Syzygies and singularities of tensor product surfaces

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Setup:

$R := \mathrm{k}[s, t, u, v]$ bigraded ring over an algebraically closed field k

$$\deg s, t = (1, 0), \quad \deg u, v = (0, 1)$$

Therefore,

$$R_{2,1} = \mathrm{Span}_{\mathrm{k}}\{s^2u, stu, t^2u, s^2v, stv, t^2v\}$$

Let $p_i \in R_{2,1}$ linearly independent, no common zeros on $\mathbb{P}^1 \times \mathbb{P}^1$

$$U := \mathrm{Span}_{\mathrm{k}}\{p_0, p_1, p_2, p_3\} \subset R_{2,1}$$

Consider the ideal $I_U = \langle p_0, p_1, p_2, p_3 \rangle \subset R$

Note that p_i's being BPF on $\mathbb{P}^1 \times \mathbb{P}^1 \iff \sqrt{I_U} = (s, t) \cap (u, v)$.

Since p_i's define regular map

$$\phi_U : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$
$$((s:t),(u:v)) \longmapsto (p_0 : p_1 : p_2 : p_3)$$

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Goals: To classify and compute

- minimal bigraded free resolution of I_U
- Hilbert function of I_U
- associated primes and primary decomposition of I_U
- implicit equation of X_U (important in geometric modeling)
- singularities of X_U

History:

The study of tensor product surfaces goes back to the last century:

- **Salmon (1882)** [8], **Edge (1931)** [7] the theory of ruled surfaces.
- **Degan (1999)** [3] studied surfaces in $\mathbb{P}^3$ of bidegree $(2, 1)$ with base points.
- **Zube (1998, 2003)** [2], [3] describes the possibilities for the singular locus.
- **Elkadi-Galligo-Lê (2004)** [8] give a geometric description of the image and singular locus for a generic U.
- **Galligo-Lê (2008)** [9] examine the nongeneric case.

Cox, Dickenstein and Schenck (2007) [1]:

Study the bigraded commutative algebra of a 3-dim BPF subspace

$$U = \mathrm{Span}_k\{p_0, p_1, p_2\} \subset R_{2,1}$$

Showing that there are two numerical types of possible bigraded minimal free resolution of I_U corresponding to:

1. generic case: $\mathbb{P}(U) \cap \Sigma_{2,1}$ is a finite set of points
2. non-generic case: $\mathbb{P}(U) \cap \Sigma_{2,1}$ is infinite (in fact, a smooth conic)

For $p \in R_{2,1}$ write

$$p = as^2u + bstu + ct^2u + ds^2v + estv + ft^2v$$

Identify $p \neq 0$ with a point $(a : b : c : d : e : f) \in \mathbb{P}^5$.

Since p_i's are LI, $\mathbb{P}(U) := \mathrm{Span}_k\{p_0, p_1, p_2, p_3\} \subset \mathbb{P}^5$.

The **Segre** map $\sigma_{2,1} : \mathbb{P}^2 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^5$

$$((x_0 : x_1 : x_2), (y_0 : y_1)) \longmapsto (x_0y_0 : x_1y_0 : x_2y_0 : x_0y_1 : x_1y_1 : x_2y_1)$$

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Let $\Sigma_{2,1} = \mathrm{im}(\sigma_{2,1}) \subset \mathbb{P}^5$

Generic case:

$$0 \leftarrow I_U \leftarrow (-2,-1)^3 \longleftarrow \begin{matrix}(-6,-1)\\ \oplus \\ (-4,-2)^3 \\ \oplus \\ (-3,-3)^2\end{matrix} \longleftarrow \begin{matrix}(-4,-3)^3 \\ \oplus \\ (-6,-2)^2\end{matrix} \longleftarrow (-6,-3) \leftarrow 0$$

Non-generic case:

$$0 \leftarrow I_U \leftarrow (-2,-1)^3 \longleftarrow \begin{matrix}(-6,-1)\\ \oplus \\ (-4,-2)^3 \\ \oplus \\ (-2,-3)\end{matrix} \longleftarrow \begin{matrix}(-4,-3)^2 \\ \oplus \\ (-6,-2)^2\end{matrix} \longleftarrow (-6,-3) \leftarrow 0$$

Our Results:
Extend the work of CDS to the setting of a 4-dim subspace of $R_{2,1}$

A Key Difference:
For a BPF 3-dim subspace U, there can never be a linear syzygy

Not the case for 4-dim subspaces! For example,

$$U = \mathrm{Span}_{\mathsf{k}}\{s^2u, s^2v, t^2u, t^2v + stv\}$$

is 4-dim and BPF, but has a **unique** linear syzygy of bidegree $(0, 1)$:

$$v \cdot (s^2u) - u \cdot (s^2v) = 0.$$

Observation: linear syzygies give lots of structure!

The following tables describe our classification:

We prove there are six numerical types of possible bigraded min free res.

Type	Lin. Syz.	Emb. Pri.	Sing. Loc.	Example
1	none	$\mathfrak{m}$	T	$s^2u+stv, t^2u, s^2v+stu, t^2v+stv$
2	none	$\mathfrak{m}, P_1$	$C \cup L_1$	$s^2u, t^2u, s^2v+stu, t^2v+stv$
3	1 type $(1,0)$	$\mathfrak{m}$	L_1	$s^2u+stv, t^2u, s^2v, t^2v+stu$
4	1 type $(1,0)$	$\mathfrak{m}, P_1$	L_1	$stv, t^2v, s^2v-t^2u, s^2u$
5a	1 type $(0,1)$	P_1, P_2	$L_1 \cup L_2 \cup L_3$	$s^2u, s^2v, t^2u, t^2v+stv$
5b	1 type $(0,1)$	P_1	$L_1 \cup L_2$	$s^2u, s^2v, t^2u, t^2v+stu$
6	2 type $(0,1)$	none	$\emptyset$	s^2u, s^2v, t^2u, t^2v

- Type refers to the graded Betti numbers of the bigraded minimal free resolution for I_U
- Sing. Loc. $=$ codimension one singular locus of X_U, T $=$ a twisted cubic curve, C $=$ a smooth plane conic, L_i $=$ a line
- $\mathfrak{m} = \langle s, t, u, v\rangle$ or $P_i = \langle l_i, s, t\rangle$ where l_i is a linear form in u, v

Classification of the minimal bigraded free resolutions of I_U:

Type	Bigraded Minimal Free Resolution of I_U
1	$0 \leftarrow I_U \leftarrow (-2,-1)^4 \longleftarrow \begin{matrix}(-2,-4)\\ \oplus \\ (-3,-2)^4 \\ \oplus \\ (-4,-1)^2\end{matrix} \longleftarrow \begin{matrix}(-3,-4)^2 \\ \oplus \\ (-4,-2)^3\end{matrix} \longleftarrow (-4,-4) \leftarrow 0$
2	$0 \leftarrow I_U \leftarrow (-2,-1)^4 \longleftarrow \begin{matrix}(-2,-3)\\ \oplus \\ (-3,-2)^4 \\ \oplus \\ (-4,-1)^2\end{matrix} \longleftarrow \begin{matrix}(-3,-3)^2 \\ \oplus \\ (-4,-2)^3\end{matrix} \longleftarrow (-4,-3) \leftarrow 0$
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(i,j) denotes the rank one free module $R(i,j)$

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5	$0 \leftarrow I_U \leftarrow (-2,-1)^4 \longleftarrow \begin{matrix} (-2,-2) \\ \oplus \\ (-3,-2)^2 \\ \oplus \\ (-4,-1)^2 \end{matrix} \longleftarrow (-4,-2)^2 \leftarrow 0$
6	$0 \leftarrow I_U \leftarrow (-2,-1)^4 \longleftarrow \begin{matrix} (-2,-2)^2 \\ \oplus \\ (-4,-1)^2 \end{matrix} \longleftarrow (-4,-2) \leftarrow 0$

For Types 3, 4, 5 and 6, we determine all the differentials

Example: Let

$$U = \mathrm{Span}_k\{s^2u, s^2v, t^2u, t^2v + stv\}$$

There is a **unique** linear syzygy of bidegree $(0, 1)$:

$$v \cdot (s^2u) - u \cdot (s^2v) = 0.$$

- U is of type 5a in our classification:

Type	Lin. Syz.	Emb. Pri.	Sing. Loc.	Example
5a	1 type $(0, 1)$	P_1, P_2	$L_1 \cup L_2 \cup L_3$	$s^2u, s^2v, t^2u, t^2v + stv$

- The embedded primes of I_U are

$$\langle s, t, u\rangle, \ \langle s, t, v\rangle$$

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$$\langle s, t, u\rangle, \ \langle s, t, v\rangle$$

- The bigraded Betti numbers of I_U are

$$0 \leftarrow I_U \leftarrow R(-2,-1)^4 \leftarrow \begin{array}{c} R(-2,-2) \\ \oplus \\ R(-3,-2)^2 \\ \oplus \\ R(-4,-1)^2 \end{array} \leftarrow R(-4,-2)^2 \leftarrow 0$$

- Using approximation complexes, the image of ϕ_U is the hypersurface

$$X_U = \mathbf{V}(x_0x_1^2x_2 - x_1^2x_2^2 + 2x_0x_1x_2x_3 - x_0^2x_3^2).$$

- The reduced codimension one singular locus of X_U is

$$\mathbf{V}(x_0,x_2) \cup \mathbf{V}(x_1,x_3) \cup \mathbf{V}(x_0,x_1).$$

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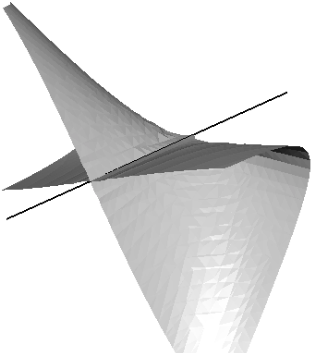

If $x_0 \neq 0$, then $x_1^2 x_2 - x_1^2 x_2^2 + 2x_1 x_2 x_3 - x_3^2 = 0$

x_2-axis is a reduced codimension one singular locus

Remark: Our Classification Table indicates if U has a linear syzygy, then the codimension one singular locus of X_U is either empty or a union of lines.

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Implicit Equation of X_U:

Botbol-Busé-Chardin-Dickenstein-Dohm-Jouanolou [2], [4], [5]

If U is BPF, then the implicit equation for X_U may be extracted from the differentials of a certain complex: the "determinant" of the complex $\mathcal{Z}_\bullet^\mu$ is a power of the implicit equation for X_U.

Gelfand-Kapranov-Zelevinsky [1]

This "determinant" can be obtained as the **gcd** of the maximal minors of the first differential of $\mathcal{Z}_\bullet^\mu$.

Our Work: pure syzygies (bidegree $(0,*)$ or $(*,0)$) can greatly simplify the process of finding the implicit equation.

Theorem

If U is BPF, then the implicit equation for X_U is determinantal, obtained as a specific 4×4 minor of the first differential of the approximation complex $\mathcal{Z}^2_\bullet$.

Approximation Complexes [4], [5], [6]:

$$I_U = \langle p_0, \dots, p_3 \rangle \subseteq R = \mathrm{k}[s, t, u, v]$$

Let $S := R[x_0, \dots, x_3]$, regard p_i as elements of S.

Consider two Koszul complexes

$$K_\bullet(\underline{x}; S): \quad 0 \longrightarrow S \xrightarrow{d_4^x} S^4 \xrightarrow{d_3^x} S^6 \xrightarrow{d_2^x} S^4 \xrightarrow{d_1^x} S \longrightarrow 0$$

$$K_\bullet(\underline{p}; S): \quad 0 \longrightarrow S \xrightarrow{d_4^p} S^4 \xrightarrow{d_3^p} S^6 \xrightarrow{d_2^p} S^4 \xrightarrow{d_1^p} S \longrightarrow 0$$

The approximation complex $\mathcal{Z}_\bullet$

Modules $\mathcal{Z}_n := Z_n(\underline{p}; S) = \ker(d_n^p) \subseteq K_n(\underline{p}; S)$

Differentials $d_\bullet^x$

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Example: Consider $\{s^2u, s^2v, t^2u, t^2v + stv\}$.

$$\mathcal{Z}_\bullet : \qquad \cdots \longrightarrow Z_1(\underline{p}; S) \xrightarrow{d_1^x} Z_0(\underline{p}; S) \longrightarrow 0$$

$$\begin{aligned}
\mathcal{Z}_0 &= S \\
\mathcal{Z}_1 &= \mathrm{Span}_S \left\{ \begin{bmatrix} -v \\ u \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -t^2 \\ 0 \\ s^2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -st - t^2 \\ 0 \\ s^2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -sv - tv \\ tu \end{bmatrix}, \begin{bmatrix} -tv \\ 0 \\ -sv \\ su \end{bmatrix} \right\} \\
&= \mathrm{Span}_S\{-ve_0 + ue_1, -t^2e_0 + s^2e_2, \ \ldots \} \subset S^4
\end{aligned}$$

For example, $d_1^x(-ve_0 + ue_1) = -vx_0 + ux_1$

For $\mu \in \mathbb{N}$, let $\mathcal{Z}_\bullet^\mu$ be the degree μ strand in s, t, u, v.

Thus, a basis for $\mathcal{Z}_1^{\mu=2}$ is

$$\{\{s, t, u, v\} \cdot (ue_1 - ve_0), -t^2 e_0 + s^2 e_2, \ldots\}$$

And a basis for $\mathcal{Z}_0^{\mu=2} = R_2$ is

$$\{s^2, st, t^2, su, sv, tu, tv, u^2, uv, v^2\}$$

Thus $d_1^X : \mathcal{Z}_1^2 \longrightarrow \mathcal{Z}_0^2$ is a 10×8 matrix

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Thus $d_1^x : \mathcal{Z}_1^2 \longrightarrow \mathcal{Z}_0^2$ is a 10×8 matrix

$$\begin{array}{c} s^2 \\ st \\ t^2 \\ su \\ sv \\ tu \\ tv \\ u^2 \\ uv \\ v^2 \end{array}\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & x_2 & -x_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & x_1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -x_0 & x_1 & \cdot & \cdot \\ x_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -x_3 \\ -x_0 & \cdot & \cdot & \cdot & \cdot & \cdot & -x_2 & \cdot \\ \cdot & x_1 & \cdot & \cdot & \cdot & \cdot & x_3 & x_1 + x_3 \\ \cdot & -x_0 & \cdot & \cdot & \cdot & \cdot & -x_2 & -x_2 \\ \cdot & \cdot & x_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -x_0 & x_1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -x_0 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Any maximal minor must involve exactly two of the last three rows:
If it involves only one row, then there is an entire zero column, or two LD columns.
It cannot involve all three rows, the submatrix consisting of the last three rows only has rank two.

Thus, gx_1^2, gx_0x_1, gx_0^2 appear among the maximal minors, where g is a 6×6 minor involving the top seven rows of the matrix.

Deleting the last 3 rows we obtain

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & x_2 & -x_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & x_1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -x_0 & x_1 & \cdot & \cdot \\ x_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -x_3 \\ -x_0 & \cdot & \cdot & \cdot & \cdot & \cdot & -x_2 & \cdot \\ \cdot & x_1 & \cdot & \cdot & \cdot & \cdot & x_3 & x_1 + x_3 \\ \cdot & -x_0 & \cdot & \cdot & \cdot & \cdot & -x_2 & -x_2 \end{bmatrix}$$

Remove the zero columns.

We obtain a 7×6 matrix:

$$\begin{bmatrix} \cdot & \cdot & x_2 & -x_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & x_1 & \cdot & \cdot \\ \cdot & \cdot & -x_0 & x_1 & \cdot & \cdot \\ x_1 & \cdot & \cdot & \cdot & \cdot & -x_3 \\ -x_0 & \cdot & \cdot & \cdot & -x_2 & \cdot \\ \cdot & x_1 & \cdot & \cdot & x_3 & x_1 + x_3 \\ \cdot & -x_0 & \cdot & \cdot & -x_2 & -x_2 \end{bmatrix}$$

With a similar reasoning, we can delete the first 3 rows.

$$\begin{bmatrix} x_1 & \cdot & \cdot & \cdot & \cdot & -x_3 \\ -x_0 & \cdot & \cdot & \cdot & -x_2 & \cdot \\ \cdot & x_1 & \cdot & \cdot & x_3 & x_1 + x_3 \\ \cdot & -x_0 & \cdot & \cdot & -x_2 & -x_2 \end{bmatrix}$$

Deleting the zero columns, we obtain the implicit equation as a 4×4 minor!

$$X_U = \mathbf{V}(x_0x_1^2x_2 - x_1^2x_2^2 + 2x_0x_1x_2x_3 - x_0^2x_3^2)$$

Lemma

If $p_0, \ldots, p_n \in R$ are **LI** with a linear syzygy of degree $(1,0)$, then after a suitable change of variables

$$\mathrm{Span}_k\{p_0, p_1, p_2, \ldots, p_n\} = \mathrm{Span}_k\{sp, tp, p_2, \ldots, p_n\}$$

for some polynomial p.

Proof: Suppose there exists a linear syzygy of degree $(1,0)$. Then

$$\sum_{i=0}^{n}(a_i s + b_i t)p_i = 0 = s \cdot \sum_{i=0}^{n} a_i p_i + t \cdot \sum_{i=0}^{n} b_i p_i = sf + tg,$$

where $f = \sum_{i=0}^{n} a_i p_i$, $g = \sum_{i=0}^{n} b_i p_i$.
Since $sf + tg = 0$, then $f = tp, g = -sp$ for some $p \in R$.

By linear independence of p_i's, at least one pair of the $a_i s + b_i t$ is LI
WLOG asssume

$$det \begin{bmatrix} a_0 & a_1 \\ b_0 & b_1 \end{bmatrix} \neq 0$$

Then we cans solve for p_0, p_1 in terms of $p_2, \ldots, p_n, sp, tp$.
Hence $\mathrm{Span}_k\{p_0, p_1, p_2, \ldots, p_n\} = \mathrm{Span}_k\{sp, tp, p_2, \ldots, p_n\}$.

Note: If there is a linear syzygy of degree $(0, 1)$, then

$$\mathrm{Span}_k\{p_0, p_1, p_2, \ldots, p_n\} = \mathrm{Span}_k\{up, vp, p_2, \ldots, p_n\}$$

If there is a linear syzygy of bidegree $(1,0)$:

Classification of the resolutions

If U is base point free and $I_U = \langle ps, pt, p_2, p_3 \rangle$ with p irreducible, then I_U has a resolution of Type 3.

Proof Using suitable linear change of variables, we may assume $p = su + tv$, and

$$I_U = \langle s(su+tv), t(su+tv), stL_1, s^2L_1 + t^2L_2 \rangle,$$

where $L_1 = au + v$ and $L_2 = bu + cv$ are LI.

Now let

$$I_W = \langle s(su+tv), t(su+tv), stL_1 \rangle.$$

The minimal free resolution of I_W is

$$0 \longleftarrow I_W \longleftarrow (-2,-1)^3 \xleftarrow{\begin{bmatrix} t & 0 \\ -s & sL_1 \\ 0 & p \end{bmatrix}} (-3,-1) \oplus (-3,-2) \longleftarrow 0$$

To obtain a mapping cone resolution, we need to compute $I_W : p_3$

$$0 \longrightarrow R/I_W : p_3 \xrightarrow{p_3} R/I_W \longrightarrow R/\langle I_W, p_3 \rangle \longrightarrow 0$$

We first find the primary decomposition for I_W.

- If $a = 0$, then $L_1 = v$,

$$\begin{aligned} I_W &= \langle s(su+tv), t(su+tv), stv \rangle \\ &= \langle u, v\rangle \cap \langle s, t\rangle^2 \cap \langle u, t\rangle \cap \langle su+tv, (s,v)^2\rangle \end{aligned}$$

Thus

$$\begin{aligned} I_W : p_3 &= I_W : s^2L_1 + t^2L_2 \\ &= \langle u, t\rangle : s^2L_1 + t^2L_2 \ \cap \langle su+tv, (s,v)^2\rangle : s^2L_1 + t^2L_2 \\ &= \langle u, t\rangle \cap \langle su+tv, (s,v)^2\rangle \\ &= \langle su+tv, uv^2, tv^2, stv, s^2t\rangle \end{aligned}$$

- If $a \neq 0$, rescale u and t by a so $L_1 = u + v$. Then

$$\begin{aligned} I_W &= \langle s(su+tv), t(su+tv), st(u+v)\rangle \\ &= \langle u, v\rangle \cap \langle s, t\rangle^2 \cap \langle v, s\rangle \cap \langle u, t\rangle \cap \langle u+v, s-t\rangle \end{aligned}$$

Thus

$$\begin{aligned} I_W : p_3 &= I_W : s^2L_1 + t^2L_2 \\ &= \langle v, s\rangle : s^2L_1 + t^2L_2 \ \cap \ \langle u, t\rangle : s^2L_1 + t^2L_2 \\ &\quad \cap \langle u+v, s-t\rangle : s^2L_1 + t^2L_2 \\ &= \langle v, s\rangle \cap \langle u, t\rangle \cap \langle u+v, s-t\rangle \\ &= \langle su+tv, uv(u+v), tv(u+v), tv(s-t), st(s-t)\rangle \end{aligned}$$

$$0 \leftarrow I_w : p_3 \longleftarrow \begin{array}{c} (-1,-1) \\ \oplus \\ (0,-3) \\ \oplus \\ (-1,-2) \\ \oplus \\ (-2,-1) \\ \oplus \\ (-3,0) \end{array} \longleftarrow \begin{array}{c} (-1,-3)^2 \\ \oplus \\ (-2,-2)^2 \\ \oplus \\ (-3,-1)^2 \end{array} \longleftarrow \begin{array}{c} (-2,-3) \\ \oplus \\ (-3,-2) \end{array} \leftarrow 0$$

Geometry and the Segre-Veronese variety:

Proposition

If U is BPF, then the ideal I_U

1. has a unique linear syzygy of bidegree $(0,1)$ iff $\mathbb{P}(U) \cap \Sigma_{2,1}$ contains a $\mathbb{P}^1$ fiber of $\Sigma_{2,1}$.
2. has a pair of linear syzygies of bidegree $(0,1)$ iff $\mathbb{P}(U) \cap \Sigma_{2,1} = \Sigma_{1,1}$.
3. has a unique linear syzygy of bidegree $(1,0)$ iff $\mathbb{P}(U) \cap Q$ contains a $\mathbb{P}^1$ fiber of Q.

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