Tropical Severi Varieties and Applications

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Background/Motivation

"Asymptotic behavior of a hypersurface, $X = V(f)$ in $(\mathbb{C}^*)^n, f \in \mathbb{C}[\mathbb{Z}^n]$

is described by *Newton*(f)."

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$y(t) = y_0 t^v + l.o.t.,$

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$(t \to \infty)$

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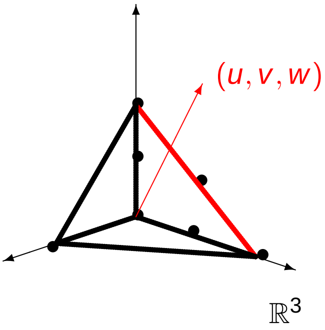

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What if X is not a hypersurface?

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- Found a **partial description of** $\mathrm{Trop}(\mathrm{Sev}(\Delta, \delta))$ in terms of the polygon, Δ.
- Applications:
  1. *Mikhalkin's Correspondence theorem* in terms of Tropical Intersection Theory
  2. Relation with *Secondary Fans*

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Main Theorems

Let $r = dim(\mathrm{Sev}(\Delta, \delta))$.

1. If $rank(\omega) > r$, $\omega \notin \mathrm{Trop}(\mathrm{Sev}(\Delta, \delta))$.
2. If $rank(\omega) = r$ and $\omega \in \mathrm{Trop}(\mathrm{Sev}(\Delta, \delta))$, Δ_ω is simple-nodal.
3. If $\omega \in \mathrm{Trop}(\mathrm{Sev}(\Delta, \delta))$ is regular with the maximal rank r,

$$\mathrm{in}_\omega \mathrm{Sev}(\Delta, \delta) \leftrightarrow_{set} \mathbb{V}_{\Delta_\omega}$$

$$m_{\mathrm{Sev}(\Delta,\delta)}(\omega) = I(\mathbb{V}) \cdot \widetilde{\prod} length(Edges(\Delta_\omega)).$$

4. Let $\boldsymbol{p} = \{p_1, \ldots, p_r\} \subset ((\mathbb{K}^*)^2)^r$, generic ($\mathbb{K} = \cup_{n \geq 1} \mathbb{C}(t^{1/n})$).

$$\mathrm{Trop}(\mathcal{L}(\boldsymbol{p})) \cap \mathrm{Trop}(\mathrm{Sev}(\Delta, \delta)) \leftrightarrow \{\text{tropical curves passing points in} \boldsymbol{Val(p)}$$

$$\boldsymbol{m}(\omega; \mathrm{Trop}(\mathcal{L}(\boldsymbol{p})), \mathrm{Trop}(\mathrm{Sev}(\Delta, \delta))) = \prod 2area(Triangles).$$

5. If $\exists \omega \in \mathrm{Trop}(\mathrm{Sev}(\Delta, \delta))$ with maximal rank, not extending to a concave function on Δ, then $\mathrm{Trop}(\mathrm{Sev}(\Delta, \delta))$ cannot be a subfan of $SecFan(\Delta, \Delta \cap \mathbb{Z}^2)$.

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Example of $\mathrm{Trop}(\mathrm{Sev}(\Delta, \delta))$

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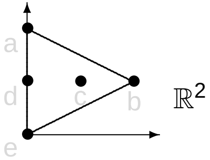

Δ_ω:

$\prod Edges(\Delta_\omega) = 2$

$f = ay^2 + bx^2y + cxy + dy + e \in \mathbb{P}^4_{[a:\cdots:e]}$

$\overline{Sev(\Delta, 1)}$

$= \{f \in \mathbb{P}^4 : f$ defines a curve with one node$\}$

$= V(16b^2d^2 - 8bc^2d + c^4 - 64ab^2e)$

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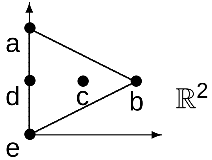

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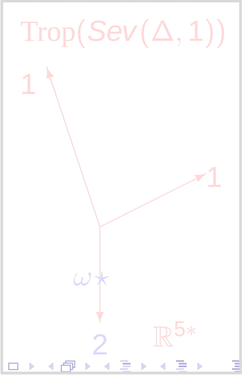

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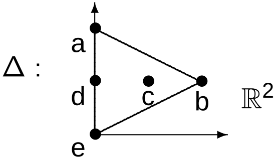

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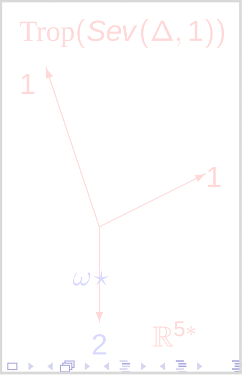

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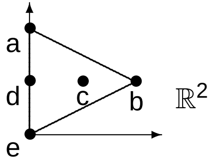

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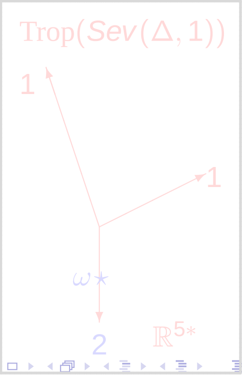

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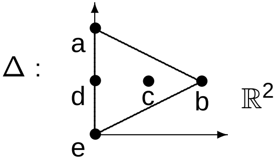

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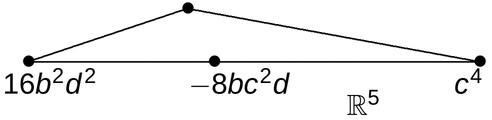
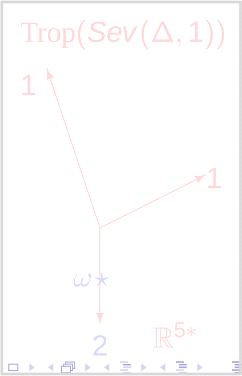
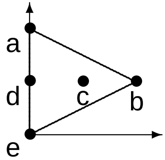




Example of $\mathrm{Trop}(\mathrm{Sev}(\Delta, \delta))$

Δ :



$\mathbb{R}^2$

Δ_ω:

$\prod Edges(\Delta_\omega) = 2$

$f = ay^2 + bx^2y + cxy + dy + e \in \mathbb{P}^4_{[a:\cdots:e]}$

$Sev(\Delta, 1)$

$= \overline{\{f \in \mathbb{P}^4 : f \text{ defines a curve with one node}\}}$

$= V(16b^2d^2 - 8bc^2d + c^4 - 64ab^2e)$

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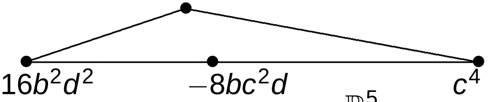
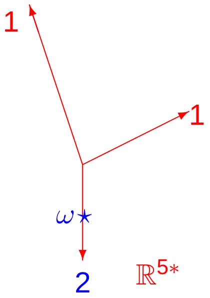


$\mathbb{R}^5$

$\mathrm{Trop}(Sev(\Delta, 1))$



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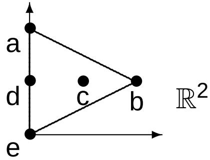

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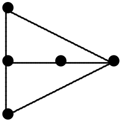

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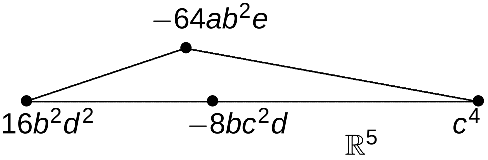
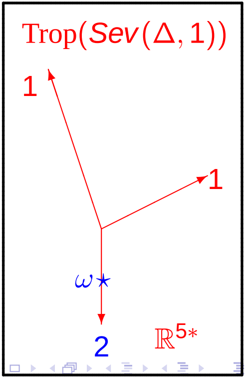
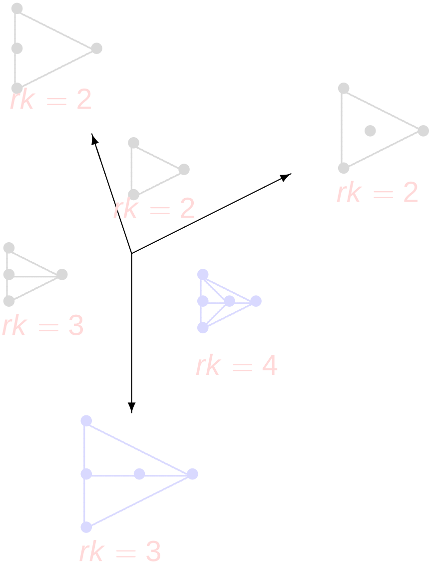




$\mathrm{Trop}(\mathrm{Sev}(\Delta, \delta))$ vs. Subdivisions of Δ

$Sev(\Delta, \delta = 1)$:



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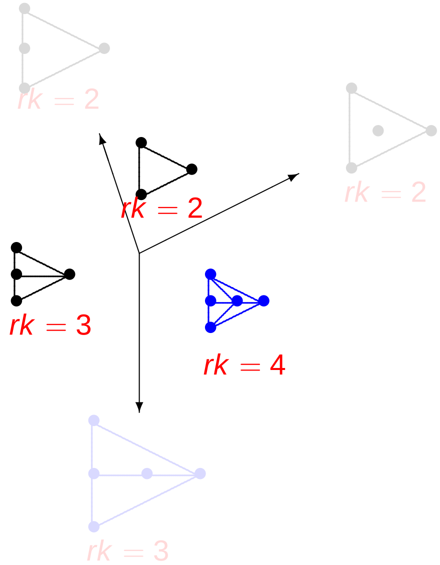

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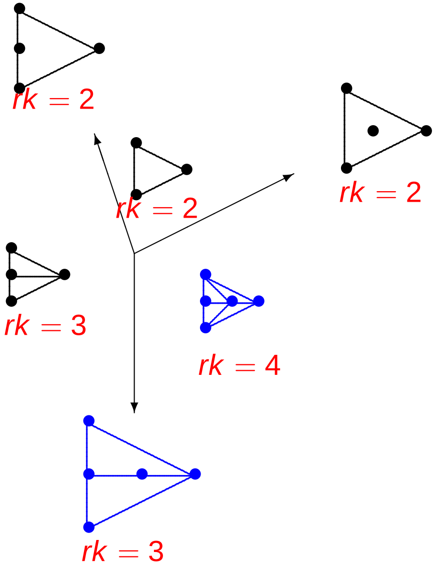
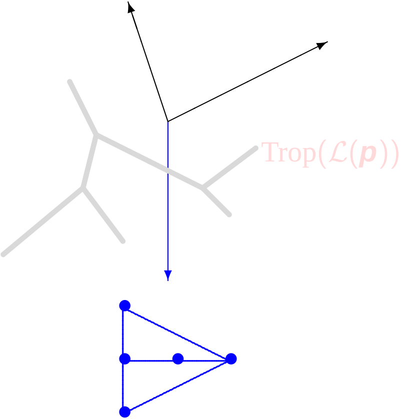


First Application: Degree of $\mathrm{Sev}(\Delta, \delta)$



$$\boldsymbol{m}(\omega; \mathrm{Trop}(\mathcal{L}(\boldsymbol{p})), \mathrm{Trop}(\mathrm{Sev}(\Delta, \delta)))$$

$$= \boldsymbol{m}_{\mathcal{L}(\boldsymbol{p})}(\omega) \cdot \boldsymbol{m}_{Sev(\Delta,1)}(\omega) \cdot \xi(\omega; \mathcal{T}_1, \mathcal{T}_2)$$

$$= 1 \cdot 2 \cdot 2 = 4$$

(combinatorial formula for ξ

is also found in (Y,11))

Mikhalkin's multiplicity of τ_ω

$:= \prod_{\Delta_\omega} 2area(Triangles)$

$= 2 \cdot 2 = 4$

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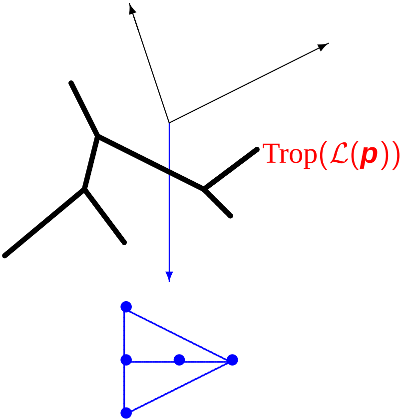

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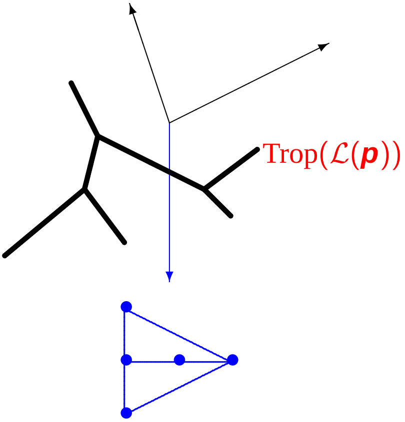

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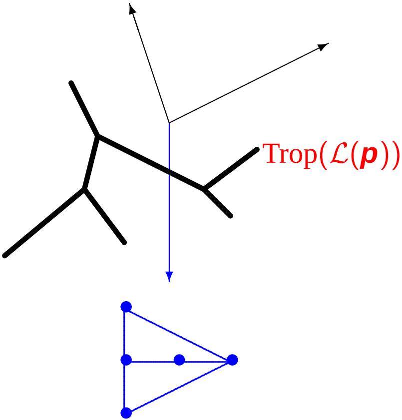

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Second Application: Secondary Fans

- *SecFan*$(\Delta, \mathcal{A})$ is a complete fan in $\mathbb{R}^{\mathcal{A}}$ parameterizing regular marked subdivisions of $(\Delta, \mathcal{A})$. (Discriminantal variety, Chow quotient, etc)
- $\mathrm{Trop}($*Sev*$(\Delta, 1))$ is a subfan of *SecFan*$(\Delta, \Delta \cap \mathbb{Z}^2)$
- What about $\mathrm{Sev}(\Delta, \delta)$ for general δ?
- A counterexample is found by E.Katz(2008).

Theorem (Y,11)

If $\exists\, \omega \in \mathrm{Trop}(\mathrm{Sev}(\Delta, \delta))$ *with maximal rank which does not extend to a concave function on* Δ*, then* $\mathrm{Trop}(\mathrm{Sev}(\Delta, \delta))$ *cannot be a subfan of SecFan*$(\Delta, \Delta \cap \mathbb{Z}^2)$.

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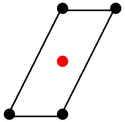
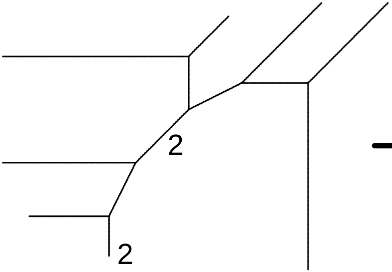


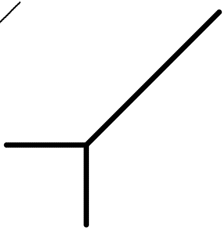
[Yang1] J.Yang, Initial Schemes of Very Affine Severi Varieties. arXiv:1108.5839
[Yang2] J.Yang, Some Parameter Spaces of Curves on Toric Surfaces, Their tropicalizations and Degrees. In preparation.

Thank you!

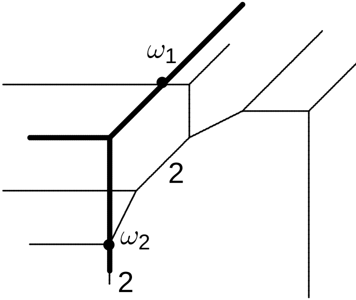
Tropical Product



$\mathcal{T}_1$

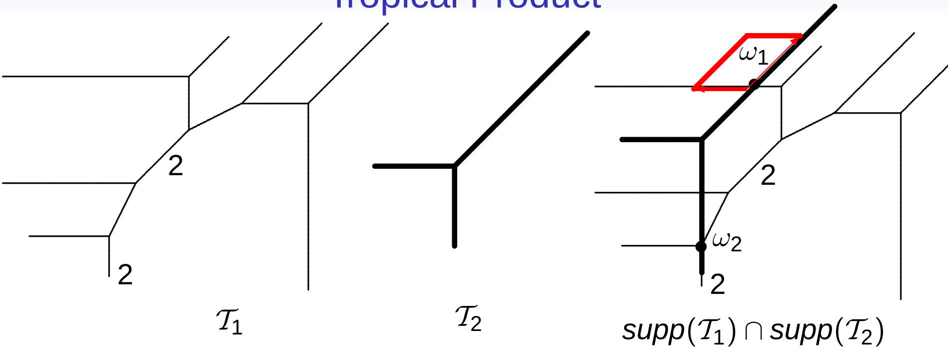


$\mathcal{T}_2$



$supp(\mathcal{T}_1) \cap supp(\mathcal{T}_2)$

Tropical Product



The tropical intersection multiplicity of $\mathcal{T}_1$ and $\mathcal{T}_2$ at ω is

$$\boldsymbol{m}(\omega) = \boldsymbol{m}(\omega; \mathcal{T}_1, \mathcal{T}_2) := \boldsymbol{m}_{\mathcal{T}_1}(\omega) \cdot \boldsymbol{m}_{\mathcal{T}_2}(\omega) \cdot \xi(\omega; \mathcal{T}_1, \mathcal{T}_2),$$

where $\xi(\omega; \mathcal{T}_1, \mathcal{T}_2)$, is the volume of the parallelepiped constructed by the fundamental cells of the lattices $\mathbb{L}_i \cap \mathbb{Z}^n$, $\quad (i = 1, 2)$.

Algebraic Geometry	Tropical Geometry
very affine varieties	tropical varieties: balanced weighted rational polyhedral complex $\subset \mathbb{R}^n$
$X \subset (\mathbb{C}^*)^n$ $\longrightarrow$	$\mathrm{Trop}(X)$, tropicalization of X • $dim(X) = dim(\mathrm{Trop}(X))$ • $\mathrm{Trop}(X_1 \cup X_2) = \mathrm{Trop}(X_1) + \mathrm{Trop}(X_2)$ • $\mathrm{Trop}(X_1 \cap gX_2) = \mathrm{Trop}(X_1) \cdot \mathrm{Trop}(X_2)$ for generic $g \in (\mathbb{C}^*)^n$

$\mathrm{in}_\omega \mathrm{Sev}(\Delta, \delta)$ vs $\mathbb{V}_{\partial\Delta_{\omega,nodal}}$

Definition

Let $\mathcal{S}(\Delta) : \Delta_1 \cup \cdots \cup \Delta_m$ be a nodal subdivision of Δ.
$f \in \mathbb{V}_{\partial\Delta_{\omega,nodal}} \subset \mathbb{P}_\Delta \Leftrightarrow$

- $s \in Edges(\mathcal{S}(\Delta)) \Rightarrow f_s$ is a pure power of a binomial, $x^a y^b(\alpha x^c + \beta y^d)^{|s|}$;
- Δ_i is a trangle $\Rightarrow f_{\Delta_i}$ defines a rational curve which is unibranch at each intersection point with the boundary divisors of the toric surface X_{Δ_i};
- Δ_j is a parallelogram $\Rightarrow f_{\Delta_j}$ has the form $x^k y^l(\alpha x^a + \beta y^b)^p(\gamma x^c + \delta y^d)^q$.

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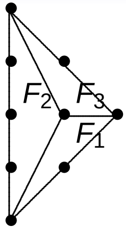
Theorem

$\mathbb{V}_{\mathcal{S}(\Delta)_{\omega,nodal}}$ *is a translation of a closed subgroup* $\mathbb{G}_{\mathcal{S}(\Delta)_{\omega,nodal}}$ *of an algebraic torus.*

$$M_{\partial\mathcal{S}(\Delta)} = \begin{pmatrix} & \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 \\ F_1 & -2 & 1 & 1 & 0 & 0 & 0 \\ F_2 & 0 & -1 & 0 & 4 & 1 & 0 \\ F_3 & 0 & 0 & -1 & 0 & -1 & 2 \end{pmatrix}$$



$$M_{\partial\Delta,\mathbb{P}^1} = \begin{pmatrix} & \alpha_1 & \beta_1 & \alpha_2 & \beta_2 & \alpha_3 & \beta_3 \\ s_{12} & 1 & 2 & -1 & -2 & 0 & 0 \\ s_{13} & 1 & 0 & 0 & 0 & -1 & 0 \\ s_{23} & 0 & 0 & -1 & 2 & 1 & -2 \end{pmatrix}$$

The Smith Normal Forms of $M_{\partial\mathcal{S}(\Delta)}$ and $M_{\partial\mathcal{S}(\Delta),\mathbb{P}^1}$ coincide to each other as :

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix}.$$

Thus $\mathbb{V}_{\mathcal{S}(\Delta)}$ is a union of two translations of 3-dimensional subtorus of $\mathbb{T}_\Delta$.