Extensions of Toric Varieties

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Notation

Let S be a subsemigroup of $\mathbb{N}^d$ generated minimally by $\mathbf{m}_1, \ldots, \mathbf{m}_n$.

When $\mathbf{m} \in S$, we define $\delta(\mathbf{m})$ to be the minimum of all the sums

$s_1 + \cdots + s_n$ where $s_1, \ldots, s_n \in \mathbb{N}$ and $\mathbf{m} = s_1\mathbf{m}_1 + \cdots + s_n\mathbf{m}_n$.

I_S and V_S denotes the toric ideal and toric variety of S.

Extension

Denote by $S_{\ell,\mathbf{m}}$ the affine semigroup generated by $\ell\mathbf{m}_1, \ldots, \ell\mathbf{m}_n$ and $\mathbf{m}$, where ℓ is a positive integer. We say that the affine toric variety $V_{S_{\ell,\mathbf{m}}} \subset \mathbb{A}^{n+1}$ is an **extension** of $V_S \subset \mathbb{A}^n$, if $\mathbf{m} \in S$, and ℓ is a positive integer relatively prime to a component of $\mathbf{m}$. A projective toric variety $\overline{E} \subset \mathbb{P}^{n+1}$ will be called an **extension** of another one $\overline{X} \subset \mathbb{P}^n$ if its affine part E is an extension of an affine part X of $\overline{X}$.

Remarks

1. Notice that $I_S \subset I_{S_{\ell,\mathbf{m}}}$ and $I_{\overline{S}} \subset I_{\overline{S}_{\ell,\mathbf{m}}}$.
2. The question of whether or not $I_{S_{\ell,\mathbf{m}}}$ (resp. $I_{\overline{S}_{\ell,\mathbf{m}}}$) has a minimal generating set containing a minimal generating set of I_S (resp. $I_{\overline{S}}$) is not trivial.
3. This definition generalizes the one given for monomial curves by Arslan-Mete (2007) and Sahin (2009).
4. Thoma (1996) studied special extensions where $\ell = k\delta(\mathbf{m})$.
5. Morales (1991) used this idea to produce Noetherian symbolic blow-ups.

Extensions of a toric curve

Take $S = \mathbb{N}\{1, 4, 5\}$ and $V_S = \{(v, v^4, v^5) \mid v \in K\}$. If $\ell = 1$ and $\mathbf{m} = 10$ we have $V_{S_{1,10}} = \{(v, v^4, v^5, v^{10}) \mid v \in K\}$ where $S_{1,10} = \mathbb{N}\{1, 4, 5, 10\}$. Thus the projective closure of $V_{S_{1,10}}$ is a projective extension of the projective closure of V_S. But the relation between the semigroups is changed, as the new ones are just $\overline{S} = \mathbb{N}\{(5, 0), (4, 1), (1, 4), (0, 5)\}$ and

$$\overline{S}_{1,10} = \mathbb{N}\{(10, 0), (9, 1), (6, 4), (5, 5), (0, 10)\}.$$

Although $I_{S_{1,10}} = I_S + \langle x_3^2 - x_4 \rangle$, no minimal generating set of $I_{\overline{S}}$ extends to a minimal generating set of $I_{\overline{S}_{1,10}}$, since $\mu(I_{\overline{S}}) = \mu(I_{\overline{S}_{1,10}})(= 5)$.

Proposition

If the toric variety $V_{S_{\ell,\mathbf{m}}} \subset \mathbb{A}^{n+1}$ is an extension of $V_S \subset \mathbb{A}^n$, then $I_{S_{\ell,\mathbf{m}}} = I_S + \langle F \rangle$, where $F = x_{n+1}^{\ell} - x_1^{s_1} \cdots x_n^{s_n}$. Moreover, if $\mathcal{G}$ is a reduced Gröbner basis for I_S with respect to a term order $\succ$, then $\mathcal{G} \cup \{F\}$ is a reduced Gröbner basis for $I_{S_{\ell,\mathbf{m}}}$ with respect to a term order refining $\succ$ and making x_{n+1} the biggest variable.

Corollary

If $V_S \subset \mathbb{A}^n$ is a set theoretic complete intersection, arithmetically Cohen-Macaulay (Gorenstein), so are its extensions $V_{S_{\ell,\mathbf{m}}} \subset \mathbb{A}^{n+1}$.

BAD extensions are nicer globally!

Proposition

If $\mathcal{G}$ is a reduced Gröbner basis for $I_{\overline{S}}$ with respect to a term order $\succ$ making x_0 the smallest variable and $\ell \geqslant \delta(\mathbf{m})$, then $\mathcal{G} \cup \{F\}$ is a reduced Gröbner basis for $I_{\overline{S}_{\ell,\mathbf{m}}}$ with respect to a term order refining $\succ$ and making x_{n+1} the biggest variable and thus $I_{\overline{S}_{\ell,\mathbf{m}}} = I_{\overline{S}} + \langle F \rangle$, where
$F = x_{n+1}^{\ell} - x_0^{\ell-\delta(\mathbf{m})} x_1^{s_1} \cdots x_n^{s_n}$.

Corollary

If $V_{\overline{S}} \subset \mathbb{P}^n$ is a set theoretic complete intersection, arithmetically Cohen-Macaulay (Gorenstein), so are its extensions $V_{\overline{S}_{\ell,\mathbf{m}}} \subset \mathbb{P}^{n+1}$ provided that $\ell \geqslant \delta(\mathbf{m})$.

NICE extensions are nicer locally!

Proposition

If $\mathcal{G}$ is a minimal standard basis of I_S with respect to a negative degree reverse lexicographic ordering $\succ$ and $\ell \leqslant \delta(\mathbf{m})$, then $\mathcal{G} \cup \{F\}$ is a minimal standard basis of $I_{S_{\ell,\mathbf{m}}}$ with respect to a negative degree reverse lexicographic ordering refining $\succ$ and making x_{n+1} the biggest variable.

Since ${I_{S_{\ell,\mathbf{m}}}}^* = {I_S}^* + \langle F^* \rangle$ we have

Theorem

If $V_S \subset A^n$ has a Cohen-Macaulay tangent cone at 0, then so have its extensions $V_{S_{\ell,\mathbf{m}}} \subset A^{n+1}$, provided that $\ell \leqslant \delta(\mathbf{m})$.

Example

One can produce Cohen-Macaulay tangent cones using arithmetically Cohen-Macaulay projective toric varieties, since $I_S = {I_S}^*$. Therefore, all of their affine nice extensions will have Cohen-Macaulay tangent cones and local rings with non-decreasing Hilbert functions. The affine cone $V_S \subset \mathbb{A}^4$ over the twisted cubic with $S = \{(3,0),(2,1),(1,2),(0,3)\}$ and its nice extensions illustrate this point.

Proposition

If the local ring of $V_S \subset A^n$ is of homogeneous type, then its extensions will also have local rings of homogeneous type if and only if $\ell \leqslant \delta(\mathbf{m})$.

Example

Similarly, the local ring of the affine cone of a projective toric variety is always of homogeneous type, again by $I_S = {I_S}^*$ the betti numbers coincide. Thus, its affine nice extensions will have homogeneous type local rings which are not necessarily homogeneous. Take for example $S = \{(3,0),(2,1),(1,2),(0,3)\}$, $\ell = 1$ and $\mathbf{m} = (0, 3s)$ for any $s > 1$. Then, although $I_{S_{\ell,\mathbf{m}}} = I_S + \langle x_4^s - x_5 \rangle$ is not homogeneous, its local ring is of homogeneous type.

Theorem

If $V_S \subset A^n$ has a local ring with non-decreasing Hilbert function, then so have its extensions $V_{S_{\ell,\mathbf{m}}} \subset A^{n+1}$, provided that $\ell \leqslant \delta(\mathbf{m})$.

Example

If $S = \{(6,0),(0,2),(7,0),(6,4),(15,0)\}$,

$I_S = \langle x_1x_2^2 - x_4, x_3^3 - x_1x_5, x_1^5 - x_5^2\rangle$. Since $V_S \subset \mathbb{A}^5$ is a toric surface of codimension 3, I_S is a c.i. and thus the local ring of V_S is Gorenstein. But, ${I_S}^* = \langle x_5^2, x_4, x_3^3x_5, x_3^6, x_1x_5\rangle$ and thus the tangent cone at the origin is not Cohen-Macaulay. Nevertheless, its Hilbert function H_S is non-decreasing: $H_S(0) = 1, H_S(1) = 4, H_S(2) = 8, H_S(3) = 13, H_S(r) = 6r - 6,$ for $r \geqslant 4$.

All nice extensions of V_S will be toric surfaces with local rings of dimension 2 and embedding codimension 4 whose Hilbert functions are non-decreasing even though their tangent cones are not Cohen-Macaulay.