morphism $\pi_\Sigma : X_\Sigma \to X_\tau$ is contained in the open affine subvariety $U_\sigma \subset X_\Sigma$. It is isomorhic to X_{σ,N_σ}. And, the restriction $\pi_\Sigma|_{Z^\Gamma_\Sigma} : Z^\Gamma_\Sigma \to Z^\Gamma$ is the normalization map.*
2. *If Σ' is any regular subdivision of Σ, then $\pi_{\Sigma'} : X_{\Sigma'} \to X_\tau$ is an embedded resolution of $Z^\Gamma \subset X_\tau$.*

In general, $\dim \sigma = \dim Z^\Gamma$ and the open set $U_\sigma \cong T \times X_{\sigma,N_\sigma}$ where T is a torus of complementary dimension. Moreover, Z^Γ_Σ meets the torus at the unit $\mathbf{1} \in T$.

Two examples

Example (Hypersurfaces in $\mathbb{A}^n$)

When Z^Γ is a hypersurface in $\mathbb{A}^n$, $\ker(\varphi \otimes \mathbb{R})$ is a hyperplane in $N_\mathbb{R}$. So, we can take Σ to be the fan obtained by slicing the positive orthant along that hyperplane. This Σ is the fan of the normalized blowup of the term ideal of the ideal I_{Z^Γ}.

Example (Curves in $\mathbb{A}^n$)

Let $\varphi : \mathbb{N}^n \to \Gamma$ be given by the matrix $\begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$. Then $Z^\Gamma \cong \operatorname{Spec} \mathbb{C}[t^{a_1}, t^{a_2}, \dots, , t^{a_n}]$, $X_\tau = \mathbb{A}^n$, $\sigma = \mathbb{R}_{\geq 0} \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$, $N_\sigma = \mathbb{Z} \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$, $X_{\sigma, N_\sigma} \cong \mathbb{A}^1$, we can take Σ to be the stellar subdivision along the ray σ. Here, X_Σ is the blowup of $\overline{(x_1^{A/a_1}, x_2^{A/a_2}, \dots, x_n^{A/a_n})}$ where A is any common multiple of the a_is.

A monomial space curve formula

Theorem (—)

Let $C = \operatorname{Spec}\mathbb{C}[t^a, t^b, t^c] \subset \mathbb{A}^3 = \operatorname{Spec}\mathbb{C}[x, y, z]$ *and assume none of* a, b *and* c *are contained in the numerical semigroup generated by the other two. Let* $I = I_C$ *and let* $\{f_1, f_2, \ldots\}$ *be a minimal binomial generating set for* I *written in nondecreasing degree when* x, y *and* z *have degrees* a, b *and* c *respectively. Then, we need only the* I*-adic valuation and the valuation* ν *to compute the multiplier ideals, where* ν *is that valuation with generating sequence* $x \mapsto a$, $y \mapsto b$, $z \mapsto c$ *and* $f_1 \mapsto \deg(f_2)$.

$$\mathcal{J}(I^\lambda) = I^{(\lfloor \lambda - 1 \rfloor)} \cap \{f \in R \mid \nu(f) \geq \lfloor \nu(I)\lambda - \nu(J_{R_\nu/R}) \rfloor\}$$

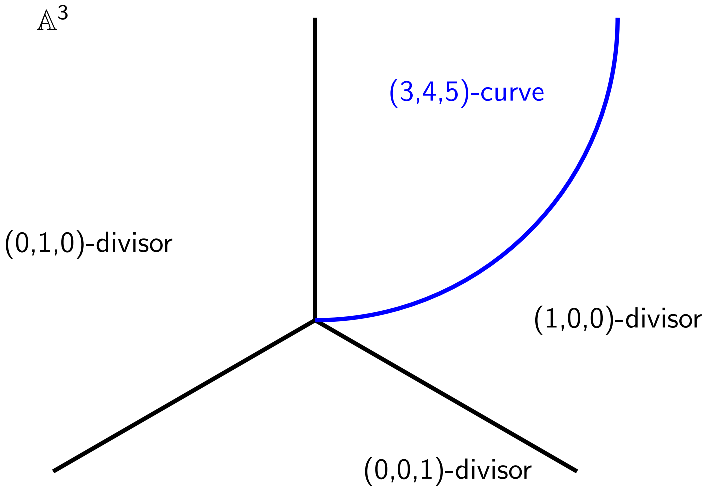
where $\nu(I) = \deg(f_2)$ *and* $\nu(J_{R_\nu/R}) = a + b + c - 1 + \deg(f_2) - \deg(f_1)$.

The (3,4,5)-curve

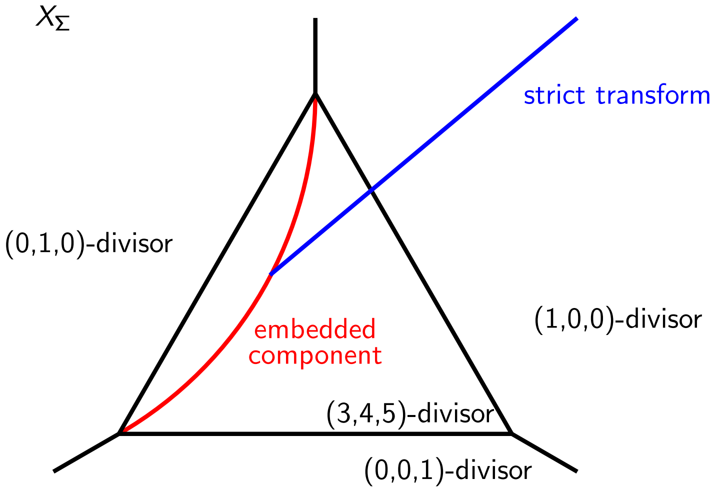
Example

Let $C = \operatorname{Spec} \mathbb{C}[t^3, t^4, t^5] \subset \mathbb{A}^3 = \operatorname{Spec} \mathbb{C}[x, y, z]$. So, $I = (y^2 - xz, x^3 - yz, z^2 - x^2y)$. Let $\mathfrak{a} = \overline{(x^{20}, y^{15}, z^{12})}$. We start with the partial embedded resolution of C to get X_Σ. That is, we blow up $\mathfrak{a}$, or equivalently, we subdivide along $\begin{bmatrix} 3 & 4 & 5 \end{bmatrix}$. We obtain a toric variety such that the strict transform of the curve is smooth and contained in the smooth locus. However, there is an embedded component supported on the intersection of the exceptional divisor and the strict transform of $V(y^2 - xz)$.

The partial embedded resolution of the (3,4,5)-curve



The partial embedded resolution of the (3,4,5)-curve

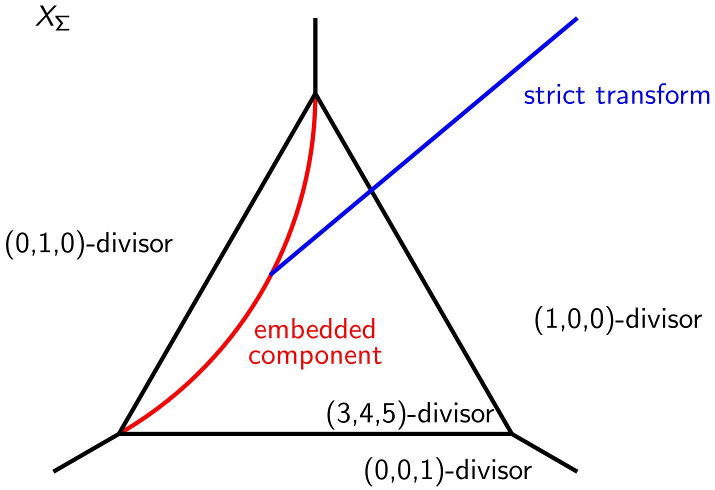


After the first blowup

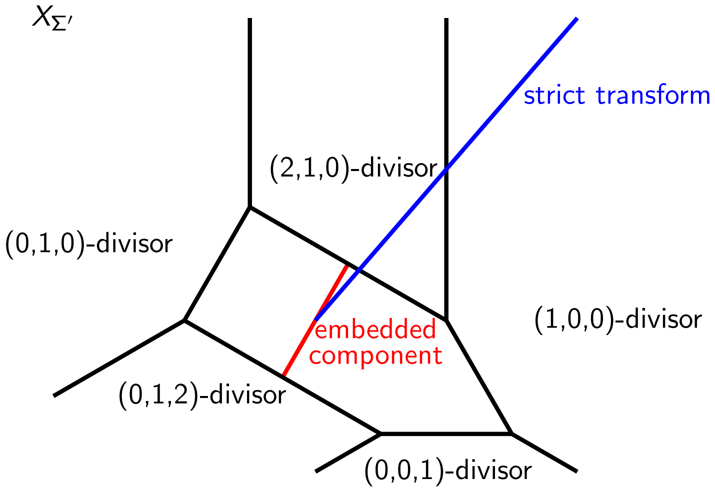
Example (The (3,4,5)-curve, continued)

On the open affine U_σ, $A_\sigma = \mathcal{O}_{U_\sigma} = \mathbb{C}\left[\frac{y}{x}, \left(\frac{y^2}{xz}\right)^{\pm 1}, \left(\frac{x^3}{yz}\right)^{\pm 1}\right]$ and $I = \left(\left(\frac{y}{x}\right)^8\left(\frac{y^2}{xz} - 1\right), \left(\frac{y}{x}\right)^9\left(\frac{x^3}{yz} - 1\right)\right)$. In particular, I is monomial in $\frac{y}{x}$, $\frac{y^2}{xz} - 1$ and $\frac{x^3}{yz} - 1$. Unfortunately, we don't have this good behavior at the other two points on the embedded component. So, we take the normalized blowup of the total transform of (y^2, xz). This creates a new toric variety $X_{\Sigma'}$ that is a partial embedded resolution of both C and $V(y^2 - xz)$. That is, any regular subdivision of Σ' yields an embedded resolution of both C and $V(y^2 - xz)$.

The second blowup



The second blowup



Monomialization (toroidalization) has been acheived

Example (The (3,4,5)-curve, continued)

On the 2-cone $\mathbb{R}^2_{\geq 0}\begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 5 \end{bmatrix}$, the coordinate ring is $\mathbb{C}\left[z, y, \frac{y^2}{z}, \frac{y^3}{z^2}, \frac{y^4}{z^3}, \frac{y^5}{z^4}, \left(\frac{xz}{y^2}\right)^{\pm 1}\right]$. And, the point of interest has maximal ideal $\left(z, y, \frac{y^2}{z}, \frac{y^3}{z^2}, \frac{y^4}{z^3}, \frac{y^5}{z^4}, \frac{xz}{y^2} - 1\right)$. On any affine chart around this point such that $\frac{x^3}{yz} - 1$ does not vanish, $I = \left(y^2\left(\frac{xz}{y^2} - 1\right), yz\right)$, is monomial in $\left\{z, y, \frac{y^2}{z}, \frac{y^3}{z^2}, \frac{y^4}{z^3}, \frac{y^5}{z^4}, \frac{xz}{y^2} - 1\right\}$.

Behavior at the other point not in U_σ is similar.

The formula for the multiplier ideals of the (3,4,5)-curve

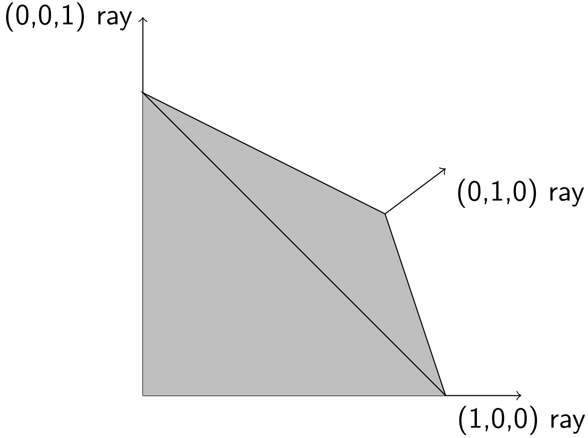
Example (The (3,4,5)-curve, continued)

Now that the total transform is monomial everywhere, it suffices to look to the Rees valuations of the total transform. For that, we need only consider $\left(\left(\frac{y}{x}\right)^8\left(\frac{y^2}{xz}-1\right),\left(\frac{y}{x}\right)^9\left(\frac{x^3}{yz}-1\right)\right)$ and Howald's formula since the generic points of the components of the total transform of I are in U_σ. The valuations we get are the I-adic one $\frac{y^2}{xz}-1, \frac{x^3}{yz}-1 \mapsto 1$ and $\nu : \frac{y}{x}, \frac{y^2}{xz}-1 \mapsto 1$. So,

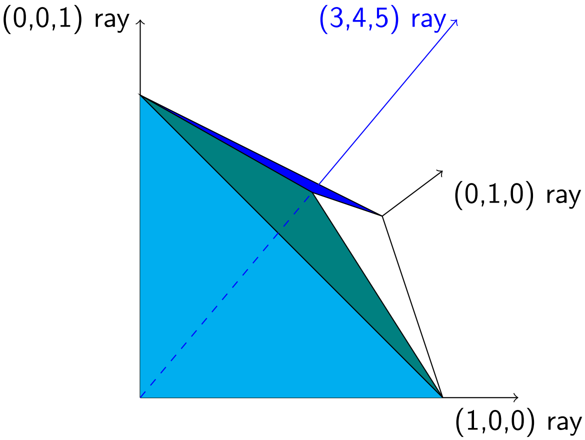
$$\mathcal{J}(I^\lambda) = I^{(\lfloor\lambda-1\rfloor)} \cap \{f \in \mathbb{C}[x,y,z] \mid \nu(f) \geq \lfloor 9\lambda - 12\rfloor\}$$

where ν is given by the generating sequence $x \mapsto 3$, $y \mapsto 4$, $z \mapsto 5$ & $y^2 - xz \mapsto 9$.

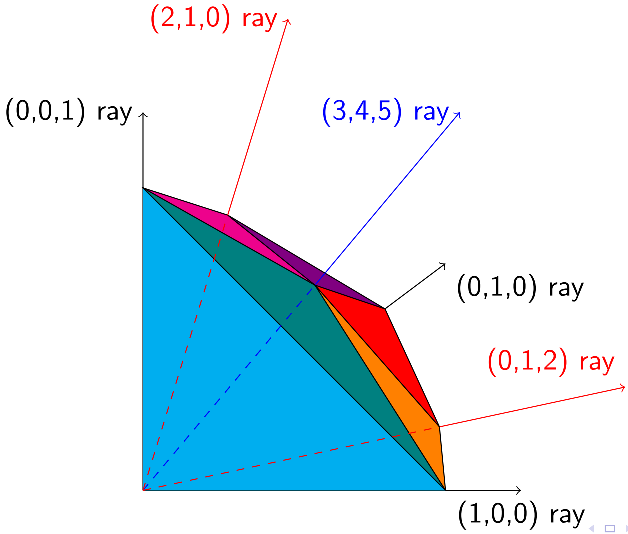
The partial embedded resolution of the (3,4,5)-curve and $V(y^2 - xz)$



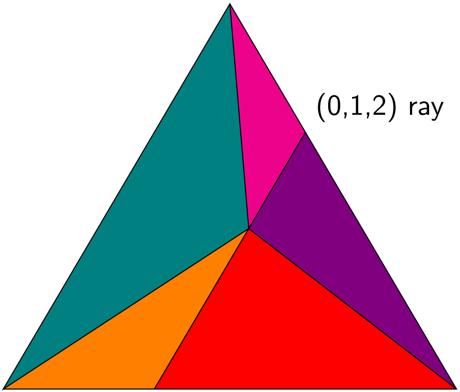
The partial embedded resolution of the (3,4,5)-curve and $V(y^2 - xz)$



The partial embedded resolution of the (3,4,5)-curve and $V(y^2 - xz)$



A cross-section of the fan



Thank you