Stable Intersections of Tropical Varieties

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Tropical Varieties

Let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be an ideal. The **tropical variety** of I is

$$\mathcal{T}(I) := \{w \in \mathbb{R}^n : \mathrm{in}_w(I) \text{ contains no monomials}\}.$$

For ideals I and J,

- $\mathcal{T}(I \cap J) = \mathcal{T}(I) \cup \mathcal{T}(J)$,
- $\mathcal{T}(I + J) \subseteq \mathcal{T}(I) \cap \mathcal{T}(J)$; inclusion may be strict

A tropical variety of a **prime** ideal

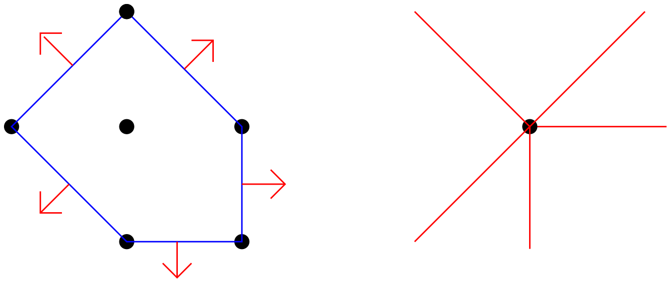
- is a rational polyhedral fan, whose dimension is equal to the Krull dimension of I
- satisfies the balancing condition, with weight of a generic point $w \in \mathcal{T}(I)$ defined as the sum of the multiplicities of monomial-free minimal associated primes of $\mathrm{in}_w(I)$.

Example

Let $I = \langle 1 + x + y + y^2 \rangle \subset \mathbb{C}[x, y]$.
The point $w = (-1, 0)$ is in $\mathcal{T}(I)$ because $\mathrm{in}_w(I) = \langle 1 + y + y^2 \rangle$ contains no monomials. It has weight 2.

Tropical Hypersurfaces

If the ideal is principal, generated by a polynomial f, then $\mathcal{T}(I)$ depends only on the Newton polytope P of f. It is the union of normal cones to edges of P. We will also denote $\mathcal{T}(\langle f \rangle)$ by $\mathcal{T}(P)$.



$$f = x + x^2 + y + xy + x^2y + xy^2$$

Is every rational balanced polyhedral fan a tropical variety?

- Yes, for fans of dimension 1 or codimension 1.
- There are *non-realizable* 2-dimensional fans in $\mathbb{R}^4$.

The problem of characterizing *realizable* fans is very difficult – as a special case, it includes characertizing representable matroids.

Definition

A *tropical k-cycle* is a pure k-dimensional rational balanced polyhedral fan with integer weigts (negative weights are allowed).

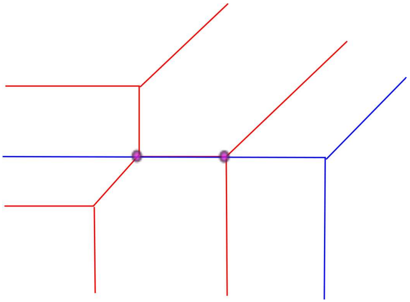
Note:

- The union of two tropical k-cyles is again a tropical k-cycles. We think of this as the **sum** of tropical cycles.
- The intersection of two tropical cycles need not be a tropical cycle.

Stable Intersections

Definition/Lemma Let $\mathcal{F}_1$, $\mathcal{F}_2$ be tropical cycles. The following sets coincide and are called the *stable intersection* of $\mathcal{F}_1$ and $\mathcal{F}_2$, denoted $\mathcal{F}_1 \cdot \mathcal{F}_2$

1. $\lim_{\varepsilon \to 0} \mathcal{F}_1 \cap (\mathcal{F}_2 + \varepsilon v)$ for a generic $v \in \mathbb{R}^n$
2. $\{w \in \mathcal{F}_1 \cap \mathcal{F}_2 : \mathrm{link}_w(\mathcal{F}_1) - \mathrm{link}_w(\mathcal{F}_2) = \mathbb{R}^n\}$



The second description is better for **computations**. We have an implementation in the software **Gfan**.

Properties

- Stable intersection of a codim-k cycle and codim-l cycle is either 0 (empty) or a codim-$(k+l)$ cycle.
- Stable intersection is associative and commutative.
- $\mathcal{F}_1 \cdot (\mathcal{F}_2 + \mathcal{F}_3) = \mathcal{F}_1 \cdot \mathcal{F}_2 + \mathcal{F}_1 \cdot \mathcal{F}_3$, where $+$ is the union.

The first two statements are difficult to prove.

Relation to Polytopes

- Let P be a polytope, and let Q be its image under orthogonal projection onto a linear subspace L. Then

$$\mathcal{T}(P) \cdot L = \mathcal{T}(Q) \quad (\text{in } L)$$

- For polytopes $P_1, \ldots, P_n \subset \mathbb{R}^n$,

$$\mathcal{T}(P_1) \cdots \mathcal{T}(P_n)$$

  is the origin with weight equal to the **mixed volume** of $P_1, \ldots, P_n$.

Stable Intersections of Tropical Varieties

Let I and J be ideals in $\mathbb{C}[x_1, \ldots, x_n]$. Let J' be obtained from J by replacing x_i with $c_i x_i$ where $c_1, \ldots, c_n$ are generic non-zero elements of $\mathbb{C}$. Then $\mathcal{T}(J) = \mathcal{T}(J')$, and

Theorem (Osserman–Payne '09, Jensen–Y.)

$$\mathcal{T}(I) \cdot \mathcal{T}(J) = \mathcal{T}(I + J')$$

In particular, stable intersections of tropical hypersurfaces are **realizable**.

Using this theorem, we can prove **Bernstein's Theorem**:
for n polynomials in n variables, the number of common roots in $(\mathbb{C}^*)^n$ is equal to the mixed volume of the Newton polytopes.

Two algebras

Algebra of Tropical Cycles

- Let T^r be the $\mathbb{R}$-vector space of codimension-r tropical cycles in $\mathbb{R}^n$, with $\mathbb{R}$ weights.
- Scalar multiplication acts on the weights.
- Addition is taking union.

Then

$$\mathcal{T}_n = T^0 \oplus T^1 \oplus \dots T^n$$

is a graded algebra with stable intersection as multiplication.

Polytope Algebra of McMullen

Let Π_n be an $\mathbb{R}$-algebra, generated as an $\mathbb{R}$-vector space by the classes of rational polytopes $[P]$, modulo relations

- $[P] = [P + v]$ for polytope $P \subset \mathbb{R}^n$ and $v \in \mathbb{R}^n$
- $[P \cup Q] + [P \cap Q] = [P] + [Q]$ if $P \cup Q$ is again a polytope.

The multiplication is Minkowski sum: $[P] \cdot [Q] = [P + Q]$.

[McMullen '89] Π_n is a graded algebra, with r-th graded piece spanned by $\{(\log[P])^r : P$ is a rational polytope in $\mathbb{R}^n\}$.

Isomorphism

Theorem (follows from McMullen '93 & Fulton–Sturmfels '97)

The graded algebras Π_n and $\mathcal{T}_n$ are isomorphic via the isomorphism

$$[P] \mapsto \exp(\mathcal{T}(P)) = 1 \oplus \mathcal{T}(P) \oplus \frac{1}{2!}\mathcal{T}(P)^2 \oplus \cdots \oplus \frac{1}{n!}\mathcal{T}(P)^n$$

In other words, the tropical hypersurface $\mathcal{T}(P)$ is the **logarithm** of the polytope P.

Corollary

The graded piece T^r of $\mathcal{T}_n$ is spanned by

$$\{\mathcal{T}(P)^r : P \text{ is a rational polytope in } \mathbb{R}^n\}.$$

Corollary

Every tropical cycle is a linear combination of ***realizable*** *tropical varieties.*