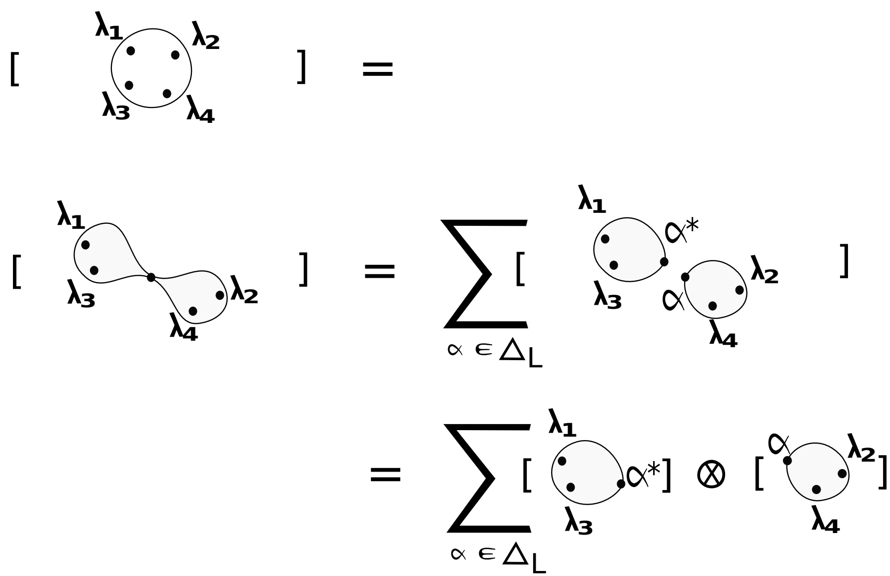
The combinatorial commutative algebra of conformal blocks



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Objects we want to study

$$V_{C,\vec{p}}(\vec{\lambda}, L)$$

Vector spaces of conformal blocks.

$$\mathcal{M}_{C,\vec{p}}(G)$$

Moduli of quasi-parabolic principal bundles on a curve C with parabolic structure at marked points $\vec{p}$.

The combinatorics behind the dimensions of these vector spaces allow us to prove:

For $(C, \vec{p})$ generic,

The projective coordinate ring of $\mathcal{M}_{C,\vec{p}}(SL_2(\mathbb{C}))$ associated to the square $\mathcal{L}^2$ of an effective line bundle is Koszul.

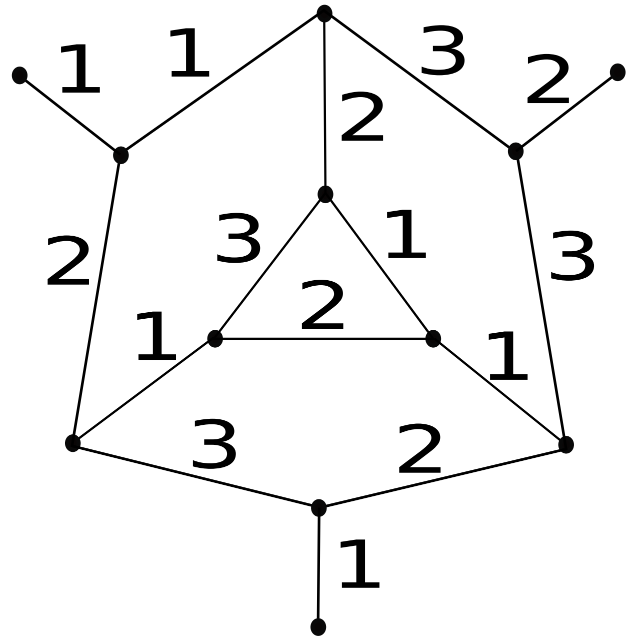
The combinatorics behind the dimensions of these vector spaces allow us to prove:

For $(C, \vec{p})$ generic,

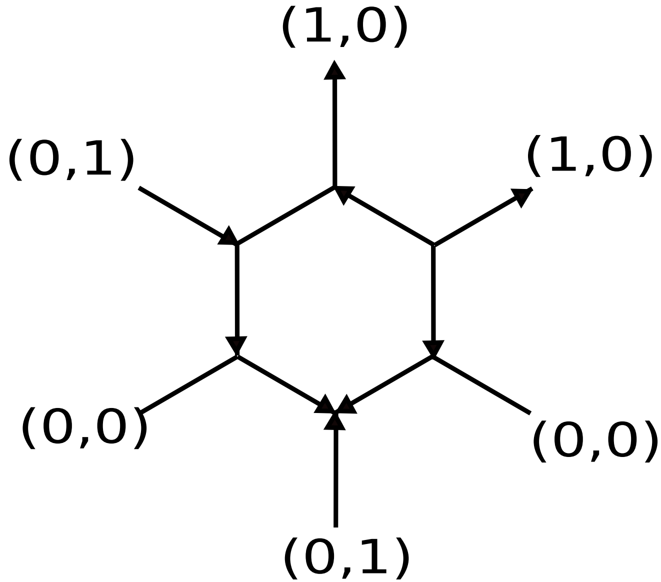
The Cox ring of $\mathcal{M}_{C,\vec{p}}(SL_2(\mathbb{C}))$ is generated in degree ≤ 2 with relations generated in degree ≤ 4.

The Cox ring of $\mathcal{M}_{\mathbb{P}^1,\vec{p}}(SL_3(\mathbb{C}))$ is generated in degree 1 with relations generated in degree ≤ 3.

… with these objects:



... with these objects:



you may recognize conformal blocks from:

Mathematical Physics: Partition functions for the WZW model of conformal field theory, TQFT, Modular functors.

Geometry: Moduli of (vector/principal) bundles, Moduli of flat connections, Gromov-Witten invariants, (non-abelian) theta functions, birational geometry of $\bar{M}_{0,n}$.

Representation Theory: Quantum Horn problem, Saturation conjectures, Kac-Moody algebras, Quantum groups at roots of unity.

Conformal Blocks: notation

G a simply connected, simple group over $\mathbb{C}$.

$Lie(G) = \mathfrak{g}$, a simple Lie algebra over $\mathbb{C}$ (e.g. $sl_m(\mathbb{C}), sp_{2n}(\mathbb{C})$).

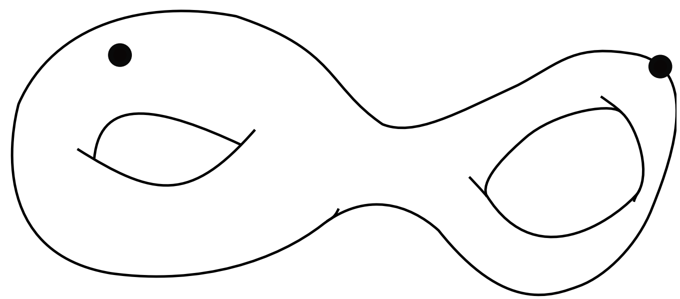
Δ- a Weyl chamber for $\mathfrak{g}$.

$\lambda \in \Delta$- dominant weights for $\mathfrak{g}$.

Δ_L - the $L-$restricted Weyl chamber.

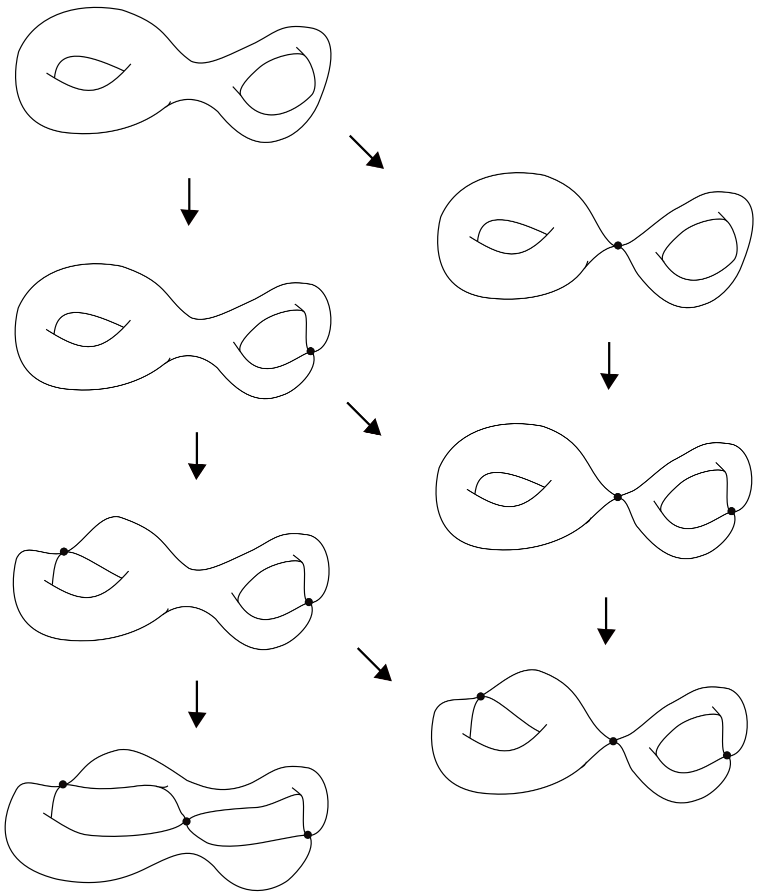
Conformal Blocks

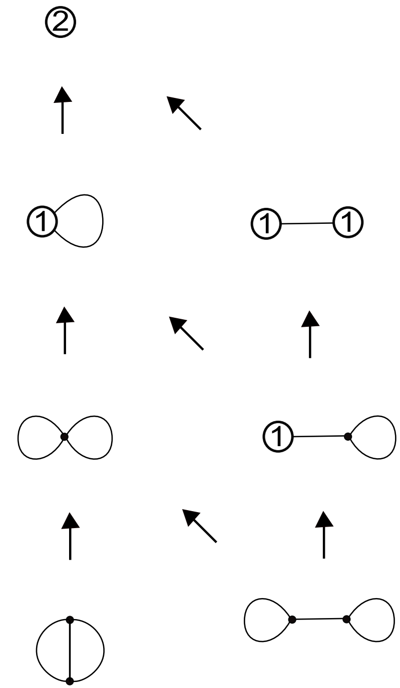
C a stable curve of genus g over $\mathbb{C}$, with n specified "marked" points, $p_1, \ldots, p_n \in C$.



This defines a point in the Deligne-Mumford stack $\bar{\mathcal{M}}_{g,n}$.

Example: $\bar{\mathcal{M}}_{2,0}$





Conformal Blocks

For any n dominant weights $\lambda_i \in \Delta_L$, there is a vector space $V_{C,\vec{p}}(\vec{\lambda}, L)$ called the Space of Conformal Blocks.

$$(C, \vec{p}), \vec{\lambda}, L \Rightarrow V_{C,\vec{p}}(\vec{\lambda}, L)$$

This is built from the representation theory of the affine Kac-Moody algebra $\hat{\mathfrak{g}}$ associated to $\mathfrak{g}$.

Properties of Conformal Blocks

1. Each space $V_{C,\vec{p}}(\vec{\lambda}, L)$ is finite dimensional

2. The dimension of $V_{C,\vec{p}}(\vec{\lambda}, L)$ depends only on the data $(\vec{\lambda}, L)$, the genus g, and the number of points n.

The dimension is given by the Verlinde formula:

$$\mathbb{V}_{g,n}(\vec{\lambda}, L) = |T_L|^{g-1} \sum_{\mu \in \Delta_L} Tr_{\vec{\lambda}}(exp(2\pi i \frac{(\mu + \rho)}{L + h^\vee})) \prod_{\alpha} |2sin(\pi \frac{(\alpha|\mu + \rho)}{L + h^\vee})|^{2-2g}$$

Moduli of bundles

Proposition [Kumar, Narisimhan, Ramanathan, Faltings, Beauville, Lazlo, Sorger] :
For every $\vec{\lambda}, L$ there is a line bundle $\mathcal{L}(\vec{\lambda}, L)$ on $\mathcal{M}_{C,\vec{p}}(G)$ such that

$$H^0(\mathcal{M}_{C,\vec{p}}(G), \mathcal{L}(\vec{\lambda}, L)) = V_{C,\vec{p}}(\vec{\lambda}, L)$$

For a line bundle $\mathcal{L}(\vec{\lambda}, L)$ we let $R_{C,\vec{p}}(\vec{\lambda}, L)$ denote the corresponding graded projective coordinate ring.

The sum of all the spaces of global sections over all these line bundles is the Cox ring, $Cox(\mathcal{M}_{C,\vec{p}}(G))$.

Innocent questions

What generates $Cox(\mathcal{M}_{C,\vec{p}}(G))$?

What relations hold between these generators?

What is the multigraded Hilbert function of $Cox(\mathcal{M}_{C,\vec{p}}(G))$?

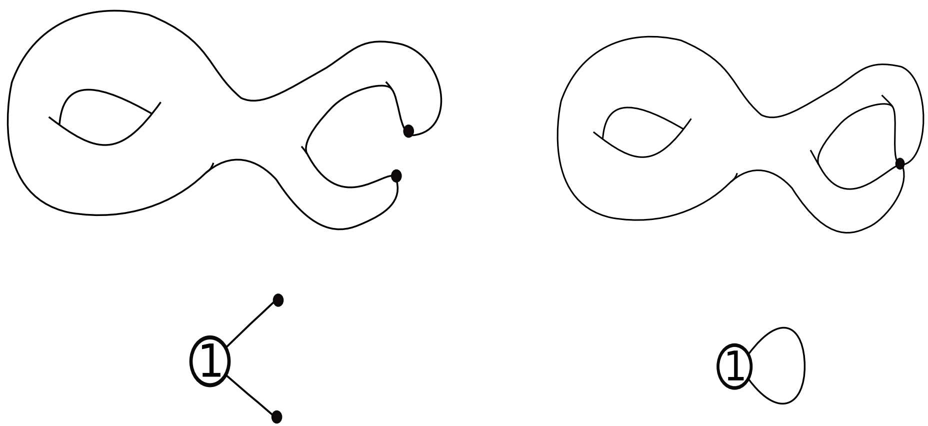
Conformal Blocks: vector bundle property

Theorem [Tsuchiya, Ueno, Yamada]:
For every $\vec{\lambda}, L$ there is a vector bundle $V(\vec{\lambda}, L)$ on $\bar{\mathcal{M}}_{g,n}$, with fiber over $(C, \vec{p})$ equal to $V_{C,\vec{p}}(\vec{\lambda}, L)$.

Tsuchiya, Ueno, and Yamada; Faltings also established the Factorization Rules:

Conformal Blocks: Factorization

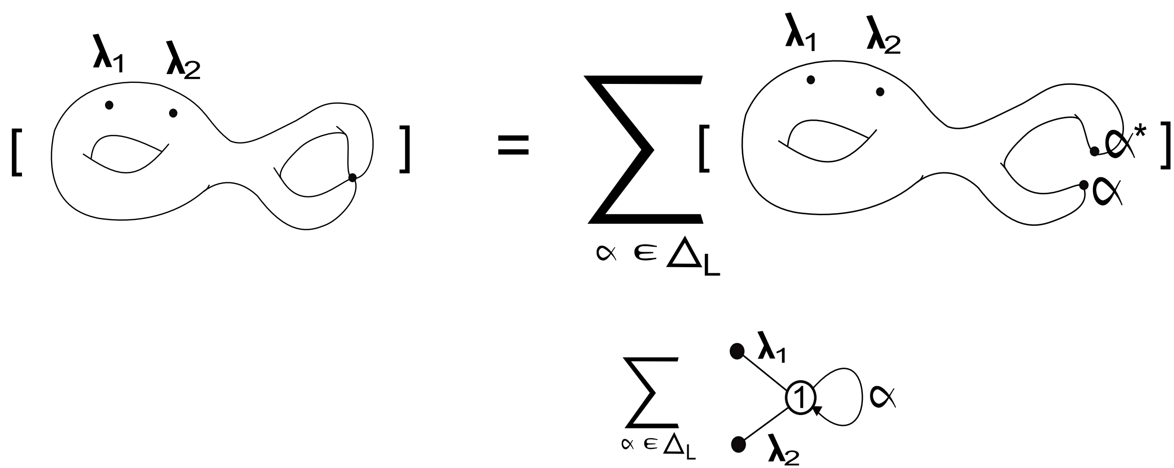
For $(C, \vec{p})$ a stable curve with a doubled point $q \in C$, there is a normalized stable marked curve $(C', q_1, q_2, \vec{p})$.



Conformal Blocks: Factorization

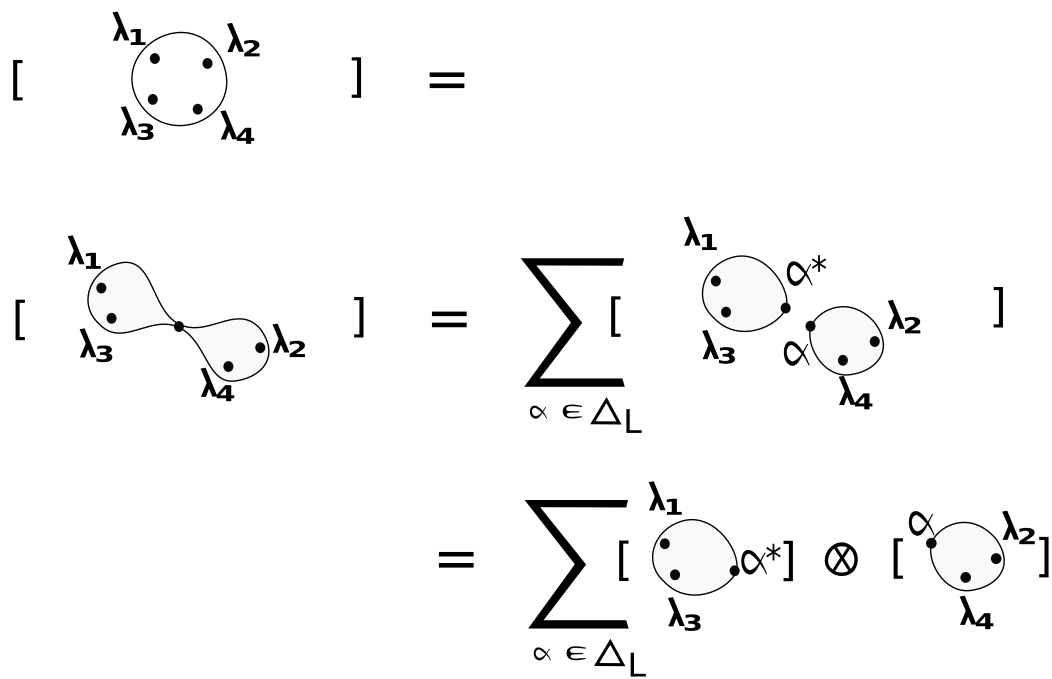
For a marked curve $(C, \vec{p})$ with a doubled point q,

$$V_{C,\vec{p}}(\vec{\lambda}, L) \cong \bigoplus_{\alpha \in \Delta_L} V_{C', q_1, q_2, \vec{p}}(\alpha, \alpha^*, \vec{\lambda}, L)$$



Conformal Blocks: Factorization

The fusion rules allow us to express $dim(V_{C,\vec{p}}(\vec{\lambda}, L))$ purely in terms of the spaces $V_{0,3}(\lambda_1, \lambda_2, \lambda_3, L)$.



$SL_2(\mathbb{C})$ case

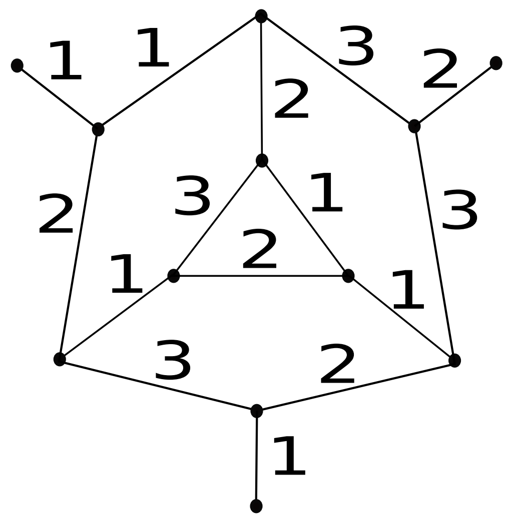
In the case $G = SL_2(\mathbb{C})$, the spaces $V_{0,3}(r_1, r_2, r_3, L)$ are dimension 1 or 0.

Proposition [Quantum Clebsch-Gordon rule] :

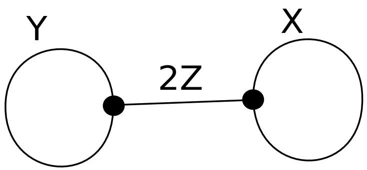
The dimension of $V_{0,3}(r_1, r_2, r_3, L)$ is either 1 or 0. It is dimension 1 if and only if $r_1 + r_2 + r_3$ is even, $\leq 2L$, and r_1, r_2, r_3 are the side-lengths of a triangle

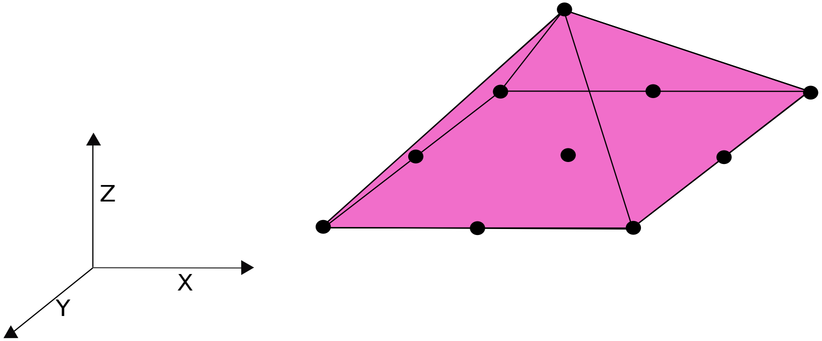
$SL_2(\mathbb{C})$ case

For Γ a trivalent graph of genus g with n marked points we define $P_\Gamma(L)$ to be the polytope given by non-negative integer weightings of the edges of Γ which satisfy the Quantum Clebsch-Gordon rules at each trinode with respect to level L.



Example: g= 2, n =0, L = 2





$SL_2(\mathbb{C})$ case

For Γ as above, we define $P_\Gamma(\vec{r}, L)$ to be the polyope given by integer weightings of Γ which satisfy the Clebsch-Gordon rules for L, with leaf weights specified to be $r_1, \ldots, r_n$.

Corollary [of the factorization rules] :

The dimension of $V_{C,\vec{p}}(\vec{r}, L)$ is equal to the number of lattice points in $P_\Gamma(\vec{r}, L)$

Note: it follows that the number of lattice points in $P_\Gamma(\vec{r}, L)$ is independent of the graph.

Multiplying conformal blocks

There is a natural multiplication operation:

$$P_\Gamma(\vec{r}, L) \times P_\Gamma(\vec{s}, K) \to P_\Gamma(\vec{r} + \vec{s}, L + K)$$

$$P_\Gamma(L) \times P_\Gamma(K) \to P_\Gamma(L + K)$$

Multiplying conformal blocks

This operation defines semigroups:

$$\mathcal{P}_\Gamma(\vec{r}, L) = \bigoplus_{N \geq 0} P_\Gamma(N\vec{r}, NL)$$

$$\mathcal{P}_\Gamma = \bigoplus_{N \geq 0} P_\Gamma(N)$$

with associated semigroup algebras $\mathbb{C}[\mathcal{P}_\Gamma(\vec{r}, L)]$, $\mathbb{C}[\mathcal{P}_\Gamma(L)]$.

Factorization in terms of commutative algebra

[M; Sturmfels, Xu] : For any C of genus g with n marked points, and Γ a trivalent graph with with first Betti number g and n leaves, there is a flat degeneration

$$Cox(\mathcal{M}_{C,\vec{p}}(SL_2(\mathbb{C})) \Rightarrow \mathbb{C}[\mathcal{P}_\Gamma]$$

$$R_{C,\vec{p}}(\vec{r}, L) \Rightarrow \mathbb{C}[\mathcal{P}_\Gamma(\vec{r}, L)]$$

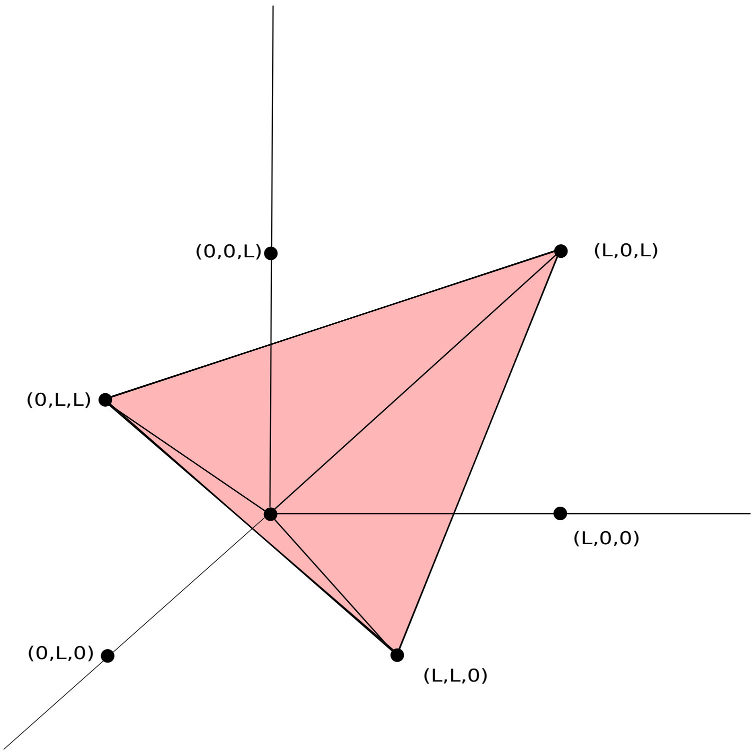
Factorization in terms of commutative algebra

For any C of genus g with n marked points, and Γ a trivalent graph with with first Betti number g and n leaves, there is a flat degeneration

$$Cox(\mathcal{M}_{C,\vec{p}}(G)) \Rightarrow [\bigotimes_{v \in V(\Gamma)} Cox(\mathcal{M}_{0,3}(G))]^{T_\Gamma}.$$

The $SL_2(\mathbb{C})$ case

$$Cox(\mathcal{M}_{0,3}(SL_2(\mathbb{C}))) = \mathbb{C}[P_3(1)]$$



$\frac{1}{2}$

The $SL_2(\mathbb{C})$ case

For any g, n, there is a special choice of Γ such that the semigroup algebra $\mathbb{C}[\mathcal{P}_\Gamma]$ is generated by elements of degree ≤ 2 and has relations generated by forms of degree ≤ 4.

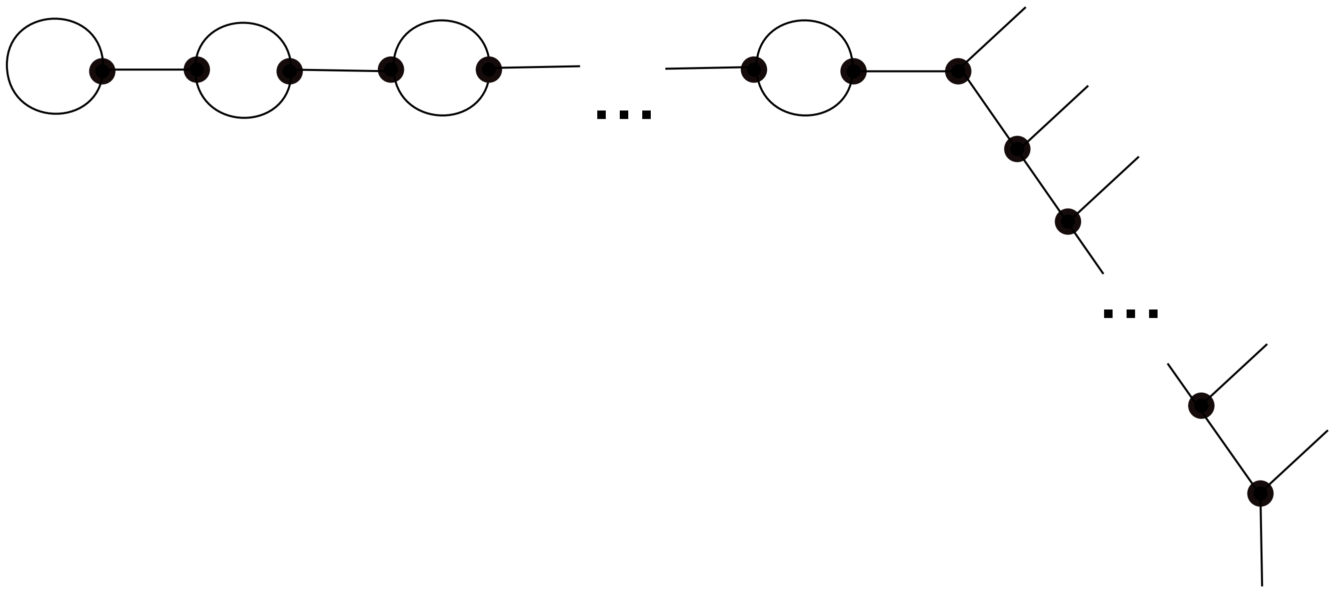
For any g, n, there is a special choice of Γ such that the semigroup algebra $\mathbb{C}[\mathcal{P}_\Gamma(2\vec{r}, 2L)]$ is generated by elements of degree 1, and its defining ideal of relations has a quadratic square-free Gröbner basis.

The $SL_2(\mathbb{C})$ case

For generic $(C, \vec{p})$, the algebra $Cox(\mathcal{M}_{C,\vec{p}}(SL_2(\mathbb{C})))$ is generated in degree ≤ 2 and has relations generated in degree ≤ 4

For generic $(C, \vec{p})$, the projective coordinate ring of $\mathcal{L}(\vec{r}, L)^{\otimes 2}$ is generated in degree 1 and is Koszul.

The special graph $\Gamma(g, n)$



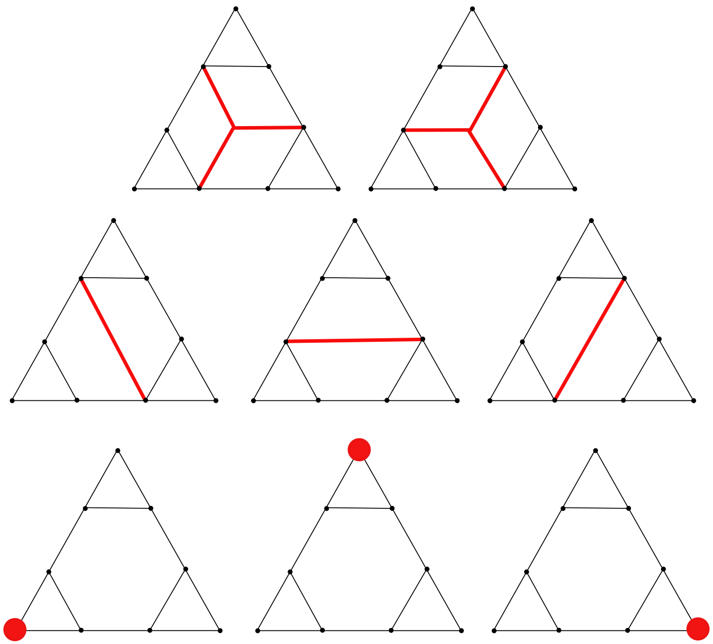
The $SL_2(\mathbb{C})$ case

Buscynska, Buscynski, Kubjas, and Michalek showed that $\mathbb{C}[\mathcal{P}_\Gamma]$ is always generated in degree $\leq g+1$.

The lift of these relations to the algebra $Cox(M_{C,\vec{p}}(SL_2(\mathbb{C})))$ was studied by Sturmfels and Xu, and also by Sturmfels and Velasco.

In particular, Sturmfels and Velasco have given a realization of $Cox(M_{C,\vec{p}}(SL_2(\mathbb{C})))$ as a quotient of the coordinate ring of an even spinor variety in the genus 0 case.

the $SL_3(\mathbb{C})$ case



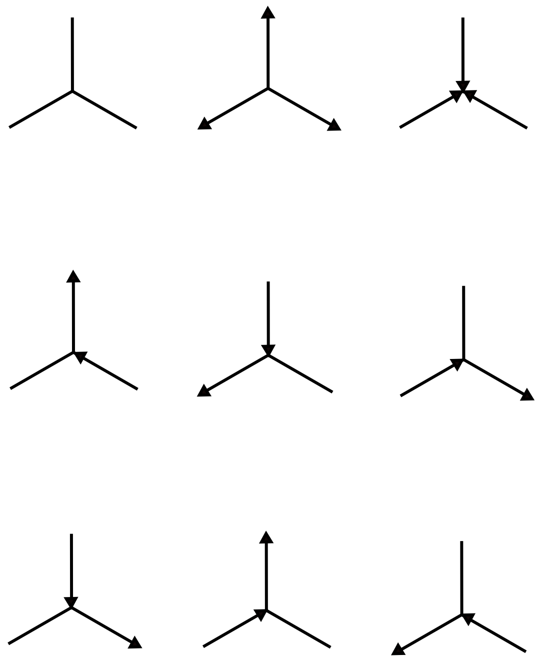
Toric degenerations also exist in the $SL_3(\mathbb{C})$ case, along with similar presentation results for $Cox(\mathcal{M}_{\mathbb{P},\vec{p}}(SL_3(\mathbb{C})))$.

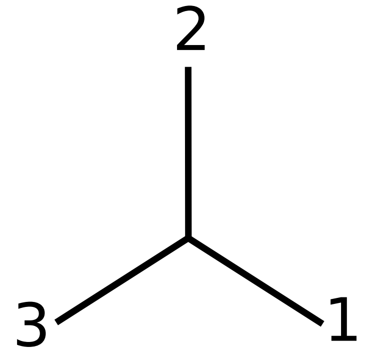
The $SL_3(\mathbb{C})$ case

Using valuations built out of representation theory data, it possible to find a presentation of $Cox(\mathcal{M}_{0,3}(SL_3(\mathbb{C})))$.

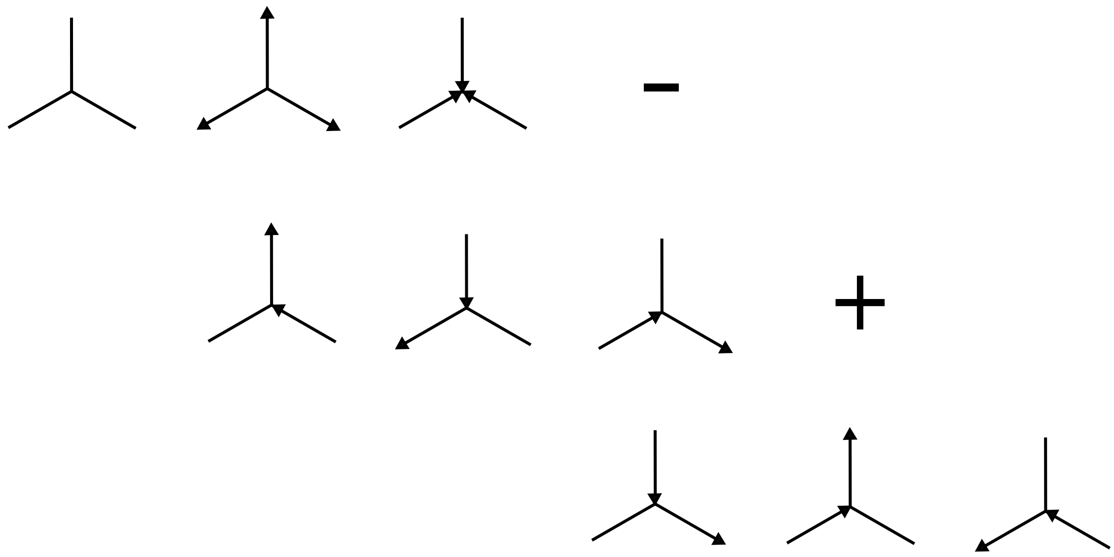
$$\mathbb{C}[Z, X, Y, P_{12}, P_{23}, P_{31}, P_{21}, P_{32}, P_{13}]/ < ZXY - P_{12}P_{23}P_{31} + P_{21}P_{32}P_{13} >$$

The algebra $Cox(\mathcal{M}_{0,3}(SL_3(\mathbb{C})))$

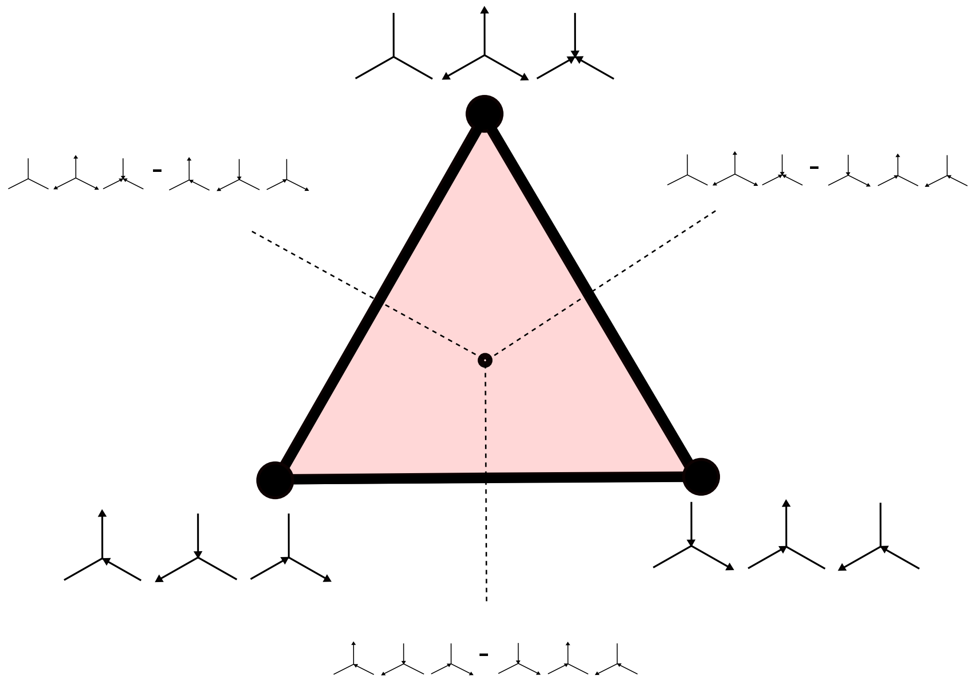




The algebra $Cox(\mathcal{M}_{0,3}(SL_3(\mathbb{C})))$

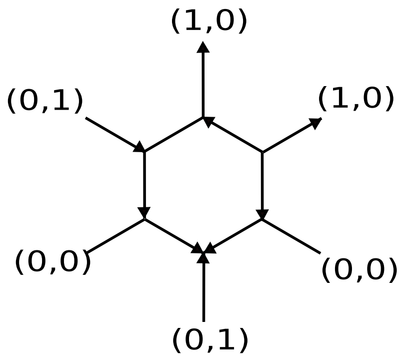


The $SL_3(\mathbb{C})$ case



The $SL_3(\mathbb{C})$ case

For any C of genus g with n marked points, and Γ a trivalent graph with with first Betti number g and n leaves, there are $3^{|V(\Gamma)|}$ toric degenerations of $Cox(\mathcal{M}_{C,\vec{p}}(SL_3(\mathbb{C})))$.



The $SL_3(\mathbb{C})$ case

For $(\mathbb{P}, \vec{p})$ generic, the algebra $Cox(\mathcal{M}_{\mathbb{P}^1, \vec{p}}(SL_3(\mathbb{C})))$ is generated in degree 1 with relations generated in degrees ≤ 3.

Polyhedral counting rules: $SL_3(\mathbb{C})$

$$\mathbb{V}_{0,3}(\lambda, \mu, \eta, L) = L - max\{\lambda_1 + \lambda_2, \mu_1 + \mu_2, \eta_1 + \eta_2, L_1, L_2\} + 1$$

$$L_1 = \frac{1}{3}(2(\lambda_1 + \mu_1 + \eta_1) + \lambda_2 + \mu_2 + \eta_2) - min\{\lambda_1, \mu_1, \eta_1\}$$

$$L_2 = \frac{1}{3}(2(\lambda_2 + \mu_2 + \eta_2) + \lambda_1 + \mu_1 + \eta_1) - min\{\lambda_2, \mu_2, \eta_2\}.$$

originally stated by Kirrilov, Mathieu, Senechal, Walton.

Thankyou!