The maximum likelihood degree of a very affine variety

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The maximum likelihood degree

- $\mathbb{P}^{n-1}$, the projective space with coordinates $p_1, \ldots, p_n$, where the p_i represents the probability of the i-th event.
- An implicit statistical model is a closed subvariety $V \subseteq \mathbb{P}^{n-1}$.
- The data comes in the form of integers $u_1, \ldots, u_n$, where the u_i is the number of times the i-th event was observed.

The maximum likelihood degree

- In order to find the values of p_i on V which best explains the given data u_i, one finds critical points of the likelihood function

$$L(p_1, \ldots, p_n) = p_1^{u_1} \cdots p_n^{u_n} / (p_1 + \cdots + p_n)^{u_1 + \cdots + u_n}.$$

- *The maximum likelihood degree* of the model is the number of critical points of $L|_V$, for sufficiently general $u_1, \ldots, u_n$.

- This number is well-defined.

Log-concavity conjectures

- A sequnce $a_0, \ldots, a_p$ is *log-concave* if for all i

$$a_{i-1}\, a_{i+1} \leq a_i^2.$$

- If there are *no internal zeroes*, log-concavity implies *unimodality*:

$$a_0 \leq \cdots \leq a_{i-1} \leq a_i \geq a_{i+1} \geq \cdots \geq a_p \quad \text{for some } i.$$

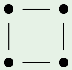
Chromatic polynomial of graphs

Let G be a graph.

- The chromatic polynomial of G is the function

$$\chi_G(q) = (\text{number of proper colorings of } G \text{ with } q \text{ colors}).$$

Example

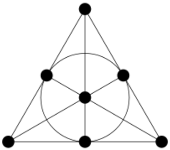


$$\begin{aligned}\chi_G(q) &= q(q-1)(q-2)(q-2) + q(q-1)(q-1)(q-1) \\ &= 1q^4 - 4q^3 + 6q^2 - 3q\end{aligned}$$

Independent sets of vectors

- Let $\mathcal{A} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a subset of a vector space V.
- Define the f-vector of $\mathcal{A}$ by

$$f_i = (\text{number of independent subsets of cardinality } i \text{ in } \mathcal{A}).$$



Example (Fano plane)

For $\mathcal{A} = \mathbb{F}_2^3 \setminus \{\mathbf{0}\}$, we have

$$f_0 = 1, \quad f_1 = 7, \quad f_2 = 21, \quad f_3 = 28.$$

Conjecture (Read-Rota 68, Mason-Welsh 69)

- *The first sequence is log-concave for any* G.
- *The second sequence is log-concave for any* $\mathcal{A}$.

- These conjectures are proved (H, H-Katz, Lenz) using the topology of hypersurface complements, CSM class, tropical geometry, and matroid theory.

- The maximum likelihood framework provides solutions to stronger conjectures (Hoggar 74, Dawson 84, Colbourn 87) with simpler proofs, in characteristic zero.

Theorem

Let M be a matroid representable over a field of characteristic zero.

1. *The h-vector of the matroid complex of M is log-concave.*
2. *The h-vector of the broken circuit complex of M is log-concave.*

- "1" implies a conjecture of Colbourn on the reliability of a network.
- "1" was conjectured by Dawson in general.
- Any matroid complex is a broken circuit complex, but not conversely.

  Therefore "2" is stronger than "1".

Log-concavity conjectures

Corollary

Let M be a matroid representable over a field of characteristic zero.

1. *The f-vector of the matroid complex of M is strictly log-concave.*
2. *The f-vector of the broken circuit complex of M is strictly log-concave.*

This should be compared to f-vectors and h-vectors of

other "nice" shellable simplicial complexes, such as...

Björner's example (from Ziegler's polytope book)

Examples 8.40. The unimodality conjecture fails for a simplicial polytope of dimension $d = 20$ with the following f-vector, for which $f_{11} > f_{12} < f_{13}$.

$$
\begin{array}{lcrl}
f_{-1} & = & 1 & \\
f_0 & = & 4203045807626 & \\
f_1 & = & 84060916163336 & \\
f_2 & = & 798578704207074 & \\
f_3 & = & 4791472253296106 & \\
f_4 & = & 20363758019368323 & \\
f_5 & = & 65164051780016980 & \\
f_6 & = & 162910744316489788 & \\
f_7 & = & 325834059588060117 & \\
f_8 & = & 529707205213463823 & \\
f_9 & = & 709935971390166248 & \\
f_{10} & = & 805494832051588614 & \\
f_{11} & = & 821976324224631043 & / \\
f_{12} & = & 821976324224611712 & \longleftarrow \\
f_{13} & = & 822000129478641948 & \backslash \\
f_{14} & = & 747383755288236256 & \\
f_{15} & = & 546761228419958342 & \\
f_{16} & = & 293715859557026466 & \\
f_{17} & = & 106920718330384544 & \\
f_{18} & = & 23458617733909980 & \\
f_{19} & = & 2345861773390998 &
\end{array}
$$

How do you prove that a sequence is log-concave?

Let's say that a homology class of an algebraic variety is *representable*

if it is the class of a subvariety.

Theorem

Write an even dimensional homology class ξ as a linear combination

$$\xi = \sum_i e_i [\mathbb{P}^{k-i} \times \mathbb{P}^i] \in H_{2k}(\mathbb{P}^m \times \mathbb{P}^n; \mathbb{Z}).$$

Then some positive multiple of ξ is representable if and only if $\{e_i\}$ form a log-concave sequence of nonnegative integers with no internal zeros.

The maximum likelihood degree

- Therefore it suffices to show that there is a subvariety of $\mathbb{P}^m \times \mathbb{P}^n$ whose homology class is given by the coefficients of $\chi_M(q+1)$.
- The idea for this comes from the *maximum likelihood estimation* in algebraic statistics.

Varchenko's conjecture

- $\mathcal{A}$ = (arrangement in $\mathbb{C}^r$ defined by linear functions $f_1, \ldots, f_n$).
- $\varphi = \prod_{i=1}^n f_i^{u_i}$, $u_i \in \mathbb{Z}$.

Conjecture (Varchenko 95)

If the arrangement is essential and the u_i are general, then

1. φ *has only finitely many critical points in* $\mathbb{C}^r \setminus \mathcal{A}$.
2. *All critical points of* φ *are nondegenerate.*
3. *The number of critical points is equal to* $(-1)^r \chi(\mathbb{C}^r \setminus \mathcal{A}) \quad \big(= \chi_{\mathcal{A}}(1)\big)$.

Varchenko's conjecture is proved by Orlik and Terao.

- If $\mathcal{A}$ is defined over the real numbers,

$$\begin{aligned}\chi_{\mathcal{A}}(1) &= \text{(the number of critical points of } \varphi) \\ &= \text{(the number of bounded regions of } \mathbb{R}^r \setminus \mathcal{A}).\end{aligned}$$

- $\mathcal{A}$ is essential iff lowest dimensional flats are zero dimensional.

- $\mathcal{A}$ is essential iff we have the embedding

$$\mathbb{C}^r \setminus \mathcal{A} \longrightarrow (\mathbb{C}^*)^n, \qquad x \longmapsto (f_1, \ldots, f_n).$$

- Any arrangement is of the form $\mathcal{A} = \mathbb{C}^k \times \mathbb{A}'$ with $\mathcal{A}'$ essential.

- Let U be an r-dimensional smooth subvariety of $(\mathbb{C}^*)^n$.
- Let $f_1, \dots, f_n$ be the coordinate functions on U.
- $\varphi = \prod_{i=1}^n f_i^{u_i}$, $u_i \in \mathbb{Z}$.

Theorem

If the u_i are sufficiently general, then

1. *φ has only finitely many critical points in U.*
2. *All critical points of φ are nondegenerate.*
3. *The number of critical points is equal to $(-1)^r \chi(U)$.*

- Next we consider the collection of all critical points of all possible master functions simultaneously:

$$\mathfrak{X}^\circ(U) = \left\{ \sum_{i=1}^{n} u_i \cdot \text{dlog}(p_i)(x) = 0 \right\} \subseteq U \times \mathbb{P}^{n-1},$$

where $u_1, \ldots, u_n$ are now homogeneous coordinates of $\mathbb{P}^{n-1}$.

- The closure $\mathfrak{X}(U) \subseteq \mathbb{P}^n \times \mathbb{P}^{n-1}$ gives the subvariety we want!
- In fact, this is truly a natural choice, because it is a geometric realization of the characteristic class of U.

Suppose U is r-dimensional and not isomorphic to a torus.

Theorem

If U is wonderful, then

$$\big[\mathfrak{X}(U)\big] = \sum_{i=0}^{r} v_i \big[\mathbb{P}^{r-i} \times \mathbb{P}^{n-1-r+i}\big] \in H_*(\mathbb{P}^n \times \mathbb{P}^{n-1}),$$

where

$$c_{SM}(\mathbf{1}_U) = \sum_{i=0}^{r} (-1)^i v_i \, [\mathbb{P}^{r-i}] \in H_*(\mathbb{P}^n).$$

If U is the complement $\mathbb{C}^r \setminus \mathcal{A}$, then (almost by definition)

$$\chi_{\mathcal{A}}(q+1) = \sum_{i=0}^{r} (-1)^i v_i \, q^{r-i}.$$

Theorem

Let U be the hypersurface $\{g=0\} \subseteq (\mathbb{C}^*)^n$ with

$$c_{SM}(\mathbf{1}_U) = \sum_{i=0}^{r} (-1)^i v_i \, [\mathbb{P}^{r-i}] \in H_*(\mathbb{P}^n).$$

If g is general with respect to Δ_g, then U is wonderful, and

$$v_i = MV_n(\underbrace{\Delta, \ldots, \Delta}_{r-i}, \underbrace{\Delta_g, \ldots, \Delta_g}_{i+1}), \qquad i = 0, \ldots, r.$$

In particular, ML-degree of U is equal to the normalized volume

$$v_r = (-1)^r \int c_{SM}(\mathbf{1}_U) = \text{Volume}(\Delta_g).$$