Orientability of real toric varieties and lower bounds for real polynomial systems

AMS MEETING, AKRON

Jenya Soprunova
(with Frank Sottile)

Kent State University

October 21, 2012

Polynomial system

$$\begin{aligned} F_1(t_1, \ldots, t_n) &= 0 \\ F_2(t_1, \ldots, t_n) &= 0 \\ &\ldots \\ F_n(t_1, \ldots, t_n) &= 0 \end{aligned}$$

Bezout Theorem

The number of solutions of a complex generic system is the product of the degrees of $F_1, F_2, \ldots, F_n$.

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For a system with real coefficients let

$$d := \# \text{ complex solutions} \qquad r := \# \text{ real solutions}$$

$$d \, mod 2 \leq r \leq d$$

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Better lower bound?

Kushnirenko's Theorem

Sparse polynomial system:

$$(\star)\ F_1(t_1, \ldots, t_n) \ = \ F_2(t_1, \ldots, t_n) \ = \ \cdots \ = \ F_n(t_1, \ldots, t_n) \ = \ 0$$

$\Delta := \mathrm{Newton\ Polytope}(F_i)$, same for all F_i
Kushnirenko: Number of complex solutions of a generic system in $(\mathbb{C}^\times)^n$ is $n!\mathrm{Vol}(\Delta)$

Kushnirenko's Theorem

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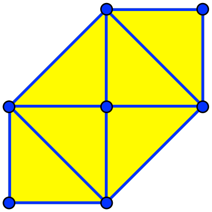
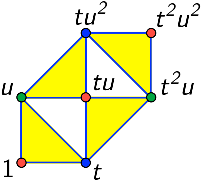


Number of complex solutions $= 2! \cdot 3 = 6$

$=$ number of triangles in a unimodular triangulation T_Δ

Hexagonal Example

Unimodular Balanced Triangulation:



$$c_0(1 + tu + t^2u^2) + c_1(t + tu^2) + c_2(u + t^2u) = 0$$

$$d_0(1 + tu + t^2u^2) + d_1(t + tu^2) + d_2(u + t^2u) = 0$$

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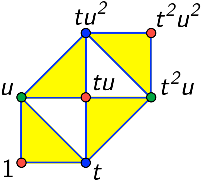

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Signature $\sigma(T_\Delta) := |\#\text{yellow triangles} - \#\text{white triangles}| = 2$

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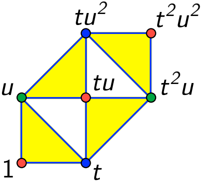

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Theorem (S., Sottile)

Such a system has at least two real solutions.

Projective toric variety

X_Δ – projective toric variety parametrized by the monomials in Δ, that is, X_Δ is the closure of the image of the map

$$\varphi_\Delta : \begin{array}{rcl} (\mathbb{C}^\times)^n & \longrightarrow & \mathbb{P}^\Delta \\ (t_1, t_2, \dots, t_n) & \longmapsto & [t^m \mid m \in \Delta \cap \mathbb{Z}^n] \end{array}$$

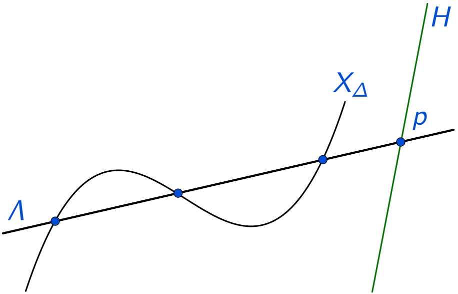
$$\sum_{m \in \Delta \cap \mathbb{Z}^m} c_m t^m = 0 \longleftrightarrow \sum_{m \in \Delta \cap \mathbb{Z}^m} c_m x_m = 0$$

$$\begin{array}{rcccc} \text{System } (\star) & \longleftrightarrow & n \text{ linear equations} & \text{on} & X_\Delta \\ \text{solutions to } (\star) & \longleftrightarrow & \Lambda & \cap & X_\Delta \\ & & \text{codim} = n & & \dim = n \end{array}$$

Projection

$E \subset \Lambda$— real hyperplane disjoint from X_Δ
$H(\simeq \mathbb{P}^n)$ — real linear subspace disjoint from E
π — linear projection with center E

$$\begin{array}{rccl} \pi : & \mathbb{P}^\Delta - E & \longrightarrow & H, \\ & x & \longmapsto & \mathrm{Span}(x, E) \cap H\,. \end{array}$$



Solutions to $(\star)$ are points $X_\Delta \cap \pi^{-1}(p)$, where $p := \pi(\Lambda)$.

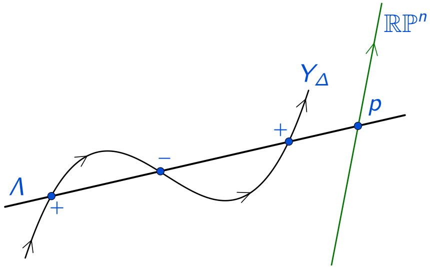
Real Degree

over $\mathbb{R}$:
Set $Y_\Delta := \overline{\varphi_\Delta((\mathbb{R}^\times)^n)}$, $\pi := \pi\big|_{Y_\Delta}$.

$$\pi\colon Y_\Delta \to \mathbb{RP}^n$$

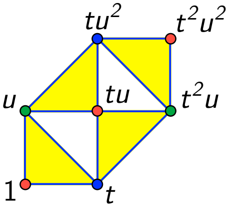
If Y_Δ and $\mathbb{RP}^n$ are oriented, $\#\pi^{-1}(p) \geq \deg \pi$ and

$$\#\text{ real solutions of } (\star) \geq \deg \pi$$



Hexagonal Example

$$7(1 + tu + t^2u^2) - 3(t + tu^2) - 2(u + t^2u) = 0$$
$$3(1 + tu + t^2u^2) + 5(t + tu^2) - 4(u + t^2u) = 0$$



$$\varphi : (t, u) \longmapsto [1 : t : u : tu : t^2u : tu^2 : t^2u^2]$$

Then the system is the linear section of $Y = \overline{\varphi((\mathbb{R}^\times)^2)}$ by

$$7(x_0 + x_3 + x_6) - 3(x_1 + x_5) - 2(x_2 + x_4) = 0$$
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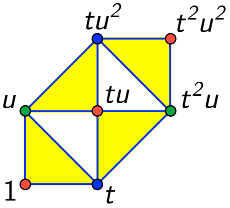

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Further, replace $y_0 = x_0+x_3+x_6$, $y_1 = x_1+x_5$, and $y_2 = x_2+x_4$:

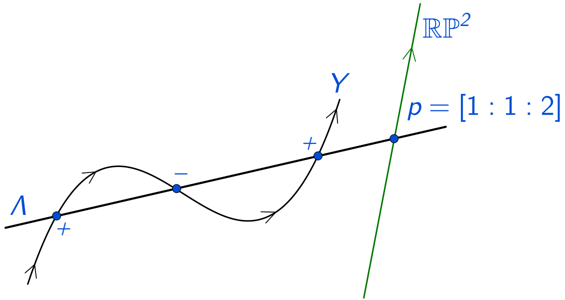
$$7y_0-3y_1-2y_2 = 0$$
$$3y_0+5y_1-4y_2 = 0$$

Then $p = [y_0 : y_1 : y_2] = [1 : 1 : 2]$.

Hexagonal Example: real degree

Real solutions are the preimages in Y of $p = [1:1:2]$ under

$$\pi : \begin{array}{ccc} \mathbb{RP}^6 & --\to & \mathbb{RP}^2 = \{[y_0 : y_1 : y_2]\} \\ [x_0 : \cdots : x_6] & \longmapsto & [x_0 + x_3 + x_6 : x_1 + x_5 : x_2 + x_4] \end{array}$$



$$\pi : Y \to \mathbb{RP}^2$$

If Y and $\mathbb{RP}^2$ were oriented, $\#\pi^{-1}(p) \geq \deg \pi$ and

$$\text{Number of real solutions} \geq \deg \pi$$

Kronecker

$$\begin{array}{ccccc} Y_\Delta^+ & \subset & S^\Delta & \overset{\pi^+}{- - - \to} & S^n \\ \downarrow & & \downarrow & & \downarrow \\ Y_\Delta & \subset & \mathbb{RP}^\Delta & \overset{\pi}{- - - \to} & \mathbb{RP}^n \end{array}$$

Proposition

If Y_Δ^+ is orientable, the number of real solutions of $(\star)$ is bounded below by $\deg \pi^+$.

Questions

- When is the real toric variety Y_Δ orientable?
- When is its double cover in the sphere Y_Δ^+ orientable?
- How to compute $\deg \pi$ (at least for some systems)?

Orientability

Let Y be an abstract real toric variety defined by a fan Σ in $\mathbb{Z}^n$.

Theorem (S., Sottile)

The smooth part of Y is orientable if and only if there exists a basis of $\{\pm 1\}^n$ such that $(-1)^v$ is a product of an odd number of basis vectors, for each primitive vector v lying on a ray of Σ.

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Let Y be a projective real toric variety defined by Δ in $\mathbb{Z}^n$ and let Y^+ be its double cover in S^Δ.

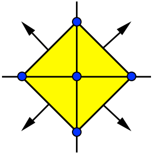
Theorem (S., Sottile)

The smooth part of Y^+ is orientable if and only if there exists a basis of $\{\pm 1\}^{n+1}$ such that $(-1)^{(v, v \cdot F)}$ is a product of an odd number of basis vectors, for each primitive vector v normal to a facet F of Δ.

Examples

Cross Polytopes

$\Delta = \langle e_1, \cdots e_n, -e_1, \cdots, -e_n \rangle$
The rays of the normal fan are the vertices $(\pm 1, \ldots, \pm 1)$ of the n-cube. Hence $(-1)^v = (-1, \ldots, -1)$, and Y_Δ is orientable.



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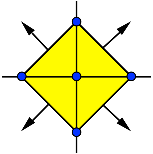
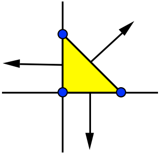


$\mathbb{RP}^2$

$v = (1,1), (-1,0), (0,-1)$
$(-1,-1)^v = (-1,-1), (-1,1), (1,-1)$
$(-1,-1) = (-1,1) \cdot (1,-1)$
$\Rightarrow \mathbb{RP}^2$ is not orientable.
$(-1,-1,-1)^{(v,b)} = (-1,-1,-1), (-1,1,1), (1,-1,1)$
$\Rightarrow S^2$ is orientable.



Y_Σ as a cell complex

Let Y_Σ be defined by a fan $\Sigma \subset \mathbb{R}^n$ with orbits $\mathcal{O}_\sigma(\mathbb{R})$.
Then $\mathcal{O}_0(\mathbb{R}) \simeq \mathbb{T}^n = (\mathbb{R}^\times)^n = (\mathbb{R}_{>0})^n \times \{\pm 1\}^n$.
$\{\pm 1\}^n \subset \mathbb{T}^n$ acts on Y_Σ permuting 2^n components of $\mathcal{O}_0(\mathbb{R})$.
Let $Y_\geq$ be the closure of one of the components of $\mathcal{O}_0(\mathbb{R})$.
Each $\mathcal{O}_\sigma(\mathbb{R})$ has a unique component (face) F_σ in $Y_\geq$.
For each σ let $\overline{\sigma} \leq \{\pm 1\}^n$ be the integer pts of σ reduced mod 2.

Proposition

The real toric variety Y_Σ is obtained as the quotient of $Y_\geq \times \{\pm 1\}^n$ by the relation

$$(p, \xi) \sim (q, \eta) \Leftrightarrow p = q \text{ and } \xi\overline{\sigma} = \eta\overline{\sigma}, \text{ where } p \in F_\sigma.$$

One can reveal the cell complex structure of Y_Δ^+ in a similar way.

Y_Σ is orientable $\Leftrightarrow$ $H_n(Y_\Sigma, \mathbb{Z})$ is nontrivial.
Our computation for both real toric varieties and spherical real toric varieties follows Nakayama and Nishimura's argument, who characterized the orientability of small covers.

Toric Varieties from Order Polytopes

P poset, $\#P = n$.

Def
The order polytope $O(P)$ is the set of all $y = (y_a \colon a \in P) \in [0,1]^n$ such that $y_a \leq y_b$ whenever $a \leq b$ in P.

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$Y_{O(P)}$ is orientable $\Leftrightarrow$ all max chains of P have odd length.

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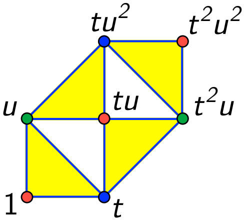
Theorem (S., Sottile)

$Y^+_{O(P)}$ is orientable $\Leftrightarrow$ all max chains of P are of the same parity.

Computing the degree for some systems

Let the projection π be defined by the balanced triangulation T_Δ.



$$\pi : \begin{array}{ccc} \mathbb{RP}^6 & \longrightarrow & \mathbb{RP}^2 = \{[y_0 : y_1 : y_2]\} \\ [x_0 : \cdots : x_6] & \longmapsto & [x_0 + x_3 + x_6 : x_1 + x_5 : x_2 + x_4] \end{array}$$

Computing the Degree: Degeneration

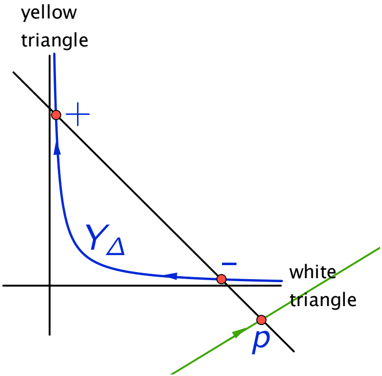
- A regular triangulation T_Δ of Δ gives rise to an $\mathbb{R}^\times$-action on $\mathbb{P}^\Delta$

$$\lim_{t\to 0} t.Y_\Delta =: Y_0 = \text{Union of coordinate planes}$$

- Each plane of Y_0 corresponds to a simplex in the regular triangulation.
- At the preimages on the planes that correspond to adjacent simplices, $d\pi$ has opposite signs.
- If the the toric degeneration $t.Y_\Delta$ does not intersect center of projection,
$\deg \pi = \sigma(T_\Delta) = |\#\text{yellow triangles} - \#\text{white triangles}|$.

Computing the Degree: Degeneration

Deform Y_Δ (or Y_Δ^+) to a union of coordinate planes (or spheres):



$\deg \pi = \sigma(T_\Delta) = |\#\text{yellow triangles} - \#\text{white triangles}|$

Difficulty: the deformation may intersect center of projection.

Theorem (Soprunova, Sottile)

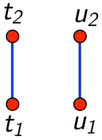
Suppose that

- Y^+_Δ is orientable
- the toric degeneration $t.Y^+_\Delta$ does not intersect center of projection π

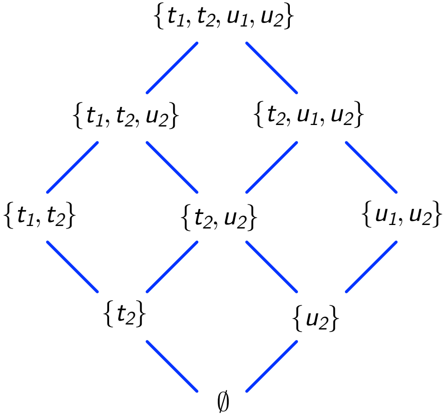
Then $\deg \pi = \sigma(T_\Delta)$ and the number of real solutions to $(\star)$ is at least $\sigma(T_\Delta)$.

Poset Example

$P =$



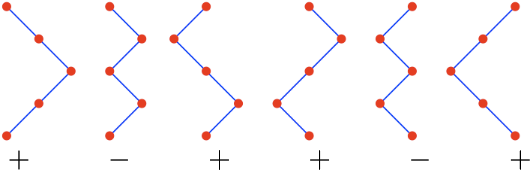
$\mathcal{J}(P) =$



$$
\begin{gathered}
c_4 t_1 t_2 u_1 u_2 + \\
c_3(t_1 t_2 u_2 + t_2 u_1 u_2) + \\
c_2(t_1 t_2 + t_2 u_2 + u_1 u_2) + \\
c_1(t_2 + u_2) + \\
c_0 = 0
\end{gathered}
$$

Poset Example, cont'd

Consider max chains of $\mathcal{J}(P)$:



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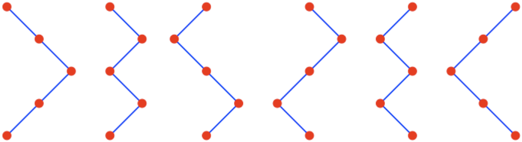

$+ \quad - \quad + \quad + \quad - \quad +$

Sign imbalance $\sigma(P) := |\#\text{even max chains } - \#\text{odd max chains }| = 2$

$$c_4 t_1 t_2 u_1 u_2 +$$
$$c_3(t_1 t_2 u_2 + t_2 u_1 u_2) +$$
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$$c_0 = 0$$

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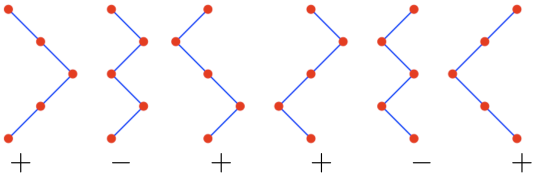

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c_0 = 0
\end{gathered}
$$

Theorem(S., Sottile)

A system of 4 such real equations has at least $\sigma(P) = 2$ real solutions.

Polynomial Systems from Posets

P - finite poset, $\#P = n$
J - upward closed subset of P

$$t^J = \prod_{a \in J} t_a \quad \text{monomial in} \quad \mathbb{R}[t_a \mid a \in P]$$

Polynomials of the form

$$(*) \sum_J c_{\#J} t^J = 0, \qquad c_{\#J} \in \mathbb{R}^\times$$

Poset Theorem

Sign-imbalance:

$\sigma(P) := |\#\text{even max chains in } \mathcal{J}(\mathcal{P}) - \#\text{odd max chains in } \mathcal{J}(\mathcal{P})|$

Theorem (Soprunova, Sottile)

Suppose the maximal chains of P are all of the same parity. Then a system of n real polynomial equations of the form $(*)$ has at least $\sigma(P)$ real solutions.

Sketch of the proof: the Order Polytope is behind this.

Poset P, $\#P = n$.

- Vertices of the order polytope $O(P)$ are char functions of upward closed subsets.
- Canonical triangulation:
  λ - linear extension, $\lambda(a_k) = k$, then

  $$0 \leq f_{a_1} \leq f_{a_2} \leq \cdots \leq f_{a_n} \leq 1$$

  defines a simplex in $O(P)$.
- This triangulation is balanced, regular, and unimodular. Its signature $= \sigma(P)$.
- If all max chains are of the same parity, $Y_{O(P)}$ is orientable.
- The degeneration $t.Y_{O(P)}$ does not intersect the center of projection.

Thank you!

Let Y be an abstract real toric variety defined by a fan Σ in $\mathbb{Z}^n$.

Theorem (S., Sottile)

The smooth part of Y is orientable if and only if there exists a basis of $\{\pm 1\}^n$ such that $(-1)^v$ is a product of an odd number of basis vectors, for each primitive vector v lying on a ray of Σ.

Wronski map

$f_1, \ldots, f_p$ - real polynomials in one variable

$$W(f_1, \ldots, f_p) = \begin{vmatrix} f_1 & \ldots & f_p \\ f_1' & \ldots & f_p' \\ \vdots & & \vdots \\ f_1^{(p-1)} & \ldots & f_p^{(p-1)} \end{vmatrix}$$

Let $\deg f_i \leq m + p - 1$, then

$$\begin{aligned} f_i &\longleftrightarrow \text{lin. form on } \mathbb{R}^{m+p} \\ W(f_1, \ldots, f_p) &\longleftrightarrow \text{vector of coeff in } \mathbb{R}^{mp+1} \end{aligned}$$

This induces the Wronski map

$$W\colon\, G(m, m+p) \longrightarrow \mathbb{RP}^{mp}$$

Eremenko-Gabrielov Theorem

Fact: In the Plucker imbedding, W is a restriction of a linear projection.

Theorem (Eremenko-Gabrielov)

$$\deg W = \sigma(G(m, m+p))$$

We recover this result degenerating the Grassmannian to a toric variety.

Non-Unimodular Case

$\sigma(P) = |\#\text{yellow odd} - \#\text{white odd}|$